Transparent boundary conditions based on the pole condition

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Abstract

Transparent boundary conditions for polygonal two-dimensional domains based on the pole condition approach are presented. The discretization of the exterior is done by infinite trapezoids, which allows to define a generalized distance variable. Taking the Laplace transform of the solution w.r.t the distance variable, incoming and outgoing solutions can be distinguished by the location of the singularities. Using special ansatz and test functions, the condition on the location of the singularities yields a new algorithmic realization of transparent boundary conditions.

Introduction

For the simulation of wave propagation phenomena on unbounded domains it is common to introduce an artificial boundary. At this boundary transparent boundary conditions have to be specified.

In [1] we have presented the pole condition approach for a class of one-dimensional test problems. Here we show how to extend this ansatz to the two-dimensional case, with an artificial boundary enclosing a convex polygonal computational domain.

Our treatment of the boundary is based on the pole condition developed by F. Schmidt [2] in the early 90s. The algorithmic realization of the pole condition in [1] yields local and efficient transparent boundary conditions. In numerical experiments spectral convergence in the number of auxiliary boundary values is observed.

1 Problem class

We consider

$$p(\partial_t)u = \Delta u + d \cdot \nabla u - k^2 u, \text{ on } \mathbb{R}^2, \ t > 0, \qquad (1)$$

where $p(\partial_t) = i\partial_t$ (Schrödinger equation), $p(\partial_t) = \partial_t$ (heat/drift diffusion equation) or $p(\partial_t) = \partial_t^2$ (Klein-Gordon equation), with appropriate initial values. The restriction to two space dimensions is not essential. All what is required is a coordinate that measures the distance to the artificial boundary.

2 Exterior discretization

Suppose that the computational domain is enclosed by a polygonal convex artificial boundary. The exterior is then discretized by infinite trapezoids in such a way that there is a globally continuous parametrization of the distance variable ξ, i.e. for fixed ξ_0 the line at distance ξ_0 must be a closed curve, c.f. Fig. 1.

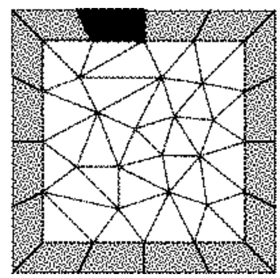

Figure 1: Interior discretization by triangles and exterior discretization by semi-infinite trapezoids.

By a bilinear map B each trapezoid is mapped to the unit infinite half-strip $[0,1]\times[0,\infty]$. In variational form the Laplace operator is transformed to

$$\int_{trapezoid} \nabla u \cdot \nabla \phi dx = \int_0^1 \int_0^\infty J^{-T}\nabla u \cdot J^{-T}\nabla\phi |J| d\xi\, d\hat{y}$$

and the mass matrix is

$$\int_{trapezoid} u\phi dx = \int_0^1 \int_0^\infty u\phi |J| d\xi\, d\hat{y}$$

where J is the Jacobian matrix of B, $|J|$ the determinant of J, and ϕ is a test function. Choosing

$$\phi_s(\hat{y}, \xi) = \phi(\hat{y}) \exp(-s\xi)$$

for some complex parameter $-s$, the integral over ξ yields the Laplace transform $U(s,t)$ of the solution $u(\xi, \hat{y})$ with respect to the distance variable. As J is linear in ξ and η each term in the stiffness and mass matrix can be transformed to the Laplace domain.

3 Pole condition

The pole condition states that a solution is outgoing if its Laplace transform $U(s,y,t)$ with respect to the distance variable is holomorphic in some half-plane. The choice of this half-plane depends on the type of equation. Details may be found in [1]. To turn the pole condition into an elegant and useful algorithm, a Möbius transform is used that maps the half-plane of the complex plane, where $U(s, \hat{y}, t)$ is analytic, onto the unit disc. Thus in the Möbius transformed coordinate

$$\tilde{s} = \frac{s + s_0}{s - s_0}$$

$U(\tilde{s}, \hat{y}, t)$ is a holomorphic function on the unit disc, which can be approximated by the following power-series expansion

$$U(\tilde{s}, \hat{y}, t) = (\tilde{s} - 1)\Big(\frac{u(x_0, \hat{y})}{2s_0} + (\tilde{s} - 1)\sum_{n\geq 0} a_n(\hat{y}, t)\tilde{s}^{\,n}\Big).$$

Here $u(x_0, \hat{y})$ is the Dirichlet boundary value of the interior solution at the artificial boundary. This way $U(s)$ has the correct behavior for large arguments s. It holds true that

$$\lim_{s\to\infty} sU(s) = u(x_0, \hat{y})$$

if $u(x_0, \hat{y})$ exists. Truncating the series expansion and matching moments equations for the coefficients $a_n(\hat{y}, t)$ can be deduced. The complex parameter s_0 defines the half-plane where U is holomorphic. An alternative way to obtain governing equations for the $a_n(\hat{y}, t)$ is discussed in [3].

4 Numerical example

Consider the Schrödinger equation

$$4i\partial_t u = \Delta u;\ 0 < t < 3$$

on the square $[-4, 4] \times [-4, 4]$ with initial data given by two Gaußian wave-packets traveling to the east boundary and the south-west corner, that hit the boundary at time $t \approx 0.5$ The parameter $s_0 = -1-i$. Space discretization is done by finite elements of degree 3. Time discretization is done by the trapezoidal rule. Fig. 2 shows the evolution of the l_2-error measured against the analytic reference solution for various numbers of expansion coefficients L.

Fig 3 shows the convergence in the number of expansion coefficients at different times t. This experiment indicates super algebraic convergence in the number of modes, as was already observed in the one-dimensional case [1].

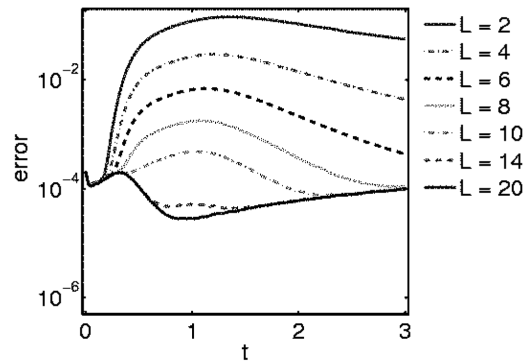

Figure 2: Evolution of the error

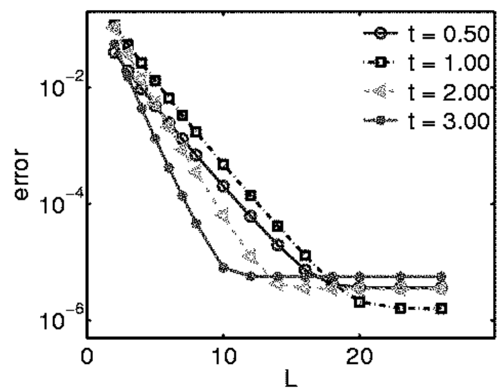

Figure 3: Convergence in the number Hardy modes.

References

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