

# A Model for Nonlinear Interactions of Internal Gravity Waves with Saturated Regions

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## 1 **Abstract**

2 A model for interactions between non-hydrostatic gravity waves and deep convective narrow  
3 hot towers is presented. The starting point of the derivation are the conservation laws for  
4 mass, momentum and energy for compressible flows combined with a bulk micro-physic  
5 model. Using multiscale asymptotics, a set of leading order equations is extracted, valid for  
6 the specific scales of the investigated regime. These are a timescale of 100 s, a horizontal  
7 and vertical lengthscale of 10 km for the wave dynamics plus a second horizontal lengthscale  
8 of 1 km for the narrow hot towers. Because of the comparatively short horizontal scales,  
9 Coriolis effects are negligible in this regime. The leading order equations are then closed  
10 by applying conditional averages over the hot tower lengthscale, leading to a closed model  
11 for the wave-scale that retains the net effects of the smaller scale dynamics. By assuming a  
12 systematically small saturation deficit in the ansatz, the small vertical displacements arising  
13 in this regime suffice to induce leading order changes of the saturated area fraction. The latter  
14 is the essential parameter in the model arising from the micro-physics.

## Zusammenfassung

Der Artikel präsentiert ein Modell zur Beschreibung der Wechselwirkung von nichthydrostatischen Schwerewellen mit hochreichenden Konvektionswolken. Der Ausgangspunkt der Herleitung sind die Erhaltungsgleichungen für Masse, Impuls und Energie für kompressible Strömungen, kombiniert mit einem „bulk“-Modell für die Mikrophysik. Mittels Mehrskalenasymptotik wird ein System von Gleichungen für die Dynamik führender Ordnung extrahiert, welches gerade für die spezifizierten Skalen Gültigkeit besitzt. Diese Skalen sind eine Zeitskala von 100 s, eine horizontale und vertikale Längenskala von 10 km für die Dynamik der Wellen sowie eine zweite horizontale Längenskala von 1 km für die Konvektionswolken. Aufgrund der verhältnismäßig kurzen horizontalen Längenskalen sind Coriolis-Effekte im betrachteten Regime vernachlässigbar. Das resultierende System wird durch bedingte Mittelung über die Längenskala der Konvektionswolken geschlossen und in ein Modell für die Wellenskala überführt, welches jedoch die effektiven Beiträge der kleinskaligen Dynamik beinhaltet. Im Ansatz wird ein systematisch kleines Sättigungsdefizit angenommen, so dass die im betrachteten Regime auftretenden kleinen vertikalen Auslenkungen zu Änderungen des Flächenanteils gesättigter Bereiche in führender Ordnung führen können. Dieser Flächenanteil ist der wesentliche aus der Mikrophysik abgeleitete Parameter.

## 32 **1 Introduction**

33 Interactions between internal gravity waves and moisture in the atmosphere give rise to several  
34 important effects. It is known that tropical deep moist convection generates gravity waves,  
35 see for example FOVELL et al. (1991); LANE et al. (2001), and that these waves yield a  
36 major contribution to overall wave-drag (KIM et al. (2003)). Parameterizations of this “gravity  
37 wave-drag” (see LINDZEN (1981)) are important for global circulation models to produce  
38 realistic flows. In this context, SURGI (1989) finds that using a stability frequency that  
39 includes a modification due to moisture yields improved results. On the other hand, patterns of  
40 convection are also strongly affected by gravity waves. EINAUDI and LALAS (1975); CHIMONAS  
41 et al. (1980) find that waves can trigger deep convection by helping to overcome convective  
42 inhibition. MAPES (1993) investigates mechanisms through which gravity waves contribute to  
43 the organization of convection.

44 Most studies investigating the interactions between gravity waves and moisture rely on the  
45 numerical solution of “full physics” models, usually the compressible or anelastic equations  
46 coupled with a bulk micro-physics model, see DURRAN and KLEMP (1983); CLARK et al.  
47 (1986); LANE et al. (2001); MIGLIETTA and ROTUNNO (2005) for example. However, because  
48 of the complexity of the involved models, it is very difficult to extract and understand the  
49 essential interaction mechanisms from the equations. For this purpose it is usually beneficial  
50 to study reduced models that describe only a certain subset of effects considered to be important  
51 while neglecting others. Because of the reduced complexity of the equations, such models are  
52 often much more accessible to mathematical analysis and can provide valuable insight into  
53 the essential dynamics of the studied subject. Approaches to study the interaction between  
54 gravity waves and moisture that feature simplified models can be found, for example, in  
55 EINAUDI and LALAS (1973), who introduce a model for gravity wave propagation in a saturated  
56 atmosphere, or in BARCILON et al. (1979, 1980); JUSEM and BARCILON (1985), who utilize

57 a switching mechanism between a dry and a reduced moist stability frequency depending on  
58 vertical displacement. The introduction of a reduced “effective stability” by moisture is also  
59 indicated in DURRAN and KLEMP (1982).

60 This paper presents a reduced model for nonlinear interactions between non-hydrostatic, non-  
61 rotating gravity waves and saturated areas in an atmosphere containing deep moist convective  
62 towers. It is an extension of the linear model derived and analyzed in RUPRECHT et al. (2010).  
63 Interesting results from the analysis of the linear model are, for example, the introduction of a  
64 lower cut-off horizontal wavenumber by moisture in addition to the well-known upper cut-off  
65 as well as a reduction of the modulus of the group velocity. Both papers rely on the ansatz  
66 introduced in KLEIN and MAJDA (2006), where the conservation laws for mass, momentum,  
67 and energy of compressible flows combined with a bulk micro-physics model constitute a set of  
68 governing equations from which reduced models for specific scales are derived using multiscale  
69 asymptotics. In contrast to RUPRECHT et al. (2010), where the saturated areas constitute a  
70 passive background modulating the propagation characteristics of gravity waves, the derivation  
71 in the present paper uses a slightly modified ansatz, leading to nonlinear coupling between the  
72 saturated area fraction and the wave-scale dynamics.

73 The final model is introduced and explained in subsection 1.1 while details of the derivation  
74 are presented later in section 2. The present paper provides only the derivation, leaving a  
75 thorough analysis of the model’s properties as well as numerical integrations for future work.

## 76 **1.1 Summary of the Model**

77 All equations below feature nondimensional quantities. Throughout this paper, superscripts  
78 indicate terms of a specific order in the employed expansion, for example  $\mathbf{u}^{(0)}$  is the order unity  
79 term in the expansion of horizontal velocity.

80 The prognostic variables in the final model are the horizontal velocity  $\mathbf{u}^{(0)}$ , the wave-scale  
81 vertical velocity  $\bar{w}^{(0)}$ , the wave-scale potential temperature  $\bar{\theta}$ , the function  $\pi = p^{(3)}/\rho^{(0)}$  where

82  $p^{(3)}$  is the pressure and  $\rho^{(0)}$  the background density, the conditional averages of the tower-scale  
 83 perturbations of vertical velocity and potential temperature  $w'$  and  $\theta'$  and finally the local vertical  
 84 displacement  $\xi_{\text{us}}$  and the area fraction of saturated regions  $\sigma$ . See section 2 for details on the  
 85 definition of the tower-scale quantities.  $\Theta_z^{(2)}$  is the moist-adiabatic background stratification.

**Linearized anelastic moist dynamics:**

$$\begin{aligned}
 & \mathbf{u}_\tau^{(0)} + \nabla_x \pi = 0 \\
 & \bar{w}_\tau^{(0)} + \pi_z = \bar{\theta}^{(3)} \\
 & \bar{\theta}_\tau^{(3)} + (1 - \sigma)\Theta_z^{(2)}\bar{w}^{(0)} = \Theta_z^{(2)}w' \\
 & \rho^{(0)}\nabla_x \cdot \mathbf{u}^{(0)} + \left(\rho^{(0)}\bar{w}^{(0)}\right)_z = 0
 \end{aligned} \tag{1.1}$$

**Averaged nonlinear tower-scale dynam-**

**ics:**

$$\begin{aligned}
 & w'_\tau + \frac{\sigma_\tau}{1 - \sigma}w' = \theta' \\
 & \theta'_\tau + \sigma\Theta_z^{(2)}w' + \frac{\sigma_\tau}{1 - \sigma}\theta' = \sigma(1 - \sigma)\Theta_z^{(2)}\bar{w}^{(0)} \\
 & \sigma_\tau = \xi_{\text{us},\tau}\Psi \\
 & \xi_{\text{us},\tau} = \bar{w}^{(0)} - \frac{w'}{1 - \sigma}
 \end{aligned} \tag{1.2}$$

88 The function  $\Psi$  characterizes the sensitivity of the saturated area fraction  $\sigma$  with respect to the  
 89 displacement  $\xi_{\text{us}}$ . From its definition (2.36) it follows that it is always positive. Hence upward  
 90 motion, that is  $\xi_{\text{us},\tau} > 0$ , increases the saturated area fraction while downward motion diminishes  
 91 it. The wave-scale equations (1.1) are the non-hydrostatic anelastic equations, linearized around  
 92 a background state with zero velocity and a stratification given by  $\Theta_z^{(2)}$ . However, the effective  
 93 stratification in (1.1)<sub>3</sub> is  $\Theta_z^{(2)}$  reduced by a factor of  $(1 - \sigma)$ . Further, the net wave-scale dynamics  
 94 result in a source term on the right hand side of (1.1)<sub>3</sub> related to release and consumption of latent  
 95 heat. Note that because  $\bar{w}^{(0)}$  appears on the right hand side of (1.2)<sub>2</sub> and (1.2)<sub>4</sub> there exists a

96 bi-directional coupling between wave- and tower-scale.

97 If  $\Psi \equiv 0$ , it follows from (1.2)<sub>2</sub> that  $\sigma = \text{const}$  and (1.1), (1.2) reduce to the model analyzed  
98 in RUPRECHT et al. (2010).

## 99 2 Model Derivation

100 The derivation relies on the techniques outlined in detail in KLEIN (2004, 2008, 2010) and in-  
101 volves five steps. First, characteristic length- and time scales of the analyzed regime have to be  
102 identified. Then, corresponding coordinates resolving these scales are introduced by rescaling a  
103 set of “universal” coordinates  $\mathbf{x}$ ,  $z$  and  $t$  by powers of the generic asymptotic expansion parame-  
104 ter  $\varepsilon$ , arising from the distinguished limit introduced in MAJDA and KLEIN (2003). Expansions  
105 in powers of  $\varepsilon$  are then inserted into the governing equations, which, following KLEIN and MA-  
106 JDA (2006), are the conservations laws for mass, momentum and energy for compressible flows  
107 combined with the bulk model for warm micro-physics (2.1) adopted from GRABOWSKI (1998).  
108 Collecting all leading order equations then results in a set of equations, which finally require  
109 some form of closure in order to obtain a closed set of equations.

$$\begin{aligned} q_{v,t} + \mathbf{u} \cdot \nabla_{\parallel} q_v + wq_{v,z} &= \\ & C_{\text{ev}} - C_{\text{d}} \\ q_{c,t} + \mathbf{u} \cdot \nabla_{\parallel} q_c + wq_{c,z} &= \\ & C_{\text{d}} - C_{\text{ac}} - C_{\text{cr}} \\ q_{r,t} + \mathbf{u} \cdot \nabla_{\parallel} q_r + wq_{r,z} + \frac{1}{\rho} (\rho q_r V_{\text{T}})_z &= \\ & C_{\text{ac}} + C_{\text{cr}} - C_{\text{ev}}. \end{aligned} \tag{2.1}$$

111 Here,  $q_v$ ,  $q_c$  and  $q_r$  denote the nondimensional mixing ratios of vapor, cloud water, and rain  
112 water while  $C_{\text{d}}$ ,  $C_{\text{ev}}$ ,  $C_{\text{ac}}$ ,  $C_{\text{cr}}$  represent the conversion between these species by conden-  
113 sation/evaporation of cloud water/vapor, evaporation of rain water, auto conversion of cloud

114 droplets into rain, and collection of cloud water by falling rain.  $V_T$  is the terminal velocity  
 115 of rain droplets.

## 116 2.1 Scales and Expansions

The derivation features a horizontal and vertical lengthscale of 10 km respectively, corresponding to the regime of non-rotating, non-hydrostatic internal waves, cf. chapter 8 in GILL (1982), as well as a second horizontal length scale of 1 km, corresponding to the order of magnitude of the diameter of deep convective hot towers indicated in LEMONE and ZIPSER (1980); STEVENS (2005). The employed time scale is 100 s, compatible with the typical value of the Brunt-Väisälä frequency of  $N \sim 0.01 \text{ s}^{-1}$  quoted in GILL (1982). Equation (2.2) summarizes the employed scales and introduces the corresponding coordinates resolving them.

$$\begin{aligned}
 \text{Horizontal: } & 10 \text{ km, } 1 \text{ km} & - & \mathbf{x}, \eta = \varepsilon^{-1} \mathbf{x} \\
 \text{Vertical: } & 10 \text{ km} & - & z \\
 \text{Time: } & 100 \text{ s} & - & \tau = \varepsilon^{-1} t
 \end{aligned} \tag{2.2}$$

The order of magnitude of vertical displacements resulting from (2.2)<sub>3</sub> and a reference velocity of  $10 \text{ m s}^{-1}$  is 1 km. Thus the nondimensional vertical displacement  $\xi$  is of the order

$$\xi \sim \frac{1 \text{ km}}{10 \text{ km}} = 0.1 = \mathcal{O}(\varepsilon). \tag{2.3}$$

Hence, as  $q_{vs,z} = \mathcal{O}(1)$ , the amount  $\delta q$  of released or consumed condensate by vertical displacements is also of the order

$$\delta q \sim \xi q_{vs,z} = \mathcal{O}(\varepsilon). \tag{2.4}$$

The saturation deficit is defined as the difference between the saturation mixing ratio  $q_{vs}$  and the mixing ratio of water vapor

$$\delta q_{vs} := q_{vs} - q_v. \tag{2.5}$$

In a regime with a saturation deficit  $\delta q$  of order unity, condensation or evaporation of  $\mathcal{O}(\varepsilon)$  amounts of vapor or cloud water cannot exert a leading order effect on the size of saturated regions, see the upper illustration in figure 1. This regime is investigated in RUPRECHT et al. (2010). There, the saturated area fraction  $\sigma$  is constant over time and acts as a wave-modulating background. In the present paper, a systematically small saturation deficit is assumed, that is the vapor mixing ratio  $q_v$  is expanded as

$$q_v(\eta, x, z, \tau) = q_{vs}^{(0)}(z) + \varepsilon q_v^{(1)}(\eta, x, z, \tau) + \mathcal{O}(\varepsilon^2), \quad (2.6)$$

whereas  $q_{vs}^{(0)}$  is the leading order term of the saturation mixing ratio. From (2.6) it follows that the saturation deficit is of the order

$$\delta q_{vs} = \mathcal{O}(\varepsilon), \quad (2.7)$$

117 so that now (2.4) suffices to induce a leading order change in the size of saturated regions, see  
 118 the lower illustration in figure 1 for a sketch of this regime. The expansions of the mixing ratios  
 119 of cloud water  $q_c$  and rain water  $q_r$  read

$$120 \quad q_{c/r}(\eta, x, z, \tau) = q_{c/r}^{(0)}(\eta, x, z, \tau) + \varepsilon q_{c/r}^{(1)}(\eta, x, z, \tau) + \mathcal{O}(\varepsilon^2). \quad (2.8)$$

121 The dynamical quantities are expanded as described in RUPRECHT et al. (2010), see appendix A  
 122 for a summary. Note that the horizontal velocity is assumed to be independent from  $\eta$  at leading  
 123 order.

All resulting leading order equations are split into equations for the tower-scale averages, defined by

$$\bar{\phi}(\mathbf{x}, z, \tau) := \lim_{\eta_0 \rightarrow \infty} \frac{\int_{[-\eta_0, \eta_0]^2} \phi(\eta, \mathbf{x}, z, \tau) d\eta}{|[-\eta_0, \eta_0]^2|} \quad (2.9)$$

and perturbations  $\tilde{\phi} := \phi - \bar{\phi}$ . Here,

$$[-\eta_0, \eta_0]^2 := \{(\eta_1, \eta_2) \in \mathbb{R}^2 : |\eta_1|, |\eta_2| \leq \eta_0\} \quad (2.10)$$

denotes the square around the origin with side lengths of  $2\eta_0$ . The resulting leading order equations for the averages read

$$\begin{aligned}
\mathbf{u}_\tau^{(0)} + \nabla_x \pi &= 0 \\
\bar{w}_\tau^{(0)} + \pi_z &= \bar{\theta}^{(3)} \\
\bar{\theta}_\tau^{(3)} + (1 - \overline{H_{q_v}}) \Theta_z^{(2)} \bar{w}^{(0)} &= \overline{H_{q_v} \tilde{w}^{(0)}} \Theta_z^{(2)} \\
\nabla_x \cdot (\rho^{(0)} \mathbf{u}^{(0)}) + (\rho^{(0)} \bar{w}^{(0)})_z &= 0.
\end{aligned} \tag{2.11}$$

The derivation of (2.11) is essentially identical to the one in RUPRECHT et al. (2010). As the present paper focusses on the modified micro-physics, the reader is referred there for details but some key-steps can be found in appendix A. The switching function in the present derivation is defined according to saturation not at leading order but at order  $\mathcal{O}(\varepsilon)$ , that is

$$H_{q_v} := \begin{cases} 1 : q_v^{(1)} = q_{vs}^{(1)} \\ 0 : q_v^{(1)} < q_{vs}^{(1)}. \end{cases} \tag{2.12}$$

124 Note that the large-scale equations (2.11) are essentially identical to the ones in RUPRECHT et al.  
125 (2010), the difference between the two models arises due to the different effective small-scale  
126 equations derived from the micro-physical model, which determine  $\overline{H_{q_v} \tilde{w}^{(0)}}$  on the right hand  
127 side of (2.11)<sub>3</sub>.

128 As the following derivations deal solely with the dynamics on the tower-scale resolved by  $\eta$ ,  
129 the wave-scale related arguments  $\mathbf{x}$  and  $z$  are omitted for the sake of a more compact notation.  
130 Unless explicitly mentioned otherwise, the occurring quantities can nevertheless depend on both  
131 wave-scale coordinates, too.

## 132 2.2 Leading Order Micro-Physics Equation

The leading order equations emerging from the micro-physics model (2.1) split into the case with saturation not only at order  $\mathcal{O}(1)$  but also at order  $\mathcal{O}(\varepsilon)$  and the weak under-saturated case.

**Regime I (saturation):**

$$q_v^{(1)} = q_{vs}^{(1)} \tag{2.13}$$

$$q_{c,\tau}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_\eta q_c^{(1)} = -w^{(0)} q_{vs,z}^{(0)}$$

**Regime II (weak under-saturation):**

$$q_{v,\tau}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_\eta q_v^{(1)} = -w^{(0)} q_{vs,z}^{(0)} \tag{2.14}$$

$$q_{c,\tau}^{(1)} + \mathbf{u}^{(0)} \cdot \nabla_\eta q_c^{(1)} = 0$$

The derivation exploits that  $q_{vs}^{(0)}, q_{vs}^{(1)}$  both depend on  $z$  only, cf. KLEIN and MAJDA (2006), and assumes  $q_c^{(0)} = q_r^{(0)} = 0$  at  $\tau = 0$ . Because of this assumption and  $\delta q_{vs}^{(0)} = 0$ , the terms  $C_{ev}^{(0)}$  and  $C_{cr}^{(0)}$  arising in RUPRECHT et al. (2010) and describing evaporation of rain water and collection of cloud water by falling rain vanish and do not occur here. Introduce the material derivative

$$D_\tau = \partial_\tau + \mathbf{u}^{(0)} \cdot \nabla_\eta, \tag{2.15}$$

the leading order vertical displacement  $\xi^{(0)}$  defined as the solution of

$$D_\tau \xi^{(0)} = w^{(0)} = \bar{w}^{(0)} + \tilde{w}^{(0)} \tag{2.16}$$

and the first order total water mixing ratio

$$q_T := q_{vs}^{(0)} + \varepsilon \left( q_v^{(1)} + q_c^{(1)} \right). \tag{2.17}$$

Letting

$$q_{T,0}(\eta) := q_T(\eta, 0) \tag{2.18}$$

denote some prescribed initial distribution of  $q_T$ , combining (2.13) and (2.14) and integrating in time yields

$$q_T(\eta, \tau) = q_{T,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau') - \varepsilon q_{vs,z}^{(0)} \xi^{(0)}(\eta, \tau), \tag{2.19}$$

133 exploiting that  $\mathbf{u}^{(0)}$  does not depend on  $\eta$  and using (2.16). Thus, for the employed scaling,

134 the leading order vapor mixing ratio  $q_{vs}^{(0)}$  in a rising parcel acts as an “infinite” reservoir from

135 which, in regime I,  $\mathcal{O}(\varepsilon)$  amounts of vapor condensate into cloud water  $q_c^{(1)}$  or, in regime II, are  
136 deposited into  $q_v^{(1)}$ . Once  $q_v^{(1)}$  reaches the saturation threshold  $q_{vs}^{(1)}$ , the regime switches from II  
137 to I and further ascend starts generating  $q_c^{(1)}$ . Vice versa, in a descending parcel,  $q_v^{(1)}$  and  $q_c^{(1)}$   
138 respectively are deposited back into  $q_{vs}^{(0)}$ . In this case, at first the cloud water evaporates and once  
139  $q_c^{(1)}$  is depleted, the regime switches from I to II and  $q_v^{(1)}$  starts to decrease.

140 The equations for the perturbations  $\tilde{w}^{(0)}$ ,  $\tilde{\theta}^{(3)}$  obtained by subtracting (2.11) from (A10) read

$$\begin{aligned}
& \tilde{w}_\tau^{(0)} + \nabla_\eta \cdot (\mathbf{u}^{(0)} \tilde{w}^{(0)}) = \tilde{\theta}^{(3)} \\
& \tilde{\theta}_\tau^{(3)} + \nabla_\eta \cdot (\mathbf{u}^{(0)} \tilde{\theta}^{(3)}) + \tilde{w}^{(0)} \Theta_z^{(2)} = \\
& H_{qv} (\bar{w}^{(0)} + \tilde{w}^{(0)}) \Theta_z^{(2)} \\
& - \overline{H_{qv} (\bar{w}^{(0)} + \tilde{w}^{(0)})} \Theta_z^{(2)},
\end{aligned} \tag{2.20}$$

142 utilizing that  $\mathbf{u}^{(0)}$  by assumption independent of  $\eta$ .

### 143 2.3 Displacement in Regime II

In regime I where  $H_{qv} = 1$ , (2.20)<sub>2</sub> becomes

$$\begin{aligned}
& \tilde{\theta}_\tau^{(3)} + \nabla_\eta \cdot (\mathbf{u}^{(0)} \tilde{\theta}^{(3)}) = \\
& \bar{w}^{(0)} \Theta_z^{(2)} - \overline{H_{qv} (\bar{w}^{(0)} + \tilde{w}^{(0)})} \Theta_z^{(2)},
\end{aligned} \tag{2.21}$$

while in regime II, that is for  $H_{qv} = 0$ , it reads

$$\begin{aligned}
& \tilde{\theta}_\tau^{(3)} + \nabla_\eta \cdot (\mathbf{u}^{(0)} \tilde{\theta}^{(3)}) + \tilde{w}^{(0)} \Theta_z^{(2)} = \\
& - \overline{H_{qv} (\bar{w}^{(0)} + \tilde{w}^{(0)})} \Theta_z^{(2)}.
\end{aligned} \tag{2.22}$$

Thus,  $\tilde{w}^{(0)}$  and  $\tilde{\theta}^{(3)}$  evolve purely due to large-scale forcing in regime I but oscillate with frequency  $\sqrt{\Theta_z^{(2)}}$  in regime II. Assuming that  $\tilde{w}^{(0)}$  and  $\tilde{\theta}^{(3)}$  are initially constant on the  $\eta$ -scale in regime II, then at  $\tau > 0$  they are constant in all areas that never belonged to regime I up to this

time as identical oscillations with frequency  $\sqrt{\Theta_z^{(2)}}$  are performed at every point. Hence

$$\begin{aligned} \nabla_\eta \tilde{w}^{(0)}(\eta, \tau) &= \nabla_\eta \tilde{\theta}^{(3)}(\eta, \tau) = 0 \\ \text{if } H_{q_v}(\eta, \tau') &= 0 \quad \forall \tau' \leq \tau. \end{aligned} \tag{2.23}$$

Note that  $\sqrt{\Theta_z^{(2)}}$  is the maximum frequency of waves in the large scale system (2.11), so (2.20) describes small-scale and high frequency oscillations. If at some point  $\eta$  the regime switches from I to II,  $\tilde{w}^{(0)}$  and  $\tilde{\theta}^{(3)}$  start oscillating with initial values equal to the final values in the regime I dynamics. As the interface between regime I and II moves, spatial variations in  $\tilde{w}^{(0)}$ ,  $\tilde{\theta}^{(3)}$  are generated in formerly saturated regions that become under-saturated, even for constant initial values. Now the ad-hoc assumption is made that these small-scale, high frequency oscillations can be neglected, that is

$$\tilde{w}^{(0)}(\eta, \tau) \equiv \tilde{w}_{\text{us}}(\tau), \quad \tilde{\theta}^{(3)}(\eta, \tau) \equiv \tilde{\theta}_{\text{us}}(\tau) \tag{2.24}$$

144 in regime II, whereas  $\tilde{w}_{\text{us}}$ ,  $\tilde{\theta}_{\text{us}}$  are the constant values in all the time under-saturated areas. See  
 145 figure 2 for a sketch. This approximation is motivated not by meteorological considerations but  
 146 rather by the resulting mathematical simplifications, see (2.35), that allow to close the model  
 147 by analytical means. An improved representation presumably still allowing for an analytically  
 148 computed closure could, for example, attempt to capture the excited oscillation by adding a  
 149 term  $A_{1/2} \exp(i\sqrt{\Theta_z^{(2)}}\tau)$  to  $\tilde{w}_{\text{us}}$  and  $\tilde{\theta}_{\text{us}}$  in (2.24), where  $A_{1/2}$  are determined by the time of  
 150 the regime switch. Further, the comparatively simple structure of (2.21) and (2.22) might even  
 151 allow to employ analytical solutions in the closure. The present derivation, however, utilizes  
 152 the simple approximation (2.24) and improvements of the representation of the  $\tilde{w}^{(0)}$  and  $\tilde{\theta}^{(3)}$  in  
 153 under-saturated regions are left for future work. Note that, aside from the adopted asymptotic  
 154 limit, (2.24) is the only ad-hoc assumption used in the derivation.

Define the displacement in under-saturated areas  $\xi_{\text{us}}$  by replacing  $\tilde{w}^{(0)}$  by  $\tilde{w}_{\text{us}}$  in (2.16), that

is

$$D_\tau \xi_{\text{us}}(\tau) = \partial_\tau \xi_{\text{us}}(\tau) = \bar{w}^{(0)}(\tau) + \tilde{w}_{\text{us}}(\tau). \quad (2.25)$$

Introduce the function

$$q_*(\eta, \tau) := \varepsilon^{-1} \left[ q_{\text{T}}(\eta, \tau) - \left( q_{\text{vs}}^{(0)} + \varepsilon q_{\text{vs}}^{(1)} \right) \right]. \quad (2.26)$$

The initial distribution  $q_{\text{T},0}$  prescribes the initial distribution of  $q_*$  via

$$q_{*,0} := q_*(\eta, 0) = \varepsilon^{-1} \left[ q_{\text{T},0}(\eta) - \left( q_{\text{vs}}^{(0)} + \varepsilon q_{\text{vs}}^{(1)} \right) \right]. \quad (2.27)$$

It follows from (2.17) that

$$q_* = \begin{cases} q_{\text{e}}^{(1)} & : H_{\text{qv}} = 1 \\ q_{\text{v}}^{(1)} - q_{\text{vs}}^{(1)} & : H_{\text{qv}} = 0, \end{cases} \quad (2.28)$$

hence  $q_*$  is positive in regime I and negative in regime II. As  $q_{\text{vs}}^{(0)}$ ,  $q_{\text{vs}}^{(1)}$  depend on  $z$  only, it follows from (2.19) that when employing approximation (2.24),  $q_*$  is given by

$$q_*(\eta, \tau) = q_{*,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau') - q_{\text{vs},z}^{(0)} \xi_{\text{us}}(\tau). \quad (2.29)$$

Further, we have

$$\nabla_\eta q_*(\eta, \tau) = \nabla_\eta q_{*,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau'), \quad (2.30)$$

155 hence the shape of  $q_*$  is not altered. It is horizontally advected by  $\mathbf{u}^{(0)}$  and increased or reduced  
 156 by  $\eta$ -independent values  $q_{\text{vs},z}^{(0)} \xi_{\text{us}}$ , see figure 3. In order to close the model, we employ  $q_*$  as a  
 157 level set function approximately tracking saturated and non-saturated regions. This is outlined in  
 158 subsection 2.4.

## 159 2.4 Closing the Model

Denote by  $A_i$  an individual saturated spot, that is a connected set of points in the  $\eta$ -plane for which  $q_*$  is positive in the interior and zero at the boundary. Further, denote the boundary by  $\partial A_i$ , the velocity at which  $\partial A_i$  is moving by  $\mathbf{v}_i$  and finally the outward pointing normal vector

on  $\partial A_i$  by  $\mathbf{n}$ , see figure 4. Then the evolution equation for the level set function tracking the boundary, cf. WILLIAMS (1985); OSHER and FEDKIW (2002), reads

$$\partial_\tau q_* + \mathbf{v}_i \cdot \nabla_\eta q_* = 0. \quad (2.31)$$

Using (2.29) it follows that

$$\left( \mathbf{u}^{(0)} - \mathbf{v}_i \right) \cdot \nabla_\eta q_* = -q_{vs,z}^{(0)} \left( \bar{w}^{(0)} + \tilde{w}_{us} \right). \quad (2.32)$$

Note that because  $\partial A_i$  is a contour line of  $q_*$  and  $q_*$  is positive inside  $A_i$  and negative outside,  $\mathbf{n}$  points in the opposite direction as  $\nabla_\eta q_*$ , hence

$$\mathbf{n} = -\frac{\nabla_\eta q_*}{|\nabla_\eta q_*|}. \quad (2.33)$$

By employing (2.30) and (2.33), (2.32) becomes

$$\mathbf{v}_i \cdot \mathbf{n} = \mathbf{u}^{(0)} \cdot \mathbf{n} - q_{vs,z}^{(0)} \frac{\bar{w}^{(0)} + \tilde{w}_{us}}{|\nabla_\eta q_{*,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau')|}. \quad (2.34)$$

The boundary  $\partial A_i$  is characterized by the condition

$$q_*(\eta, \tau) = 0 \Leftrightarrow q_{*,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau') = q_{vs,z}^{(0)} \xi_{us}(\tau), \quad (2.35)$$

cf. (2.29), and is thus fully determined by the displacement and the initial distribution  $q_{*,0}$ . This is the important simplification obtained by introducing approximation (2.24). For given  $q_{*,0}$ , define

$$\begin{aligned} \Psi_i(\xi_{us}, \tau) &:= \oint_{\partial A_i} \mathbf{v}_i \cdot \mathbf{n} dS \\ &= -q_{vs,z}^{(0)} \oint_{\partial A_i} \frac{1}{|\nabla_\eta q_{*,0}(\eta - \int_0^\tau \mathbf{u}^{(0)} d\tau')|} dS, \end{aligned} \quad (2.36)$$

using (2.34) and that  $\mathbf{u}^{(0)}$  is independent of  $\eta$ , hence the surface integral over  $\mathbf{u}^{(0)} \cdot \mathbf{n}$  vanishes. Note that for a fixed value of  $\xi_{us}$ , it is  $\Psi_i(\xi_{us}, \tau) = \Psi_i(\xi_{us}, 0)$ , because the saturated spot  $A_i$  as well as  $q_*$  are simply advected horizontally with  $\mathbf{u}^{(0)}$ . The function  $\Psi_i$  determines how sensitive the size of the saturated spot depends on the displacement  $\xi_{us}$ . Steep gradients of  $q_*$  lead to a

weak sensitivity while small gradients result in a strong dependence of  $|A_i|$  on  $\xi_{\text{us}}$ , cf. figure 3. In the limit  $|\nabla_{\eta} q_*| \rightarrow \infty$ , the coupling vanishes and the linear model from RUPRECHT et al. (2010) is retrieved. Note that the saturation mixing ratio decreases with height, hence  $q_{\text{vs},z}^{(0)} < 0$ , so that

$$\Psi_i(\xi_{\text{us}}, \tau) \geq 0 \quad (2.37)$$

160 holds for any value of  $\xi_{\text{us}}$  or  $\tau$  as well as any initial distribution  $q_{*,0}$ .

By integrating (2.20) over a saturated spot  $A_i$  moving at velocity  $\mathbf{v}_i$ , the following balances are obtained

$$\begin{aligned} & \frac{\partial}{\partial \tau} \int_{A_i} \tilde{w} d\eta + \oint_{\partial A_i} \tilde{w} (\mathbf{u}^{(0)} - \mathbf{v}_i) \cdot \mathbf{n} dS = \int_{A_i} \tilde{\theta} d\eta \\ & \frac{\partial}{\partial \tau} \int_{A_i} \tilde{\theta} d\eta + \oint_{\partial A_i} \tilde{\theta} (\mathbf{u}^{(0)} - \mathbf{v}_i) \cdot \mathbf{n} dS + \Theta_z^{(2)} \int_{A_i} \tilde{w} d\eta \\ & = \Theta_z^{(2)} \bar{w}^{(0)} \int_{A_i} (H_{\text{qv}} - \overline{H_{\text{qv}}}) d\eta \\ & + \Theta_z^{(2)} \int_{A_i} (H_{\text{qv}} \tilde{w}^{(0)} - \overline{H_{\text{qv}} \tilde{w}^{(0)}}) d\eta. \end{aligned} \quad (2.38)$$

see for example THOMAS and LOMBARD (1979). Now, using (2.24) and (2.34) yields

$$\begin{aligned} & \partial_{\tau} \int_{A_i} \tilde{w}^{(0)} d\eta - \tilde{w}_{\text{us}} (\bar{w}^{(0)} + \tilde{w}_{\text{us}}) \Psi_i = \int_{A_i} \tilde{\theta}^{(3)} \\ & \partial_{\tau} \int_{A_i} \tilde{\theta}^{(3)} d\eta - \tilde{\theta}_{\text{us}} (\bar{w}^{(0)} + \tilde{w}_{\text{us}}) \Psi_i + \Theta_z^{(2)} \int_{A_i} \tilde{w}^{(0)} d\eta \\ & = \Theta_z^{(2)} \bar{w}^{(0)} \int_{A_i} (H_{\text{qv}} - \overline{H_{\text{qv}}}) d\eta \\ & + \Theta_z^{(2)} \int_{A_i} (H_{\text{qv}} \tilde{w}^{(0)} - \overline{H_{\text{qv}} \tilde{w}^{(0)}}) d\eta, \end{aligned} \quad (2.39)$$

exploiting again that  $\mathbf{u}^{(0)}$  is independent from  $\eta$ . Let  $D(\eta_0) = [-\eta_0, \eta_0]^2$  denote a square containing a finite number of saturated spots  $A_1, \dots, A_{n(\eta_0)}$ , cf. figure 5. Introduce the weighted averages

$$\begin{aligned}
w' &:= \overline{H_{q_v} \tilde{w}^{(0)}} \\
\theta' &:= \overline{H_{q_v} \tilde{\theta}^{(3)}} \\
\sigma &:= \overline{H_{q_v}} \\
\Psi &:= \lim_{\eta_0 \rightarrow \infty} \frac{\sum_{i=1}^{n(\eta_0)} \Psi_i}{\int_{D(\eta_0)} \mathbf{1} d\eta}.
\end{aligned} \tag{2.40}$$

Note that because the spots  $A_i$  are advected with  $\mathbf{u}^{(0)}$ , the choice of spots located in a finite square  $D(\eta_0)$  changes with time. Hence the series in (2.40)<sub>4</sub> is rearranged depending on  $\tau$ . Assuming that it converges absolutely, however, for a given  $\xi_{\text{us}}$ , the limit  $\Psi$  of every rearrangement is the same and thus independent of time. Because according to (2.37) all terms of the series are positive, absolute convergence immediately follows from convergence. Sum up the balances (2.39), use that  $H_{q_v}$  is the characteristic function of the union of all  $A_i$ , hence

$$\sum_{i=1}^{n(\eta_0)} \int_{A_i} f d\eta = \int_{D(\eta_0)} H_{q_v} f d\eta \tag{2.41}$$

for any function  $f$ , and then apply the limit  $\eta_0 \rightarrow \infty$  to obtain

$$\begin{aligned}
w'_\tau - \tilde{w}_{\text{us}} \left( \bar{w}^{(0)} + \tilde{w}_{\text{us}} \right) \Psi &= \theta' \\
\theta'_\tau - \tilde{\theta}_{\text{us}} \left( \bar{w}^{(0)} + \tilde{w}_{\text{us}} \right) \Psi + \sigma \Theta_z^{(2)} w' & \\
= \sigma (1 - \sigma) \Theta_z^{(2)} \bar{w}^{(0)}. &
\end{aligned} \tag{2.42}$$

An expression for the evolution of  $\sigma$  can be derived from (2.34) in a similar way, employing (2.41) with  $f \equiv 1$  to get

$$\partial_\tau \sigma = \left( \bar{w}^{(0)} + \tilde{w}_{\text{us}} \right) \Psi. \tag{2.43}$$

161 Finally, the constant velocity  $\tilde{w}_{\text{us}}$  can be computed from  $w'$  via

$$\begin{aligned}
\tilde{w}_{\text{us}} &= \lim_{\eta_0 \rightarrow \infty} \frac{\int_{D(\eta_0)} (1 - H_{\text{qv}}) \tilde{w}^{(0)} d\eta}{\int_{D(\eta_0)} (1 - H_{\text{qv}}) d\eta} \\
&= \left( \bar{\tilde{w}}^{(0)} - w' \right) \frac{1}{1 - \sigma} \\
&= -w' \frac{1}{1 - \sigma}.
\end{aligned}
\tag{2.44}$$

The final model, consisting of (1.1) and (1.2), is obtained by combining (2.25), (2.42) and (2.43)

with (2.44).

### 3 Summary

This paper presents a reduced model for interactions of non-hydrostatic, non-rotating gravity waves with saturated areas in tropical deep convective clouds. The derivation features two horizontal length-scales, one related to the wave-dynamics and one related to the typical diameter of the hot towers. The conservation laws for mass, momentum and energy for compressible flows together with a bulk micro-physics model for vapor, cloud water and rain water mixing ratios are employed as governing equations. From these, using multiscale asymptotics, a set of leading order equations for the specific length- and timescales of the investigated regime is devised. This set of leading order equations is then turned into a closed model for the wave-scale dynamics by applying conditional averages over the tower-scale, eliminating the explicit dependence on the small scale coordinate while retaining the net effects of the micro-physics on the larger scale. A level-set approach is employed in the closure to track the growing and shrinking saturated regions. The essential moisture-related parameter in the final model is the saturated area fraction in horizontal slices on the lengthscale of the convective clouds. It evolves according to the vertical displacement generated by the wave-scale and net micro-scale vertical velocity. However, one ad-hoc approximation is introduced in the closure procedure at this stage: Tower-scale gradients of vertical velocity and potential temperature are neglected that arise in

182 areas becoming saturated and under-saturated again. The scope of this paper is the derivation of  
 183 the model and a detailed analysis is left for future work.

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 188 and Gravity Waves’’.

## 189 **A Key-Steps in Derivation**

190 This appendix briefly repeats the essential intermediate steps arising in the derivation of the  
 191 equations for the dynamical quantities. First, the governing non-dimensional equations are  
 192 quoted, then the employed asymptotic expansions of the dynamical quantities are presented and  
 193 finally the leading order dynamical equations are given. Additional details on the governing  
 194 equations, the expansions and the key-steps can be found in KLEIN and MAJDA (2006);  
 195 RUPRECHT et al. (2010); RUPRECHT (2010).

### 196 **A.1 Nondimensional Governing Equations**

The non-dimensional conservation laws for mass, momentum, energy (expressed as potential temperature), quoting from KLEIN and MAJDA (2006), read

$$\begin{aligned}
 \rho_t + \nabla_{||} \cdot (\rho \mathbf{u}) + (\rho w)_z &= 0 \\
 \mathbf{u}_t + \mathbf{u} \cdot \nabla_{||} \mathbf{u} + w \mathbf{u}_z + \epsilon f (\Omega \times \mathbf{v})_{||} + \epsilon^{-4} \rho^{-1} \nabla_{||} p &= 0 \\
 w_t + \mathbf{u} \cdot \nabla_{||} w + w w_z + \epsilon f (\Omega \times \mathbf{v})_{\perp} + \epsilon^{-4} \rho^{-1} p_z &= -\epsilon^{-4} \\
 \theta_t + \mathbf{u} \cdot \nabla_{||} \theta + w \theta_z &= \epsilon^2 \left( \tilde{S}_{\theta}^{\epsilon} + S_{\theta}^{q,\epsilon} \right)
 \end{aligned} \tag{A1}$$

where

$$S_{\theta}^{q,\epsilon} = \Gamma^{**} L^{**} q_{vs}^{**} \frac{\theta}{p} \left( \epsilon^{-n} \hat{C}_d - \hat{C}_{ev} \right) \quad (\text{A2})$$

is the source term related to evaporation and condensation, while  $\tilde{S}_{\theta}^{\epsilon}$  is a given external source of energy like, for example, radiation. The latter is set to zero in the present derivation. Note that as the derivation features two horizontal coordinates, the gradient transforms like

$$\nabla_{||} \mapsto \nabla_{\mathbf{x}} + \epsilon^{-1} \nabla_{\eta}. \quad (\text{A3})$$

## 197 A.2 Asymptotic Expansions

The vertical velocity is expanded as

$$w(\mathbf{x}, z, t; \epsilon) = w^{(0)}(\eta, \mathbf{x}, z, \tau) + \mathcal{O}(\epsilon), \quad (\text{A4})$$

while the horizontal velocity is assumed to be independent from the micro-scale coordinate at leading order and therefore expanded as

$$u(\mathbf{x}, z, t; \epsilon) = \mathbf{u}^{(0)}(\mathbf{x}, z, \tau) + \mathcal{O}(\epsilon). \quad (\text{A5})$$

The potential temperature is expanded around a background stratification  $\bar{\theta}(z) = 1 + \epsilon^2 \Theta^{(2)}(z)$  as

$$\begin{aligned} \theta(\mathbf{x}, z, t; \epsilon) &= 1 + \epsilon^2 \Theta^{(2)}(z) \\ &+ \epsilon^3 \theta^{(3)}(\eta, \mathbf{x}, z, \tau) + \mathcal{O}(\epsilon^4). \end{aligned} \quad (\text{A6})$$

Finally, pressure and density are expanded as

$$\begin{aligned} (p, \rho)(\mathbf{x}, z, t; \epsilon) &= (p^{(0)}, \rho^{(0)})(z) + \epsilon (p^{(1)}, \rho^{(1)})(z) \\ &+ \epsilon^2 (p^{(2)}, \rho^{(2)})(z) \\ &+ \epsilon^3 (p^{(3)}, \rho^{(3)})(\eta, \mathbf{x}, z, \tau) + \mathcal{O}(\epsilon^4). \end{aligned} \quad (\text{A7})$$

After a few transformations employing the key equations, cf. the appendix in RUPRECHT et al. (2010),

$$\nabla_{\eta} p^{(3)} = 0 \quad (\text{A8})$$

and

$$\hat{L}H_{\text{qv}}C_{\text{d}}^{(0)} = H_{\text{qv}} \left( \bar{w}^{(0)} + \tilde{w}^{(0)} \right) \Theta_z^{(2)}, \quad (\text{A9})$$

whereas  $\hat{L}$  incorporates a number of  $\mathcal{O}(1)$  constants arising during the non-dimensionalization, the leading order equations for the dynamic quantities  $\mathbf{u}$ ,  $w$ ,  $\theta$ ,  $p$  and  $\rho$  read

$$\begin{aligned} \mathbf{u}_{\tau}^{(0)} + \nabla_x \pi + \nabla_{\eta} \left( p^{(4)} / \rho^{(0)} \right) &= 0 \\ w_{\tau}^{(0)} + \mathbf{u}^{(0)} \cdot \nabla_{\eta} w^{(0)} + \pi_z &= \theta^{(3)} \\ \rho^{(0)} \nabla_{\eta} \cdot \mathbf{u}^{(1)} + \rho^{(0)} \nabla_x \cdot \mathbf{u}^{(0)} + \left( \rho^{(0)} w^{(0)} \right)_z &= 0 \\ \theta_{\tau}^{(3)} + \mathbf{u}^{(0)} \cdot \nabla_{\eta} \theta^{(3)} + w^{(0)} \Theta_z^{(2)} & \\ = H_{\text{qv}} \left( \bar{w}^{(0)} + \tilde{w}^{(0)} \right) \Theta_z^{(2)}. & \end{aligned} \quad (\text{A10})$$

Note that the term  $C_{\text{ev}}^{(0)}$  occurring in RUPRECHT et al. (2010), describing cooling by evaporation rain water, vanishes here because saturation at leading order is assumed. The  $\eta$ -gradient of  $p^{(4)}$  and the  $\eta$ -divergence of  $\mathbf{u}^{(1)}$  vanish by applying a sublinear growth condition. Averaging over  $\eta$  then yields (1.1).

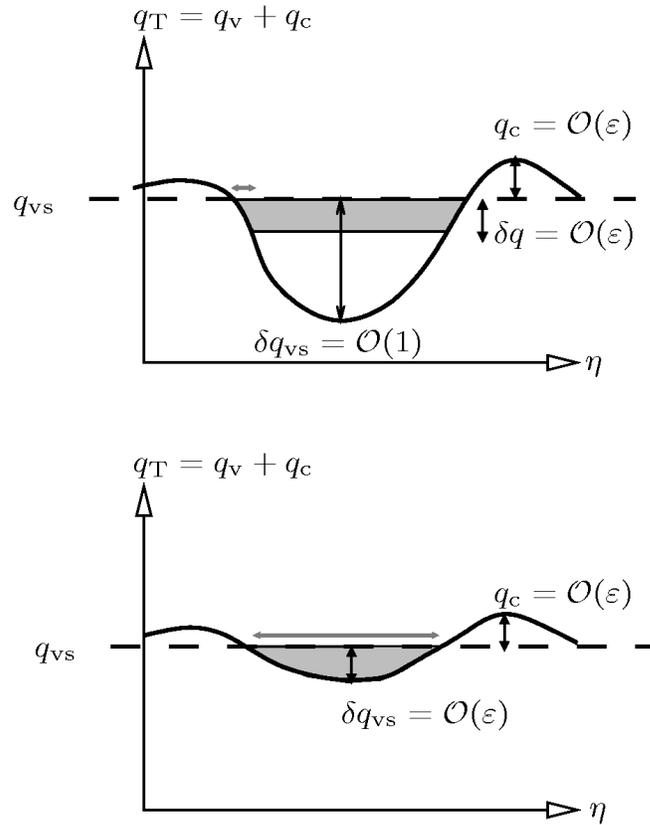
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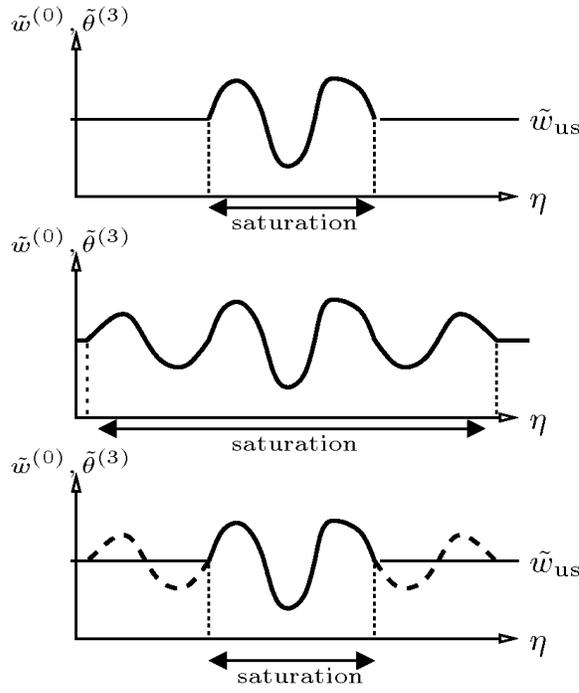
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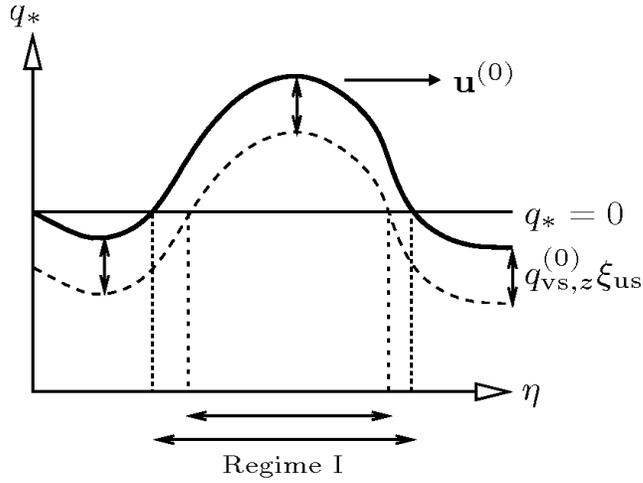
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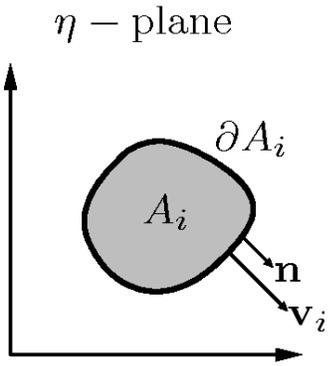
**Figure 1:** Example distribution of total water  $q_T = q_v + q_c$ , that is the sum of vapor and cloud water mixing ratio (thick solid line). The grey areas indicate the  $\mathcal{O}(\varepsilon)$  amount of condensate released by the small vertical displacements allowed for by the employed short time scale, cf. (2.4). The dashed horizontal line denotes the saturation mixing ratio. For a saturation deficit  $\delta q_{vs}$  at leading order (upper), there is only a small change in the size of the saturated area (indicated by grey double arrows), while the change is of order unity in the case of a systematically small saturation deficit (lower). In both cases, the mixing ratio of cloud water in saturated areas, that is the difference  $q_T - q_{vs}$ , is small and of order  $\mathcal{O}(\varepsilon)$ .



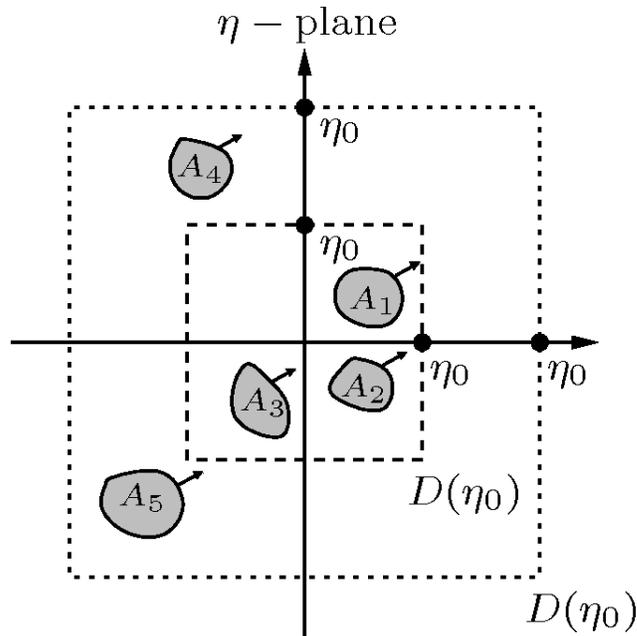
**Figure 2:** General structure of the perturbation vertical velocity  $\tilde{w}^{(0)}$  and potential temperature  $\tilde{\theta}^{(3)}$  on the  $\eta$ -coordinate. Initially, variations are present only in the saturated area, i.e. in regime I (upper). As the saturated area enlarges, variations can arise in the now saturated areas (middle). As the saturated region shrinks again, variations in the again under-saturated areas (dashed line) are neglected and the constant value  $\tilde{w}_{us}$  or  $\tilde{\theta}_{us}$  is employed (lower).



**Figure 3:** The level set function  $q_*$  tracking the interface between regime I (saturation) and II (weak under-saturation). Its gradient remains constant in time, while  $q_*$  is moved upward (if  $\xi_{us} > 0$ ) or downward (if  $\xi_{us} < 0$ ) as well as horizontally advected by  $\mathbf{u}^{(0)}$ . Regime I is identified with regions where  $q_* > 0$  while regime II is identified with areas where  $q_* < 0$ . However, because of (2.24), this involves some degree of approximation. Note that steeper gradients of  $q_*$  lead to a less sensitive dependence of the size of saturated regions on displacement.



**Figure 4:** Example of a saturated patch  $A_i$  in the  $\eta$ -plane. It is  $q_* > 0$  inside  $A_i$ ,  $q_* = 0$  on its boundary denoted by  $\partial A_i$  and  $q_* < 0$  outside of  $A_i$ . The outward directed normal vector on the boundary is denoted by  $\mathbf{n}$  and the velocity at which the boundary is moving by  $\mathbf{v}_i$ . Note that at every fixed point on  $\partial A_i$ ,  $\mathbf{n}$  and  $\mathbf{v}_i$  are collinear, but while  $\mathbf{n}$  always points out of  $A_i$ ,  $\mathbf{v}_i$  can also point into the patch, for example if the patch is shrinking.



**Figure 5:** Visualization of  $D(\eta_0)$  for two different values of  $\eta_0$ . For the smaller square it is  $n(\eta_0) = 3$ , for the larger square  $n(\eta_0) = 5$ . To every saturated spot  $A_i$  corresponds a function  $\Psi_i$ , which, for a fixed value of  $\xi_{\text{us}}$ , is constant in time. With enlarging square  $[-\eta_0, \eta_0]^2$ , more and more spots are included. As the spots are advected by  $\mathbf{u}^{(0)}$  (indicated by arrows), the choice of spots located inside any finite square change depends on  $\tau$ , corresponding to a rearrangement of the series  $\sum_{i=1}^{\infty} \Psi_i$ . In the limit, however, all spots are included and assuming absolute convergence of the series, every rearrangement converges to the same limit  $\Psi$ .