Multiscale Asymptotics Analysis for the Mesoscale Dynamics of Cloud-Topped Boundary Layers

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ABSTRACT

This paper presents the derivation of a model that captures the coupling between the dynamic and thermodynamic processes of a cloud topped boundary layer on the mesoscales using a formal multiscale asymptotic approach. The derived equations show how the anomalies in the heat, moisture, and mass budgets in the boundary layer affect boundary layer motions, and how these motions can organize and amplify (or damp) such anomalies.

The thermodynamics equations are similar to those that have been suggested in mixed layer studies, that is, the evolution of the thermodynamics variables depends upon the surface heat and moisture fluxes, cloud top radiative cooling rate, temperature and moisture jumps across the capping inversion. However, these equations are coupled to the dynamics equation through the entrainment rate at the top of the cloud deck. The entrainment rate is parameterised from results obtained in laboratory experiments and clearly shows the dependence upon the velocity perturbation which in turn strongly depends upon the horizontal gradient of the thermodynamics variables. The derived entrainment rate is applicable when the thermal jump at cloud-top is sufficiently weak and the velocity jump is of the order of the velocity perturbation.

The mathematical properties and physical characteristics of the system of equations will be explored in future papers.

1. Introduction

The atmospheric boundary layer energetically couples the atmosphere to the underlying surface, both directly through its regulation of the transfer of heat, momentum and matter (*e.g.,* water vapor), and indirectly through the modulation of radiative fluxes. Boundary layer processes thus readily imprint themselves on larger-scale circulations. For instance, boundary layer processes translate surface temperature gradients into shallow pressure anomalies which drive regions of low-level convergence and hence the climatology of precipitation (Lindzen & Nigam 1987). Boundary layer processes also determine the distribution of low-level clouds which play such a crucial role in limiting the amount of radiant energy reaching the surface ocean. For these, and similar reasons, the study of boundary layer processes, and the development of theories or models capable of encapsulating them, is a topic of enduring interest.

Bulk, or integral, models play a special role in the study of boundary layer processes. Bulk models do not resolve the vertical structure of the boundary layer, but rather predict the evolution of integral quantities, such as the boundary-layer budgets of boundary-layer mass, momentum, energy, and perhaps material quantities as well. Such models are useful in their own right. They also provide a framework for understanding the behavior of more complex models. Of the variety bulk models that have been proposed (Stevens *et al.* 2005), a particularly interesting one is the mixed layer model of Lilly (1968), as this provides an elegant framework for coupling the diversity of physical processes thought to control the distribution of marine-stratiform cloudiness within the marine boundary layer.

Like many bulk-models, the mixed-layer model of Lilly is usually justified by assuming that the processes within the boundary layers are occurring on spatial scales much smaller, and temporal scales much shorter, than the scales of processes within the environment in which they are embedded. For instance, the large-scale divergence, which plays an important role in controlling boundary layer depth, or sea-surface temperature gradients which may generate boundary layer pressure gradients, are assumed to be decoupled from processes within the boundary layer. Most studies with Lilly's mixed layer theory have an essentially thermodynamic character as they focus on the budgets of thermal energy, moisture, and mass, and their controls on cloud amount without exploring how the development of clouds, or cloud-scale processes, couples with mesoscale fluid motions within the boundary layer. To the extent bulk models have been coupled to the dynamical evolution of the layer the emphasis has been on the interaction between boundary layer processes and much larger-scale circulations (Schubert *et al.* 1979).

The interplay between dynamics and thermodynamic anomalies on a more intermediate scale is the issue that interests us. Specifically, we wish to explore how anomalies in the heat, moisture, and mass budgets in the stratocumulus-topped boundary layer affect boundary layer motions, and how these motions in turn help organize and amplify (or damp) such anomalies. For example, does the local development of precipitation within the cloud layer perturb the flow in a manner that reinforces the conditions that lead to the precipitation in the first place, or does the ensuing flow disperse the original anomaly? Likewise does the local development of cloudiness generate flow anomalies that support the development of further cloudiness, or is the initial development of cloud essentially a thermodynamic process with no contributing dynamic feedbacks?

To explore these questions we use a formal asymptotic approach that admits a multiscale analysis. Our approach is based on unified mathematical framework for the derivation of reduced multiscale models of geophysical flows suggested by Klein (2004). The framework involves four key steps. First, the 3D compressible flow equations on the rotating sphere are made non-dimensional through the identification of characteristic scales. Second, universal non-dimensional parameters are identified that are independent of any specific flow phenomenon considered. Third, a distinguished limit between these parameters are chosen. Finally, multiple scales asymptotic expansions based on the small perturbation parameter is carried out. The approach has been validated by showing that it naturally re-produces various well known (single-scale) equations in geophysical fluid dynamics, and it has been employed successfully for the derivation of a variety of new multiscale models; for example, in Majda & Klein (2003); Klein (2004); Biello & Klein (2005); Mikusky (2007), and Dolaptchiev & Klein (2007); including the case of boundary layer flows in the absence of cloud-processes (Klein, Mikusky & Owinoh 2005).

This paper presents the derivation of a model that admits coupling between dynamic and thermodynamic processes on intermediate scales. Our asymptotic analysis shows that

- the thermodynamics variables are coupled to the dynamics through the pressure and entrainment terms.
- the velocity perturbations enter the thermodynamics equations through the entrainment rate and surface fluxes,
- the coupling between the two thermodynamics equations is due mainly to entrainment, radiative, and precipitation effects.

The analysis also shows that shallow-water-like wave dynamics appear if the thermal stratification capping the boundary layer is weak. This might mean that such wave-like dynamics is not important for most stratocumulus boundary layers, or that it only becomes important on larger space and longer time scales, which might be resolvable in the larger-scale models that are often forced to parameterize boundary layer processes as being essentially homogeneous over scales of hundreds of kilometers.

The outline of our presentation is as follows. Section 2 describes basic equations in dimensionless form with the appropriate space and time scales given in Section 3. The bulk evolution equations for momentum, energy, and moisture are then derived in Section 4. The equations include the surface and entrainment fluxes and other sources such as radiative and precipitation effects which are then derived asymptotically in Section 5. A summary of systems of equations based on the parameterizations are presented and discussed in Section 6 for a weak buoyancy jump and weak surface fluxes. Our emphasize throughout is on the derivation of the model equations. Their ensuing mathematical properties and physical characteristics will be explored in future work.

2. Governing Equations

Our starting point are the full compressible gas dynamics equations, for which Klein (2004) introduced the distinguished limit such that

$$\varepsilon \to 0: \quad \mathrm{M} = \frac{u_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\varrho_{\mathrm{ref}}}} \sim \varepsilon^2, \quad \mathrm{Fr} = \frac{u_{\mathrm{ref}}}{\sqrt{g h_{\mathrm{sc}}}} \sim \varepsilon^2, \quad \text{and} \quad \mathrm{Ro} = \frac{u_{\mathrm{ref}}}{\Omega h_{\mathrm{sc}}} \sim \varepsilon^{-1}, \tag{1}$$

where the Mach M, Froude Fr and Rossby number Ro are the dimensional numbers defined in terms of the reference pressure $p_{\mathrm{ref}} = 10^5$ kg m^{-1} s^{-2}, reference density $\varrho_{\mathrm{ref}} = 1.25$ kgm^{-3}, pressure scale height $h_{\mathrm{sc}} = p_{\mathrm{ref}}/g\varrho_{\mathrm{ref}} \approx 10$ km, characteristic speed $u_{\mathrm{ref}} = 10$ m s^{-1} , reference temperature $\theta_{\mathrm{ref}} = 300$ K, characteristic time $t_{\mathrm{ref}} = h_{\mathrm{sc}}/u_{\mathrm{ref}} = 10^3$ s , Earth's rotation frequency $\Omega \sim 10^{-4}$ s^{-1} and gravitational acceleration $g = 9.8$ m^{-1} s^{-2}. One may think of ε

as a parameter measuring the ratio of the gravitational versus angular accelerations:

$$\varepsilon = \sqrt[3]{\frac{a\Omega}{g}} \sim \frac{1}{8} \cdots \frac{1}{6} \tag{2}$$

with the earth's radius $a \sim 6000$ km. Reference values for ε^α and for reference quantities scaled by ε^α are provided for different orders of α in Table 1. The emergence of a wide family of meteorological equations from this starting point provides *a posteriori* support for the distinguished limit given by (1), see Klein (2010) and references therein.

In the following we adopt a Cartesian coordinate system $\mathbf{x} = (x, y, z)$ rotating with angular velocity $\boldsymbol{\Omega}$ with gravity g acting in the (vertical) z-direction. The rotation vector $\boldsymbol{\Omega}$ is assumed to take a constant value, consistent with a tangent plane approximation. If $\varrho(\mathbf{x}, t)$ and $\mathbf{v}(\mathbf{x}, t) = \mathbf{v}_{\shortparallel} + w\mathbf{k}$ denote the fluid density and velocity fields at position $\mathbf{x} = \mathbf{x}_{\shortparallel} + z\mathbf{k}$ and time t, the mass conservation equation is

$$\frac{\partial \varrho}{\partial t} + \nabla_{\shortparallel} \cdot (\varrho \mathbf{v}_{\shortparallel}) + \frac{\partial}{\partial z}(\varrho w) = 0\,. \tag{3}$$

In conservation form, the horizontal component of the momentum equation is

$$\frac{\partial}{\partial t}(\varrho \mathbf{v}_{\shortparallel}) + \nabla_{\shortparallel} \cdot (\varrho \mathbf{v}_{\shortparallel} \circ \mathbf{v}_{\shortparallel}) + \frac{\partial}{\partial z}(\varrho \mathbf{v}_{\shortparallel} w) + \varepsilon(\widehat{\boldsymbol{\Omega}} \times \varrho \mathbf{v})_{\shortparallel} + \frac{1}{\varepsilon^4}\nabla_{\shortparallel} p = 0\,, \tag{4}$$

and the vertical component is given by

$$\frac{\partial}{\partial t}(\varrho w) + \nabla_{\shortparallel} \cdot (\varrho \mathbf{v}_{\shortparallel} w) + \frac{\partial}{\partial z}(\varrho w w) + \varepsilon(\widehat{\boldsymbol{\Omega}} \times \varrho \mathbf{v})_{\perp} + \frac{1}{\varepsilon^4}\frac{\partial p}{\partial z} = -\frac{1}{\varepsilon^4}\varrho\,. \tag{5}$$

The anisotropy between horizontal and vertical motions, associated with the volumetric force due to gravity which appears on the right hand side of (5), motivates our separate treatment of the horizontal versus vertical component of momentum.

The heat and moisture budgets are described by conservation laws for the equivalent potential temperature (Emanuel 1994) and total water mixing ratio,

$$\frac{\partial}{\partial t}(\varrho\theta_e) + \nabla_{_\parallel} \cdot (\varrho \mathbf{v}_{_\parallel}\theta_e) + \frac{\partial}{\partial z}(\varrho w \theta_e) = \varrho \mathcal{S}_{\theta_e} \,, \tag{6}$$

and

$$\frac{\partial}{\partial t}(\varrho q_t) + \nabla_{_\parallel} \cdot (\varrho \mathbf{v}_{_\parallel} q_t) + \frac{\partial}{\partial z}(\varrho w q_t) = \varrho \mathcal{S}_{q_t} \,, \tag{7}$$

respectively. The source term $\mathcal{S}_{\theta_e}$ represents diabatic processes, for instance radiation which is described in Section b. The term $\mathcal{S}_{q_t}$ represents the net moisture addition (or removal) rate, for instance as a result of precipitation.

Equations (3) - (7) are closed given an equation of state, which which we take to be that of an ideal mixture of water vapor and dry air. Here we write it in terms of the equivalent potential temperature, and our small parameter ε:

$$\varrho\theta_e = (1+q_t)\left[\frac{p}{(1+R^* q_v)}\right]^{[1-\varepsilon\Gamma_\varepsilon]} \left(\frac{q_s}{q_v}\right)^{\varepsilon\Gamma_\varepsilon R^* q_v} \exp\left(L_v^* \frac{(1+R^* q_v)}{(1+q_t)} \frac{\varrho q_v}{p}\right). \tag{8}$$

In deriving (8) we have introduced the abbreviation

$$\Gamma_\varepsilon = \frac{\Gamma^*}{1+\varepsilon^{-1} c_p^* q_t} \,, \tag{9}$$

and we have extended the distinguished limits in (1) to incorporate dimensionless thermodynamic numbers, so that

$$\frac{L_v \varrho_{\mathrm{ref}}}{p_{\mathrm{ref}}} \equiv \varepsilon^{-1} L_v^*, \qquad \frac{R_v}{R_d} \equiv R^* \varepsilon^0, \qquad \frac{R_d}{c_{p_d}} \equiv \Gamma^* \varepsilon, \qquad \frac{c_l}{c_{p_d}} \equiv c_p^* \varepsilon^{-1}, \tag{10}$$

where Γ^*, L_v^*, R^* and c_p^* are order unity constants, and temperature or pressure dependencies in the original thermodynamic parameters (for instance L_v) have been neglected. The full derivation and justification of (8) is provided in an Appendix (see also Klein & Majda (2006)).

For now it is sufficient to note that while the complexity of (8) results from our retention of all the terms in the definition of θ_e, it adds nothing of substance to the leading-order systems of equations we derive; one contribution of this work is to demonstrate this point, which can be readily extended to analogous systems of equations in more common usage.

In summary, equations (3) - (8) define a closed system of equations under the distinguished limit given by (1) and (10). They form the starting point for our subsequent analysis. The equations themselves are standard, the limit is not. Our hypothesis is that the distinguished limit we introduce captures essential asymptotic behavior of the real system, and thus is meaningful.

3. Spatial and Temporal Scales

Stratocumulus evince structure on a wide range of spatial and temporal scales, particularly under the influence of remotely generated gravity waves, or in the presence of diabatic processes such as precipitation (Savic-Jovcic & Stevens *et al.* 2007). Here we explore the coupling of thermodynamic and dynamic processes on the mesoscale which we define to be a horizontal scale of about 70 km (i.e, $\varepsilon^{-1}h_{\rm sc}$). These scales are much smaller than those typically resolved by large-scale models, but much larger than the scales typically associated with the energetic eddies of the boundary layer itself. The latter scale with the boundary layer height which we take to be $500-600$ m (i.e, $\varepsilon^{\frac{3}{2}}h_{\rm sc}$). Though the cloud base height could potentially appear as another independent length scale, we will rather extract it from the thermodynamics later in Section 5a as a consequence of the present scalings.

We consider time scales associated with the horizontal advection $\varepsilon^{-1}t_{\rm ref}$ ($\sim$ 2 h) and convective time scale t_{ref} ($\sim$ 20 min) assuming a convective velocity of the order 0.5 m s^{-1} ($\varepsilon^{\frac{3}{2}} u_{\text{ref}}$) and based on the short length scale, $\varepsilon^{\frac{3}{2}} h_{\text{sc}}$. Thus we will seek asymptotic solutions in terms of the new multiple-scale co-ordinate system: $\mathbf{X} = \varepsilon \mathbf{x}_{\shortparallel}$, $\xi = \varepsilon^{-\frac{3}{2}} \mathbf{x}_{\shortparallel}$, $\eta = \varepsilon^{-\frac{3}{2}} z$, $T = \varepsilon t$ and $\tau = t$. In these expressions η is the scaled vertical co-ordinate, whereas ξ, τ are the fast and X, T the slow variables for the horizontal directions and time, respectively. Based on these scales and in terms of $\delta = \varepsilon^{\frac{1}{2}}$, the governing equations (3) - (7) rescale to

$$\delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) \varrho + \left(\nabla_{\xi} + \delta^5 \nabla_X \right) \cdot (\varrho \mathbf{v}_{\shortparallel}) + \frac{\partial}{\partial \eta} (\varrho w) = 0 \tag{11}$$

$$\begin{aligned} \delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) (\varrho \mathbf{v}_{\shortparallel}) + \left(\nabla_{\xi} + \delta^5 \nabla_X \right) \cdot (\varrho \mathbf{v}_{\shortparallel} \circ \mathbf{v}_{\shortparallel}) + \frac{\partial}{\partial \eta} (\varrho \mathbf{v}_{\shortparallel} w) \\ + \ \delta^5 (\widehat{\boldsymbol{\Omega}} \times \varrho \mathbf{v})_{\shortparallel} + \frac{1}{\delta^8} \left(\nabla_{\xi} p + \delta^5 \nabla_X p \right) = 0 \end{aligned} \tag{12}$$

$$\begin{aligned} \delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) (\varrho w) + \left(\nabla_{\xi} + \delta^5 \nabla_X \right) \cdot (\varrho \mathbf{v}_{\shortparallel} w) + \frac{\partial}{\partial \eta} (\varrho w w) \\ + \ \delta^5 (\widehat{\boldsymbol{\Omega}} \times \varrho \mathbf{v})_{\perp} + \frac{1}{\delta^8} \left(p_{\eta} + \delta^3 \varrho \right) = 0 \end{aligned} \tag{13}$$

$$\delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) (\varrho \theta_e) + \left(\nabla_{\xi} + \delta^5 \nabla_X \right) \cdot (\varrho \mathbf{v}_{\shortparallel} \theta_e) + \frac{\partial}{\partial \eta} (\varrho w \theta_e) = \delta^3 \varrho \mathcal{S}_{\theta_e} \tag{14}$$

$$\delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) (\varrho q_t) + \left(\nabla_{\xi} + \delta^5 \nabla_X \right) \cdot (\varrho \mathbf{v}_{\shortparallel} q_t) + \frac{\partial}{\partial \eta} (\varrho w q_t) = \delta^3 \varrho \mathcal{S}_{q_t}. \tag{15}$$

The parameter δ has been introduced instead of ε so as to allow a more finely grained selection of scales.

4. Averaging

In this section we derive a new set of bulk, or vertically averaged, equations describing the leading order balance of the intermediate scales selected for our analysis, with the fine and fast scales averaged over and parameterized. Equations (11)–(15) together with the equation of state (8) expressed in terms of δ are taken as a starting point. Three main steps are involved. First, we vertically average our equations; second the dependent variables are expanded in terms of the small parameter δ and balances at different orders are identified; and third the short spatial and fast temporal scales are averaged over to derive the sub-linear growth conditions that determine the large-scale, long-time evolution. Nonlinear terms that do not vanish under the averaging over fast scales are then identified and parameterizations of these terms are discussed in the subsequent section.

Vertical averaging of the equations introduces the concept of the boundary layer depth and processes that control it. We identify the boundary layer top as a semi-permeable surface, whose height we denote by H. Vertical averaging also links the vertical momentum equation to the equation of state. Because the leading order balances are hydrostatic, vertical averaging of the vertical momentum provides a relationship between pressure and density within the boundary layer, given the pressure at H, p_H. Combining this with the equation of state provides a set of diagnostic relations for pressure and density at different orders, and the thermodynamic state of the boundary layer given by θ_e and q_t. Hence, as is familiar from bulk analyses (see *e.g.,* Schubert *et al.* (1979)), one arrives at a new governing set of equations for the prognostic variables $\{H, \mathbf{v}_{\shortparallel}, \theta_e, q_t\}$, complemented by a set of diagnostic relations that describe (perturbations of) p and ϱ as a function of θ_e, q_t, p_H and η.

In what follows we outline the basic steps involved, and the technical difficulties in so far as they arise. Examples of how the analysis is performed are given for the mass balance equation, and can be extended by the interested reader naturally to the case of the other equations. In so doing some technical difficulties arise in the treatment of the pressure gradient terms in the horizontal momentum equations. Hence these issues, and how they are dealt with, are specifically addressed in a separate subsection.

a. Depth Averaging

The equations are vertically integrated through the layer from the surface at $z = z_0(x, y)$ to a free surface $z = H(x, y, t) + z_0(x, y)$. The lower boundary condition is $w = \mathbf{v}_{\shortparallel} \cdot \nabla z_0$ on $z = z_0(x, y)$ and the kinematic free surface condition in the absence of entrainment is $\frac{\partial H}{\partial t} = \mathbf{v} \cdot \mathbf{n}$ on $z = H(x, y, t) + z_0(x, y)$. The normal vector $\mathbf{n} = -\nabla(z_0(x, y) + H(x, y, t) - z)$ points upwards, and $\mathbf{v}_{\shortparallel}$ and $\nabla_{\shortparallel}$ denote the horizontal component of the velocity and gradient-operator respectively. Throughout we denote surface values by subscript 0. In subsequent analysis we ignore the variation in the topography *i.e.,* assume $z_0 = 0$.

The dimensionless free surface kinematic boundary conditions on $\eta = H$ is expressed as

$$\delta^3 \frac{\partial H}{\partial t} = (\mathbf{v} + \mathbf{E}) \cdot \mathbf{n} \tag{16}$$

which introduces the entrainment velocity, $\mathbf{E} = E\mathbf{n}$ that encapsulates the permeability of the interface at H. Here $\mathbf{n}$ is the normal to the surface $\eta = H$ so that $\mathbf{n} = \mathbf{k} - \delta^3 \nabla_{\shortparallel} H$. In terms of the multiscale co-ordinates (16) reads

$$\delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) H + \mathbf{v}_{\shortparallel} \cdot \left(\nabla_{\xi} + \delta^5 \nabla_X \right) H = (w + E) \quad \text{on} \quad \eta = H. \tag{17}$$

Since we assume a flat bottom, the lower boundary condition is given by

$$w = 0 \quad \text{on} \quad \eta = 0. \tag{18}$$

We define the vertical average over the depth H of some quantity ϕ as

$$\langle \phi \rangle = \frac{1}{H} \int_0^H \phi \, \mathrm{d}\eta. \tag{19}$$

So, for example, averaging the continuity equation (11) and making use of the boundary condition in (18) results in a revised continuity equation, one that describes the overall mass balance in the layer of depth H and makes explicit reference to the entrainment velocity, E,

$$\delta^3 \left(\frac{\partial}{\partial \tau} + \delta^2 \frac{\partial}{\partial T} \right) \left(H \langle \varrho \rangle \right) + \left(\nabla_\xi + \delta^5 \nabla_X \right) \cdot \left(H \langle \varrho \mathbf{v}_{\shortparallel} \rangle \right) = \varrho_H E \, . \tag{20}$$

b. *Leading Order Equations*

The equations are now written in terms of dependent flow variables expanded in terms of the small parameter δ. Thus generically, for a dependent variable ϕ we write:

$$\phi = \sum_{i=0} \delta^i \phi^{(i)}(\tau, \xi, \eta, T, \mathbf{X}) \tag{21}$$

Applying this expansion to the mass continuity equation for the layer in (20) results in

$$\nabla_\xi \cdot \left(H \langle \varrho \mathbf{v}_{\shortparallel} \rangle \right)^{(i)} = (\varrho_H E)^{(i)} \quad \text{for} \quad i = 0, 1, 2 \tag{22}$$

$$\frac{\partial}{\partial \tau} \left(H \langle \varrho \rangle \right)^{(i-3)} + \nabla_\xi \cdot \left(H \langle \varrho \mathbf{v}_{\shortparallel} \rangle \right)^{(i)} = (\varrho_H E)^{(i)} \quad \text{for} \quad i = 3, 4 \tag{23}$$

$$\frac{\partial}{\partial \tau} \left(H \langle \varrho \rangle \right)^{(i-3)} + \nabla_\xi \cdot \left(H \langle \varrho \mathbf{v}_{\shortparallel} \rangle \right)^{(i)} = (\varrho_H E)^{(i)} - \left(\frac{\partial}{\partial T} \left(H \langle \varrho \rangle \right)^{(i-5)} + \nabla_X \cdot \left(H \langle \varrho \mathbf{v}_{\shortparallel} \rangle \right)^{(i-5)} \right)$$
$$\text{for} \quad i = 5, 6, 7, \cdots \tag{24}$$

Here we note that the decomposition results in the initial equation being broken into a sequence of equations describing balances at different order. Compound terms of the form $(\phi\psi)^{(i)}$ are to be understood in terms of their component expansion such that:

$$(\phi\psi)^{(i)} \equiv \sum_{j=0}^{i} \phi^{(i-j)} \psi^{(j)} \, , \tag{25}$$

where here ϕ and ψ denote two different dependent variables, for instance H and $\mathbf{v}_{\shortparallel}$. Similar notation holds for terms involving more than two dependent variables.

Although (21) holds in general, for specific variables we will additionally assume that variability as a function of the independent variables only emerges at a specific order. So doing causes some terms to vanish at low order because, for instance, gradients in the balance equations are zero at that order. The assumptions we make are as follows:

$$\theta_e = 1 + \delta^3\theta_e^{(3)} + \delta^4\theta_e^{(4)} + \delta^5\theta_e^{(5)} + \delta^6\theta_e^{(6)}(\mathbf{X},T) + \delta^7\theta_e^{(7)}(\mathbf{X},\eta,T) + \delta^8\theta_e^{(8)}(\mathbf{X},\xi,\eta,T,\tau) + \cdots \tag{26}$$

$$q_t = \delta^3 q_t^{(3)} + \delta^4 q_t^{(4)} + \delta^5 q_t^{(5)} + \delta^6 q_t^{(6)}(\mathbf{X},T) + \delta^7 q_t^{(7)}(\mathbf{X},\eta,T) + \delta^8 q_t^{(8)}(\mathbf{X},\xi,\eta,T,\tau) + \cdots \tag{27}$$

$$q_v = \delta^3 q_v^{(3)} + \delta^4 q_v^{(4)} + \delta^5 q_v^{(5)} + \delta^6 q_v^{(6)}(\mathbf{X},\eta,T) + \delta^7 q_v^{(7)}(\mathbf{X},\eta,T) + \delta^8 q_v^{(8)}(\mathbf{X},\xi,\eta,T,\tau) \cdots \tag{28}$$

The form for θ_e can be justified based on the equation of state and the hydrostatic balance that emerges at low order. The structure for q_v and q_t are by assumption (*i.e.,* water vapor perturbations are small compared to unity). It will be shown in addition that the saturation vapor mixing ratio q_s follows the form given for q_v.

For the boundary layer height we assume the following dependencies at various orders,

$$H = H^{(0)} + \delta H^{(1)}(\mathbf{X},T) + \delta^2 H^{(2)}(\mathbf{X},T) + \delta^3 H^{(3)}(\mathbf{X},\xi,T,\tau) + \cdots , \tag{29}$$

and finally we assume that

$$\mathbf{v}_{\shortparallel} = \mathbf{v}_{\shortparallel}^{(0)}(\mathbf{X}) + \delta\mathbf{v}_{\shortparallel}^{(1)}(\mathbf{X},T) + \delta^2\mathbf{v}_{\shortparallel}^{(2)}(\mathbf{X},T) + \delta^3\mathbf{v}_{\shortparallel}^{(3)}(\mathbf{X},\xi,\eta,T,\tau) + \cdots \tag{30}$$

The assumptions on the scalings given in (26) - (30) are based on field observations and simulations (Stevens *e*t al. 2002; Faloona *et al.* 2005; Stevens *et al.* 2005). We will show later, for example from (72), that $\mathbf{v}_{\shortparallel}^{(0)}$ depends on the free atmosphere geostrophic pressure gradient and thus one can allow for variation in space $\mathbf{X}$ in $\mathbf{v}_{\shortparallel}^{(0)}$ so as to allow for large scale vertical motion. By continuity this velocity scaling implies that the leading order terms for w vanish, *i.e.*, $w^{(i)} = 0$ for $i < 5$.

c. *Fast Scales Averaged Equations*

We average the equations over fast temporal and small spatial scales, i.e., we average over τ and ξ, respectively. Using the over-bar to denote such averaging we have

$$\overline{\psi}(\mathbf{X}, \eta, T) = \lim_{\tau, A \to \infty} \frac{1}{\tau A} \int_{\tau, A} \psi(\xi, \mathbf{X}, \eta, \tau, T)\ d\xi d\tau. \tag{31}$$

Averaging over fast scales eliminates gradients on these scales due to the so-called sub-linear growth condition so that, for instance, the leading order terms of the mass balance equation become:

$$\overline{E}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5\,, \tag{32}$$

and

$$\frac{\partial H^{(1)}}{\partial T} + H^{(0)} \nabla_X \cdot \mathbf{v}_{\shortparallel}^{(1)} + H^{(1)} \nabla_X \cdot \mathbf{v}_{\shortparallel}^{(0)} + \mathbf{v}_{\shortparallel}^{(0)} \nabla_X H^{(1)} = \overline{E}^{(6)}\,. \tag{33}$$

d. *Pressure Gradients*

Averaging the horizontal momentum equation leads to terms of the form

$$\int_0^H \nabla_\xi p\ \mathrm{d}\eta \quad \text{and} \quad \delta^5 \int_0^H \nabla_X p\ \mathrm{d}\eta. \tag{34}$$

The fine-scale pressure gradients appear at lower order, but are eliminated by the fast scale averaging. The larger-scale pressure gradients must be evaluated. To do so we derive diagnostic equations for the pressure at the desired order starting with the equation of state and the vertically integrated vertical momentum equation, which remains hydrostatic on the scales of motion that interest us. The appropriate order of the pressure is then substituted into the above integrals and used to evaluate the vertically averaged pressure gradient terms. To make use of the vertically averaged momentum equation we will need a pressure boundary condition, p_H.

1) Pressure Above the Boundary Layer

Consider a vertical scale greater than the boundary layer depth, *i.e.,* a scale of the order of the depth of the troposphere ~ 10 km. We further assume that the nature of flow above the boundary layer is such that the horizontal scale remains 70 km or larger. Reduced equations with such scaling can be obtained using the asymptotic expansions in powers of ε (or δ). Asymptotic analysis of the continuity and momentum equations shows that the pressure in the layer is essentially hydrostatic. From the hydrostatic equation together with the equation of state, that is,

$$p_z = -\varrho \qquad \text{and} \qquad \varrho\theta_e = p^{1-\varepsilon\Gamma^*}; \tag{35}$$

we find that

$$p = \left[1 - \frac{\Gamma}{\beta\varepsilon}\log\left(1+\varepsilon^2\beta z\right)\right]^{\frac{1}{\varepsilon\Gamma^*}} + \varepsilon^4 p_g(\mathbf{X},T) + \mathcal{O}\left(\varepsilon^5\right) . \tag{36}$$

Here we have assumed the troposphere to be drier than the boundary layer and to have a potential temperature distribution

$$\theta_e = 1 + \varepsilon^2\theta_e^{(2)}(z) + \cdots , \tag{37}$$

where $\theta_e^{(2)}(z) = \beta z$ with a constant lapse rate β.

Asymptotic analysis also shows that the horizontal pressure gradient satisfies

$$\nabla_X p^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3 \qquad \text{and} \qquad \nabla_X p^{(4)} = -(\widehat{\mathbf{\Omega}} \times \varrho^{(0)} \mathbf{v}^{(0)})_{_{||}} \tag{38}$$

where $\varrho^{(0)} = \exp(-z)$. As a consequence, the pressure term p_g in (36) satisfies the geostrophic condition, that is,

$$\nabla_X p_g = -(\widehat{\mathbf{\Omega}} \times \varrho^{(0)} \mathbf{v}^{(0)})_{_{||}} . \tag{39}$$

Because the pressure p must be continuous at the top of the boundary layer, as $z \longrightarrow \delta^3 H$ then $p \longrightarrow p_H$ giving the required boundary condition for pressure. It follows that

$$p_H = \left[1 - \frac{\Gamma^*}{\beta\delta^2} \log\big(1 + \delta^7 \beta H\big)\right]^{\frac{1}{\Gamma^* \delta^2}} + \delta^8 p_g(\mathbf{X}, T) + \mathcal{O}\big(\delta^{10}\big) \tag{40}$$

which upon expansion in terms of δ implies that

$$p_H = 1 - \delta^3 H + \frac{1}{2}\delta^6 H^2 - \frac{1}{2}\delta^8 \Gamma^* H^2 + \delta^8 p_g - \frac{1}{6}\delta^9 H^3 + \frac{1}{2}\delta^{10}\beta H^2 \cdots \tag{41}$$

2) Vertical Momentum Balance and Equation of State Expansions

Expanding the rescaled governing equation for the vertical momentum balance, and given the assumed velocity structure so that $w^{(i)} = 0$ for $i < 3$ implies that hydrostatic balances hold up to tenth order:

$$p_\eta^{(i)} = 0 \qquad \text{for} \quad i = 0, 1, 2 \tag{42}$$

$$p_\eta^{(i)} + \varrho^{(i-3)} = 0 \qquad \text{for} \quad i = 3, \ldots, 10 \tag{43}$$

where the first equation simply reflects our choice of expansion wherein $\varrho^{(0)}$ is the leading order term in the density. Integrating these equations over the depth of the boundary layer and combining with the boundary condition on p_H and an expansion of the equation of state yields expressions for pressure that can be used to evaluate vertically integrated pressure gradients in terms of other known quantities.

Solving (42) together with the boundary conditions (41) implies that $p^{(0)} = 1$, $p^{(1)} = 0$ and $p^{(2)} = 0$. It follows from the equation of state (8)

$$\varrho^{(0)} = \theta_e^{(0)} = p^{(0)} = 1\,, \tag{44}$$

$$\varrho^{(1)} = \theta_e^{(1)} = p^{(1)} = 0\,, \tag{45}$$

$$\varrho^{(2)} = \theta_e^{(2)} = p^{(2)} = 0\,. \tag{46}$$

Using these expressions simplifies further expansions of the equation of state such that

$$\varrho^{(3)} = p^{(3)} + \varphi^{(3)}\,, \tag{47}$$

$$\varrho^{(4)} = p^{(4)} + \varphi^{(4)}\,, \tag{48}$$

$$\varrho^{(5)} = p^{(5)} + \varphi^{(5)} - \Gamma^* p^{(3)}\,, \tag{49}$$

$$\begin{aligned}\varrho^{(6)} = {} & p^{(6)} + \varphi^{(6)} - \Gamma^* p^{(4)} - \varrho^{(3)}{\theta_e}^{(3)} - R^* \left(\theta_e^{(3)} - \Gamma^* {L_v}^* {q_v}^{(3)}\right) {q_v}^{(3)} \\ & + (1 + \Gamma^* {c_p}^*)\, {q_t}^{(3)} p^{(3)} + (\Gamma^* {L_v}^* - R^*)\, {q_v}^{(3)} \varrho^{(3)}\,,\end{aligned} \tag{50}$$

where

$$\varphi^{(i)} = -{\theta_e}^{(i)} + {q_t}^{(i)} + (\Gamma^* L_v^* - R^*){q_v}^{(i)} + \mathcal{Q}^{(i)}, \tag{51}$$

and $\mathcal{Q}^{(3)} = 0$ and $\mathcal{Q}^{(i)}$ (for $i = 4, 5, 6$) are given in the Appendix by (A21)–(A23).

The above expressions for $\varrho^{(i)}$ through $i = 6$ allow us, through integration of (43), to derive expressions for p through $i = 9$. These being:

$$p^{(3)} = -\eta\,, \tag{52}$$

$$p^{(4)} = 0\,, \tag{53}$$

$$p^{(5)} = 0\,, \tag{54}$$

$$p^{(6)} = \frac{\eta^2}{2} + \varphi^{(3)}(H-\eta)\,, \tag{55}$$

$$p^{(7)} = \varphi^{(4)}(H-\eta)\,, \tag{56}$$

$$p^{(8)} = \varphi^{(5)}(H-\eta) + p_g - \frac{1}{2}\Gamma^*\eta^2\,, \tag{57}$$

$$\begin{aligned} p^{(9)} = & \left({q_t}^{(6)} - {\theta_e}^{(6)}\right)(H-\eta) + \left(\Gamma^* L_v^* - R^*\right)\int_\eta^H q_v^{(6)}d\eta - \frac{\eta^3}{6} + \frac{1}{2}\varphi^{(3)}(H-\eta)^2 \\ & + \mathcal{T}_1(H-\eta) + \mathcal{T}_2(H^2-\eta^2)\,, \end{aligned} \tag{58}$$

where

$$\begin{aligned} \mathcal{T}_1 =& \mathcal{Q}^{(6)} - \left(\theta_e^{(3)} - L_v^*\Gamma^* q_v^{(3)}\right)\left(\varphi^{(3)} + R^* q_v^{(3)}\right) - R^*\varphi^{(3)} q_v^{(3)}\,, \\ \mathcal{T}_2 =& \theta_e^{(3)} - \left(L_v^*\Gamma^* - R^*\right) q_v^{(3)} + \left(1 + \Gamma^* c_p^*\right) q_t^{(3)}\,. \end{aligned}$$

Based on the above the pressure gradients are written as a series expansion in δ such that

$$\begin{aligned} \nabla_X p =& \delta^6\left[\varphi^{(3)}\nabla_X H\right] + \delta^7\left[\varphi^{(4)}\nabla_X H\right] + \delta^8\left[\varphi^{(5)}\nabla_X H + \nabla_X p_g\right] \\ & + \delta^9\Big[\left({q_t}^{(6)} - {\theta_e}^{(6)} + \mathcal{T}_1\right)\nabla_X H + (H-\eta)\left(\varphi^{(3)}\nabla_X H + \nabla_X[{q_t}^{(6)} - {\theta_e}^{(6)}]\right) \\ & \quad + \mathcal{T}_2 H\nabla_X H + \left(\Gamma^* L_v^* - R^*\right)\nabla_X\int_\eta^H q_v^{(6)}d\eta\Big] + \mathcal{O}\left(\delta^{10}\right)\,. \end{aligned} \tag{59}$$

Integrating (59) over the vertical provides the desired expression for the vertically integrated pressure gradient. This is straight forward once terms involving the vertical integral of the water vapor terms of sixth order and greater are evaluated. These contribute to the expression for $p^{(9)}$. As per our definition the water vapor mixing ratio is given by

$$q_v = \begin{cases} q_s & \text{if } \eta \geq \eta_c, \\ \\ q_t & \eta < \eta_c \end{cases} \tag{60}$$

where η_c is the condensation height, an expression for it is derived in Section 5a. In Section 4b we argued that q_s varies with η already at order six, we will show in Section 5a that asymptotic expansion of saturation mixing ratio yields $q_s^{(6)} = \beta_0 + \beta_1\eta$. Hence for

$$\int_\eta^H q_v d\eta = \delta^4 q_t^{(4)}(H-\eta) + \delta^5 q_t^{(5)}(H-\eta) + \delta^6 \int_\eta^H q_v^{(6)} d\eta + \cdots \tag{61}$$

we can evaluate the last term as follows

$$\int_\eta^H q_v^{(6)} d\eta = \begin{cases} \beta_0(H-\eta) + \beta_1\left(\frac{H^2}{2} - \frac{\eta^2}{2}\right) & \text{if } \eta \geq \eta_c, \\ \\ \beta_0(H-\eta_c) + \beta_1\left(\frac{H^2}{2} - \frac{\eta_c^2}{2}\right) + q_t^{(6)}(\eta_c - \eta) & \text{if } \eta < \eta_c. \end{cases} \tag{62}$$

Taking the gradient and integrating over the boundary layer provides the desired expression,

$$\int_0^H \left(\nabla_X \int_{\eta'}^H q_v^{(6)} d\eta'\right) d\eta = \beta_0 H \nabla_X H + \beta_1 H^2 \nabla_X H + \frac{\beta_1}{6}\nabla_X \eta_c^3 \tag{63}$$

Given the expansion for H assumed in Section 4b this completes the derivation of the vertically averaged pressure gradients. The final expressions are

$$\left(\int_0^H \nabla_X p \mathrm{d}\eta\right)^{(i)} = 0 \qquad \text{for} \quad i = 0,1,2,3,4,5,6 \tag{64}$$

$$\left(\int_0^H \nabla_X p \mathrm{d}\eta\right)^{(7)} = \varphi^{(3)} H^{(0)} \nabla_X H^{(1)} \tag{65}$$

$$\left(\int_0^H \nabla_X p \mathrm{d}\eta\right)^{(8)} = H^{(0)} \nabla_X p_g + \varphi^{(3)}\left[H^{(0)}\nabla_X H^{(2)} + H^{(1)}\nabla_X H^{(1)}\right] + \varphi^{(4)} H^{(0)} \nabla_X H^{(1)} \tag{66}$$

$$\begin{aligned}\left(\int_0^H \nabla_X p \mathrm{d}\eta\right)^{(9)} =& H^{(1)} \nabla_X p_g + \varphi^{(3)}\left[H^{(0)}\nabla_X H^{(3)} + H^{(1)}\nabla_X H^{(2)} + H^{(0)}\nabla_X H^{(3)}\right] \\ &+ \varphi^{(4)}\left[H^{(0)}\nabla_X H^{(2)} + H^{(1)}\nabla_X H^{(1)}\right] + \varphi^{(5)} H^{(0)} \nabla_X H^{(1)} \\ &+ \frac{{H^{(0)}}^2}{2}\nabla_X\left({q_t}^{(6)} - {\theta_e}^{(6)}\right) + \frac{\beta_1}{6}(\Gamma^* L_v^* - R^*)\nabla_X \eta_c^3. \end{aligned} \tag{67}$$

e. Intermediate Summary of the Asymptotic Equations

At this point it proves useful to summarize the equations that we have derived on the basis of the analysis of this section. They are:

(i) Mass Balance

$$\overline{E}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5 \tag{68}$$

$$\frac{\partial H^{(1)}}{\partial T} + H^{(0)}\nabla_X \cdot \mathbf{v}_{_\shortparallel}^{(1)} + H^{(1)}\nabla_X \cdot \mathbf{v}_{_\shortparallel}^{(0)} + \mathbf{v}_{_\shortparallel}^{(0)}\nabla_X H^{(1)} = \overline{E}^{(6)} \tag{69}$$

(ii) Horizontal Momentum Balance

$$0 = (\overline{w\varrho\mathbf{v}_{_\shortparallel}})_0^{(i)} + \left[\overline{(\varrho\mathbf{v}_{_\shortparallel})_H E}\right]^{(i)} \quad \text{for} \quad i = 0, 1, 2, 3 \tag{70}$$

$$0 = (\overline{w\varrho\mathbf{v}_{_\shortparallel}})_0^{(4)} + \left[\overline{(\varrho\mathbf{v}_{_\shortparallel})_H E}\right]^{(4)} - \varphi^{(3)} H^{(0)}\nabla_X H^{(1)} \tag{71}$$

$$\begin{aligned} H^{(0)}(\widehat{\mathbf{\Omega}} \times \mathbf{v}^{(0)})_{_\shortparallel} + H^{(0)}\nabla_X p_g = (\overline{w\varrho\mathbf{v}_{_\shortparallel}})_0^{(5)} + \left[\overline{(\varrho\mathbf{v}_{_\shortparallel})_H E}\right]^{(5)} - \varphi^{(4)} H^{(0)}\nabla_X H^{(1)} \\ - \varphi^{(3)}\left(H^{(0)}\nabla_X H^{(2)} + H^{(1)}\nabla_X H^{(1)}\right) \end{aligned} \tag{72}$$

The last two sets of equations indicate that the pressure gradients due to depth fluctuations are in balance with the surface momentum and entrainment fluxes, in particular fluxes of $\mathcal{O}(\delta^4)$ and $\mathcal{O}(\delta^5)$. The implications of these fluxes on the flow are discussed in the next section. At the next order, we find an evolution equation for the first-order velocity perturbation,

$$
\begin{aligned}
H^{(0)}\frac{\partial \mathbf{v}_{_{\shortparallel}}^{(1)}}{\partial T} + H^{(0)}\mathbf{v}_{_{\shortparallel}}^{(0)}.\nabla_X \mathbf{v}_{_{\shortparallel}}^{(1)} + H^{(0)}(\widehat{\mathbf{\Omega}} \times \mathbf{v}^{(1)})_{_{\shortparallel}} + \nabla_X \Phi = (\overline{w\varrho \mathbf{v}_{_{\shortparallel}}})_0^{(6)} + \left[\overline{(\varrho \mathbf{v}_{_{\shortparallel}})_H E}\right]^{(6)} \\
- \ \varphi^{(5)} H^{(0)} \nabla_X H^{(1)} - \varphi^{(4)} \left[H^{(1)}\nabla_X H^{(1)} + H^{(0)}\nabla_X H^{(2)}\right] \\
- \ \varphi^{(3)} \left[H^{(0)}\nabla_X H^{(3)} + H^{(1)}\nabla_X H^{(2)} + H^{(2)}\nabla_X H^{(1)}\right] \\
- \ H^{(1)}(\widehat{\mathbf{\Omega}} \times \mathbf{v}^{(0)})_{_{\shortparallel}} - H^{(1)}\nabla_X p_g
\end{aligned}
\tag{73}
$$

where

$$
\Phi = \frac{{H^{(0)}}^2}{2}\left(-{\theta_e}^{(6)} + {q_t}^{(6)}\right) + \frac{\beta_1}{6}\left(\Gamma^* L_v^* - R^*\right)\eta_c^3
$$

The coupling between the thermodynamics and the momentum variables occurs through the (hydrostatic) pressure gradient. Nonlinearities in the momentum equation arise solely through this coupling as a result.

(iii) Equivalent Potential Temperature

$$
0 = (\overline{w\varrho\theta_e})_0^{(i)} + (\overline{H\left\langle \varrho \mathcal{S}_{\theta_e}\right\rangle})^{(i-3)} \quad \text{for} \quad i = 0,1,2,3,4,5 \tag{74}
$$

$$
0 = (\overline{w\varrho\theta_e})_0^{(i)} + [\overline{(\varrho\theta_e)_H E}]^{(i)} + (\overline{H\left\langle \varrho \mathcal{S}_{\theta_e}\right\rangle})^{(i-3)} \qquad \text{for} \quad i = 6,7,8,9,10 \tag{75}
$$

$$
\frac{\partial \theta_e^{(6)}}{\partial T} + \mathbf{v}_{_{\shortparallel}}^{(0)}\nabla_X \theta_e^{(6)} = \frac{(\overline{w\varrho\theta_e})_0^{(11)}}{H^{(0)}} + \frac{[\overline{(\varrho\theta_e)_H E}]^{(11)}}{H^{(0)}} + \frac{(\overline{H\left\langle \varrho \mathcal{S}_{\theta_e}\right\rangle})^{(8)}}{H^{(0)}} \tag{76}
$$

(iv) Total Moisture Content

$$
0 = (\overline{w\varrho q_t})_0^{(i)} + (\overline{H\left\langle \varrho \mathcal{S}_{q_t}\right\rangle})^{(i-3)} \quad \text{for} \quad i = 0,1,2,3,4,5 \tag{77}
$$

$$
0 = (\overline{w\varrho q_t})_0^{(i)} + [\overline{(\varrho q_t)_H E}]^{(i)} + (\overline{H\left\langle \varrho \mathcal{S}_{q_t}\right\rangle})^{(i-3)} \qquad \text{for} \quad i = 6,7,8,9,10 \tag{78}
$$

$$\frac{\partial q_t^{(6)}}{\partial T} + \mathbf{v}_{\shortparallel}^{(0)} \nabla_X q_t^{(6)} = \frac{(\overline{w \varrho q_t})_0^{(11)}}{H^{(0)}} + \frac{[\overline{(\varrho q_t)_H E}]^{(11)}}{H^{(0)}} + \frac{(\overline{H \langle \varrho \mathcal{S}_{q_t} \rangle})^{(8)}}{H^{(0)}} \tag{79}$$

5. Closure Terms

The equations described in the previous section include a variety of aggregated quantities that must be modeled or parameterized. These include surface, and entrainment fluxes, radiative transfer, and precipitation processes. Many depend on the state of the cloud layer, as determined, for instance, by the depth of the cloud layer or the liquid water path. Hence in proposing models to close our equations it is also necessary to develop consistent asymptotic relations for the input required by such models. In this section we describe asymptotically consistent parameterizations for the radiative, surface and entrainment fluxes, but first we present our asymptotic analysis of the cloud layer.

a. Liquid Water Asymptotics

The liquid water mixing ratio is given by

$$q_l = \begin{cases} q_t - q_s & \text{if } q_t > q_s, \\ 0 & \text{otherwise.} \end{cases} \tag{80}$$

Hence an asymptotic representation of q_l depends on the asymptotic representation of the saturation mixing ratio of water vapor, q_s. By definition

$$q_s(T) = \frac{1}{R^*} \frac{p_s'(T)}{p_d'} \tag{81}$$

where T is the temperature, p_s the saturation vapor pressure, p_d is the partial pressure of dry air, and R^* is the ratio of the gas constant as represented by the distinguished limit as in (10). Note that the total pressure is simply the sum of the partial pressures, and that here primes represent dimensional quantities. The saturation vapor pressure can be approximated as a function of temperature, for instance by integrating the Clausius-Clapeyron equation about a reference temperature and vapor pressure. Doing so yields the following expression for the dimensionless saturation vapor mixing ratio:

$$q_s = \frac{\delta^3 p_s^* \exp\left(\frac{A^*}{\delta^2}\left[1 - \frac{1}{T}\right]\right)}{R^* p - \delta^3 R^* p_s^* \exp\left(\frac{A^*}{\delta^2}\left[1 - \frac{1}{T}\right]\right)}. \tag{82}$$

To arrive at this equation we have introduced the distinguished limit, $p'_{s,\mathrm{ref}}/p_{\mathrm{ref}} = 0.035 \approx \delta^3 p_s^*$ and made use of the previous distinguished limits given by (10). Given the expansion for pressure, it follows that (82) can be written as an asymptotic series in δ as follows:

$$q_s = \delta^3 \frac{p_s^*}{R^*}\left[1 + \delta A^* T^{(3)} + \delta^2 A^* \left(T^{(4)} + A^* {T^{(3)}}^2\right)\right] + \delta^6 \left(\beta_0 + \beta_1 \eta\right) + \mathcal{O}\left(\delta^7\right), \tag{83}$$

where

$$\beta_0 = \frac{p_s^*}{R^*} A^* \left(T^{(5)} + A^* T^{(3)} T^{(4)} + \frac{1}{6} A^{*2} {T^{(3)}}^3 + \frac{p_s^*}{A^*}\right) \equiv q_s^{(6)}\big|_{\eta=0} \tag{84}$$

is the saturation mixing ratio at the surface and

$$\beta_1 = \frac{p_s^*}{R^*} \tag{85}$$

The expansion (83) with $T^{(3)}$, $T^{(4)}$ and $T^{(5)}$, given in the Appendix by (A19), defines the component terms $q_s^{(4)}, q_s^{(5)}$ and $q_s^{(6)}$ implicitly.

To obtain an asymptotic expression for the cloud base height η_c we assume that the cloud base appears where the saturation mixing ratio matches the total mixing ratio in the sub-cloud layer, that is $q_t(\eta_c) = q_s(\eta_c)$. Above the cloud base, we assume that all vapor in excess of saturation condenses to liquid water, that is, the total mixing ratio is given by $q_t = q_s + q_l$, which is the sum of the saturation mixing ratio q_s and the liquid water mixing ratio q_l. Therefore

$$q_t^{(6)}(\eta_c) = q_s^{(6)}(\eta_c) = \beta_0 + \beta_1\eta_c = q_{s,0}^{(6)} + \beta_1\eta_c, \tag{86}$$

and as a reminder, subscript 0 denotes values valid at the surface. Because $q_t^{(6)}$ is assumed to be independent of height (27) we find that to leading order the cloud base height is given by

$$\eta_c = \frac{1}{\beta_1}\left(q_t^{(6)} - q_s^{(6)}\right) \tag{87}$$

These expressions can now be used with the definition of the liquid water content (80) to derive an expression for the depth averaged liquid water path:

$$\begin{aligned}\langle q_l\rangle H &= \int_{\eta_c}^{H}(q_t - q_s)\mathrm{d}\eta \\ &= \delta^6\left[(q_t^{(6)} - q_{s,0}^{(6)})(H - \eta_c) - \frac{\beta_1}{2}(H^2 - \eta_c^2)\right] + \cdots \\ &= -\delta^6\frac{\beta_1}{2}(H - \eta_c)^2 + \cdots\end{aligned} \tag{88}$$

Thus

$$\left\langle q_l^{(6)}\right\rangle H^{(0)} = -\frac{\beta_1}{2}(H^{(0)} - \eta_c)^2 \tag{89}$$

which shows, as expected, that the leading order vertically averaged liquid water mixing ratio is proportional to the square of the cloud thickness.

b. Radiative Flux $\mathcal{S}_{\theta_e}$

The source term that appears in the equivalent potential temperature balance is due to both long-wave and short-wave radiative effects. In our analysis we assume a nocturnal situation for which only long-wave fluxes are important. The up and downward radiative fluxes are given by the expression

$$F_L^{\uparrow\downarrow}(z) = (1-\epsilon)F_{\mathrm{bnd}}^{\uparrow\downarrow} + \epsilon T^4, \tag{90}$$

which we have made dimensionless through the reference value of $\sigma T_{\mathrm{ref}}^4 = 460$ W m^{-2} with $\sigma = 5.67 \times 10^{-8}$ W m^{-2} K^{-4} being the Stefan-Boltzmann constant. F_{bnd} denotes the boundary long-wave flux which is taken as just above cloud top for the downward fluxes, and just below cloud base for the upward fluxes, *i.e.*, $F_L(H)$ and $F_L(\eta_c)$ respectively. The emissivity is denoted by ϵ and is taken to be independent of the direction of the radiances. It is parameterized following the suggestion of Stevens *et al.* (2005) as follows:

$$\epsilon = 1 - \exp\left(-\delta^5 a^* \int_{\eta_c}^{\eta} \varrho q_l d\eta\right). \tag{91}$$

The term in the exponent measures the extinction cross section of the liquid water, a, multiplied by the liquid water path. The parameterization of radiation hence introduces a further distinguished limit, namely that $a = \epsilon^{-5/2} a^*$.

The net long-wave radiation flux is given by $F_L = F_L^{\uparrow} - F_L^{\downarrow}$ and expansions in orders of δ lead to:

$$F^{(0)} = F_L^{\uparrow(0)}(\eta_c) - F_L^{\downarrow(0)}(H) \Rightarrow \frac{\partial F^{(0)}}{\partial \eta} = 0, \tag{92}$$

Thus to first approximation the radiative flux is constant. At next order we have the balance

$$F^{(1)} = \Delta F^{(1)}(1-a^*) \int_{\eta_c}^{\eta} q_l^{(6)}\, \mathrm{d}\eta, \tag{93}$$

where $\Delta F^{(1)} \equiv F_L^{\uparrow(1)}(\eta_c) - F_L^{\downarrow(1)}(H)$ defines the flux difference in the boundary fluxes. Therefore

$$\frac{\partial F^{(1)}}{\partial \eta} = \Delta F^{(1)} a^* q_l^{(6)}. \tag{94}$$

This expression shows that the radiative flux is responsive to changes in the modeled cloud structure through the liquid water.

Radiative flux divergences at first order influence the θ_e budget at a much lower order. This is evident from the dimensionless form of the equation for the equivalent potential temperature:

$$\frac{D}{Dt}(\varrho\theta_e) = \varrho\mathcal{S}_{\theta_e} = \frac{\sigma T_{\mathrm{ref}}^4}{c_p U_{\mathrm{ref}}\varrho_{\mathrm{ref}}\theta_{\mathrm{ref}}}\frac{\partial F}{\partial z} = \delta^{10} = \delta^7 \frac{\partial F}{\partial \eta}; \tag{95}$$

which given that terms of order $F^{(1)}$ are the leading order in the forcing implies that the following holds up to $i = 8$,

$$(\varrho\mathcal{S}_{\theta_e})^{(i)} = 0 \quad \text{for} \quad i = 0, \ldots, 7 \tag{96}$$

$$(\varrho\mathcal{S}_{\theta_e})^{(8)} = \frac{\partial F^{(1)}}{\partial \eta} = \Delta F^{(1)} a^* q_l^{(6)}. \tag{97}$$

The depth averaged source terms are thus given by

$$\left\langle \mathcal{S}_{\theta_e}{}^{(i)} \right\rangle = 0 \quad \text{for} \quad i = 0,1,2,3,4,5,6,7 \tag{98}$$

$$\left\langle \mathcal{S}_{\theta_e}{}^{(8)} \right\rangle H^{(0)} = \Delta F^{(1)} a^* \left\langle q_l^{(6)} \right\rangle H^{(0)} = -\frac{\beta_1}{2}\Delta F^{(1)} a^* (H^{(0)} - \eta_c)^2 \tag{99}$$

The asymptotics shows that the radiative fluxes is related to the thickness of the cloud and it is interactive in the sense that it evolves, through its dependence on η_c, with the total water mixing ratio $q_t^{(6)}$ and the equivalent potential temperature $\theta_e^{(6)}$.

c. Precipitation Flux $\mathcal{S}_{q_t}$

Formation of drizzle in cloudy air is an important mechanism for depleting the cloud water. Thus when there is drizzle, the total water is no longer a conserved quantity because of the reduction of the liquid water in the cloud layer and the evaporation of the precipitation in the sub-cloud layer, thus total water mixing ratio balance is

$$\frac{D}{Dt}\varrho q_t = \varrho \mathcal{S}_{q_t} = R_e - R_p \tag{100}$$

where R_p is the rate of production of precipitation and R_e is the rate of evaporation of precipitation.

Simple parameterization of R_p include $R_p = C_o(\varrho q_l)^{\alpha_p}$, *i.e.,* precipitation rate is parameterized as some fraction of the liquid water (Pawlowska & Brenguier 2003; Comstock *et al.* 2005; van Zanten *et al.* 2005). C_o is a precipitation conversion rate. Although the evaporation of precipitation below stratocumulus can be substantial, it is neglected in the present analysis.

Based on this simple parameterization mentioned above, the drizzle effect is included in the total water mixing ratio balance as

$$\left\langle \mathcal{S}_{q_t}{}^{(8)} \right\rangle H^{(0)} = \mathcal{P}^* \left\langle q_l^{(6)} \right\rangle^{\alpha_p} H^{(0)} = -\frac{\beta_1}{2}\mathcal{P}^* \left(H^{(0)} - \eta_c\right)^{2\alpha_p} \tag{101}$$

where $\mathcal{P}^*$ is a constant of order 1 representing the precipitation conversion rate. In principal the flux of precipitation acts as a source of θ_e through the change to q_t in that equation. These effects however will only appear at higher orders, and thus do not enter into the asymptotic balances we explore.

d. Parameterization of the Surface Fluxes

To the extent ocean currents are negligible the lower boundary conditions for the velocity components are zero i.e $\mathbf{v}_{\shortparallel} = \mathbf{0}$. Other surface quantities are denoted, as before, by as subscript zero, so that the surface temperature is denoted by T_0. The equivalent potential temperature at the surface $\theta_{e,0} = \theta_e(T_0, p_{s,0})$ as for a water covered surface $q_{v,0} = q_{s,0} = q_s(T_0, p_{s,0})$. Hence using the equation of state to express $\theta_{e,0}^{(i)}$ in terms of $T_0^{(i)}$ and $q_{s,0}^{(i)}$ it is straightforward to show that (82) becomes

$$\begin{aligned} q_{s,0} = \delta^3 \frac{p_s^*}{R^*}\left[1 + \delta A^* T_0^{(3)} + \delta^2 A^*\left(T_0^{(4)} + A^* {T_0^{(3)}}^2\right)\right] + \delta^6 \frac{p_s^*}{R^*} A^* \left(\frac{p_s^*}{A^*} + T_0^{(5)}\right. \\ \left. + A^* T_0^{(3)} T_0^{(4)} + \frac{1}{6} A^{*2} {T_0^{(3)}}^3\right) + \mathcal{O}\left(\delta^7\right), \end{aligned} \tag{102}$$

The parameterization of the surface fluxes is achieved by the use of the bulk transfer formulas

$$(\overline{\varrho \mathbf{v}_{\shortparallel} w})_0 = -C_D \varrho_0 \, |\mathbf{v}_{\shortparallel}| \, \mathbf{v}_{\shortparallel} \tag{103}$$

$$(\overline{\varrho w \theta_e})_0 = -C_H \varrho_0 |\mathbf{v}_{\shortparallel}| (\theta_e - \theta_{e,0}) \tag{104}$$

$$(\overline{\varrho w q_t})_0 = -C_Q \varrho_0 |\mathbf{v}_{\shortparallel}| (q_t - q_{s,0}) \tag{105}$$

where $|\mathbf{v}_{\shortparallel}| = \sqrt{u^2 + v^2}$ and the density at the surface $\varrho_0 = 1 + \delta^3 \varrho_0^{(3)} + \cdots$ as obtained from (44)–(50). The coefficients C_D, C_H and C_Q are the drag coefficients for momentum, sensible heat and moisture, respectively. These are considered here to be constant, hence stability effects are not included in the surface exchange rules. The values of the exchange coefficients range from $1.4 \times 10^{-3} - 4 \times 10^{-3}$. We explore the weak flux limit wherein all coefficients are of the same order so that which $C_{D,H,Q} \sim \delta^6 C^*$ and C^* is an $\mathcal{O}(1)$ constant.

1) Momentum Fluxes

For these limits the momentum fluxes take the form

$$(\overline{\varrho u w})_0 = -C^* \delta^6 \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \mathbf{v}_{\shortparallel}^{(0)} - C^* \delta^7 \left(\left|\mathbf{v}_{\shortparallel}\right|^{(1)} \mathbf{v}_{\shortparallel}^{(0)} + \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \mathbf{v}_{\shortparallel}^{(1)} \right) + \cdots \tag{106}$$

where $\left|\mathbf{v}_{\shortparallel}\right|^{(0)} = \sqrt{{u^{(0)}}^2 + {v^{(0)}}^2}$ and $\left|\mathbf{v}_{\shortparallel}\right|^{(1)} = \frac{u^{(0)} v^{(1)} + u^{(1)} v^{(0)}}{\sqrt{{u^{(0)}}^2 + {v^{(0)}}^2}}$. Thus

$$(\overline{\varrho \mathbf{v}_{\shortparallel} w})_0{}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5 \tag{107}$$

$$(\overline{\varrho u w})_0^{(6)} = -C^* \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \mathbf{v}_{\shortparallel}^{(0)} \tag{108}$$

and

$$(\overline{\varrho u w})_0^{(7)} = -C^* \left(\left|\mathbf{v}_{\shortparallel}\right|^{(0)} \mathbf{v}_{\shortparallel}^{(1)} + \left|\mathbf{v}_{\shortparallel}\right|^{(1)} \mathbf{v}_{\shortparallel}^{(0)} \right) \tag{109}$$

2) Total Moisture Flux

In Section 5c and (68) we found that $\mathcal{S}_{q_t} = \delta^8 \mathcal{S}_{q_t}^{(8)} + \cdots$ and $E = \delta^6 E^{(6)} + \cdots$, thus it follows from (77) and (78) that

$$(\overline{\varrho w q_t})_0{}^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5, 6, 7, 8 \tag{110}$$

For weak moisture jump

$$(\overline{\varrho w q_t})_0{}^{(i)} = 0 \quad \text{for} \quad i = 9, 10 \tag{111}$$

and similar expansion as in the momentum flux gives

$$(\overline{\varrho w q_t})_0^{(11)} = -C^* \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \left(q_t^{(5)} - q_{s,0}^{(5)} \right). \tag{112}$$

3) Equivalent Potential Temperature Flux

We found in Section 5b that $\mathcal{S}_{\theta_e} \sim \delta^8$; hence, given (75) the condition of sub-linear growth and the assumption that entrainment effects first emerge for $i = 11$, is consistent with

$$(\overline{\varrho w \theta_e})_0^{(i)} = 0 \quad \text{for} \quad i = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, \tag{113}$$

This condition is consistent with our representation of surface fluxes, for which

$$(\overline{\varrho w \theta_e})_0^{(11)} = -C^* \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \left(\theta_e^{(5)} - \theta_{e,0}^{(5)}\right), \tag{114}$$

e. *Entrainment Velocity and Entrainment Flux*

The entrainment closure is usually based on the turbulent structure of the mixed layer. However, there is lack of consensus on the entrainment rate with various authors promoting different rates (Stevens 2002). In general, entrainment rates are expressed in terms of the surface heat flux into the boundary layer, cloud-top radiative flux out of the layer, radiative flux jump occurring inside the entrainment zone and some assumption on entrainment buoyancy flux. In most studies the wind shear is usually neglected though stratocumulus clouds simulations by Moeng (2000) show that an increase in shear leads to increase in entrainment rate by a significant amount. To avoid being tangled into the entrainment debate we estimate the entrainment velocity based on results obtained from the laboratory experiments that include stratification and shear effects.

Entrainment velocity E as observed in mixing layer flow experiments with inter-facial shear flows takes the form

$$E = C_E \left|\Delta V\right| Ri_B^{-n}, \tag{115}$$

where Ri_B is the bulk Richardson number based on the inter-facial velocity jump ΔV defined as $Ri_B = \Delta b H/(\Delta V)^2$ with H being the depth of the mixed layer, and $\Delta b = g\Delta\varrho/\varrho$ is the buoyancy jump at the top of the layer. The parameter dependencies are similar to many of the entrainment laws that have been suggested in the literature (Stevens 2002), except that here we link mixing to the differences in the mean flow, as given by ΔV rather than a convective velocity scale. Laboratory experiments, e.g. by Strang & Fernando (2001) identifies three regimes with $n \approx 0$ for $Ri_B \leq 1.5$, $n \approx 2.63 \pm 0.45$ for $1.5 \leq Ri_B \leq 5$ and $n \approx 1.30 \pm 0.15$ for $5 \leq Ri_B \leq 20$. Thus for the depth $H \sim \delta^3 h_{\text{sc}}$, the weak stratification case $\Delta b \sim \delta^5 g$ and the weak surface flux case $\Delta V \sim \delta v_{\text{ref}}$ we find that $Ri_B \sim \delta^6 g h_{\text{sc}}/v_{\text{ref}}^2 \sim \delta^6 \text{Fr}^{-2} \sim \delta^{-2}$ making use of the distinguished limit (1). This value of the Richardson number falls under the third regime of Strang and Fernando's experiments. Therefore the dimensionless entrainment velocity is given by

$$E = C_E^* \delta^2 \left|\Delta V\right| \left(\delta^5 \frac{\varrho (\Delta V)^2}{H \Delta\varrho}\right)^{\frac{3}{2}}, \tag{116}$$

where we have taken the constant $C_E = 0.02 \pm 0.01 \approx \delta^2 C_E^*$ and $n = 1.5$.

The problem now reduces to finding an expression for inter-facial density jump $\Delta\varrho$. Recall that the pressure at the region above the boundary layer is given by (36) and since in this region the pressure is hydrostatic we find that

$$\varrho = -p_z = \frac{1}{(1+\varepsilon^2\beta z)} \left[1 - \frac{\Gamma}{\beta\varepsilon} \log\left(1+\varepsilon^2\beta z\right)\right]^{\frac{1}{\Gamma\varepsilon}-1} + \mathcal{O}\left(\varepsilon^5\right), \tag{117}$$

which implies that the density just above the inversion layer ϱ_{H+} is given by

$$\varrho_{H+} = 1 - \delta^3 H + \delta^5 \Gamma H + \frac{1}{2}\delta^6 H^2 - \delta^7 \beta H + \mathcal{O}\left(\delta^8\right). \tag{118}$$

From (44) to (50) we find that the density at the top of the layer is given by

$$\varrho_{H-} = 1 - \delta^3 H + \delta^5 \left(\Gamma H + \varphi^{(5)}\right) + \delta^6 \left(\frac{1}{2}H^2 + \varphi^{(6)}\right) + \mathcal{O}\big(\delta^7\big)\,. \tag{119}$$

Therefore the density jump at the inversion layer is given by

$$(\Delta\varrho)_H = \varrho_{H+} - \varrho_{H-} = -\delta^5\varphi^{(5)} - \delta^6\varphi^{(6)} + \mathcal{O}\big(\delta^7\big) \tag{120}$$

The velocity jump is given by $\Delta\mathbf{v}_{\shortparallel} = \mathbf{v}_{\shortparallel g} - \mathbf{v}_{\shortparallel} = \left(\mathbf{v}_{\shortparallel g} - \mathbf{v}_{\shortparallel}^{(0)}\right) - \delta\mathbf{v}_{\shortparallel}^{(1)} + \mathcal{O}(\delta^2)$ and, from (72), $\mathbf{v}_{\shortparallel}^{(0)} = \mathbf{v}_{\shortparallel g}$ for a weak surface momentum flux, This leads to a velocity jump $|\Delta\mathbf{v}_{\shortparallel}| = \delta\left|\mathbf{v}_{\shortparallel}^{(1)}\right| + \mathcal{O}(\delta^2)$. Therefore the entrainment velocity is given by

$$E = \delta^6 \frac{C_E^* \left(\mathbf{v}_{\shortparallel}^{(1)}\right)^4}{H^{(0)}\left(\varphi^{(5)}\right)^{\frac{3}{2}}} + \mathcal{O}\big(\delta^7\big)\,, \tag{121}$$

where $\varphi^{(5)}$ given by (51) is assumed non-negative. Equation (121) states that the leading order entrainment, $E^{(6)}$, depends mainly on $\mathbf{v}_{\shortparallel}^{(1)}$ which in turn depends on the thermodynamics variables $\theta^{(6)}$ and $q_t^{(6)}$ as per (73). The evolution of these variables, given by (76) and (79), depend on radiative flux and drizzle respectively, in addition to the surface fluxes. Thus we can conclude that entrainment rate given by (121) is based on the strength of the radiative driving of the layer and on contributions from other energetic sources (for instance surface fluxes, wind-shear at the boundary layer top, drizzle).

For weak temperature jumps $(\Delta\theta_e)_H^{(5)}$ and weak moisture jumps $(\Delta q_t)_H^{(5)}$, the entrainment fluxes first appear at order 11, thus,

$$(\overline{E(\varrho\theta_e)_H})^{(11)} = -E^{(6)}\Delta(\varrho\theta_e)_H^{(5)}. \qquad \text{and} \qquad (\overline{E(\varrho q_t)_H})^{(11)} = -E^{(6)}\Delta(\varrho q_t)_H^{(5)}. \tag{122}$$

6. Summary of the Closed Systems of Reduced Equations

The above discussion suggests that the structure of the resultant equations depend on the magnitude of temperature jump, moisture jump and the drag coefficients. The case of weak inter-facial stability is interesting because it allows for entrainment effects to emerge at lower orders, and is discussed further below. For such a case the equivalent potential temperature has an asymptotic expansion of the form

$$\theta_e = 1 + \delta^5 \theta_e^{(5)} + \delta^6 \theta_e^{(6)}(\mathbf{X}, T) + \delta^7 \theta_e^{(7)}(\mathbf{X}, \eta, T) + \delta^8 \theta_e^{(8)}(\mathbf{X}, \xi, \eta, T, \tau) + \cdots \tag{123}$$

The momentum balance equation (72) thus reduces to

$$(\widehat{\mathbf{\Omega}} \times \mathbf{v}^{(0)})_{\shortparallel} + \nabla_X p_g = 0, \tag{124}$$

and this places a constraint on the background flow $\mathbf{v}_{\shortparallel}^{(0)}$. Equation (124), also implies that $\nabla_X \cdot \mathbf{v}_{\shortparallel}^{(0)} = 0$.

Given the above, and closure terms following the discussion in Section 5 results in the following system of equations for the evolution of the intermediate or mesoscale boundary layer flows:

$$\frac{\partial H}{\partial T}^{(1)} + \mathbf{v}_{\shortparallel}^{(0)} \cdot \nabla_X H^{(1)} + H^{(0)} \nabla_X \cdot \mathbf{v}_{\shortparallel}^{(1)} - \frac{C_E^* \left(\mathbf{v}_{\shortparallel}^{(1)}\right)^4}{H^{(0)} \left(\varphi^{(5)}\right)^{\frac{3}{2}}} = 0, \tag{125}$$

$$\frac{\partial \mathbf{v}_{\shortparallel}^{(1)}}{\partial T} + \mathbf{v}_{\shortparallel}^{(0)} \cdot \nabla_X \mathbf{v}_{\shortparallel}^{(1)} + (\widehat{\mathbf{\Omega}} \times \mathbf{v}^{(1)})_{\shortparallel} + \varphi^{(5)} \nabla_X H^{(1)} + \frac{1}{H^{(0)}} \nabla_X \Phi + \frac{C^*}{H^{(0)}} \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \mathbf{v}_{\shortparallel}^{(0)} = 0, \tag{126}$$

$$\begin{aligned} \frac{\partial \theta_e}{\partial T}^{(6)} + \mathbf{v}_{\shortparallel}^{(0)} . \nabla_X \theta_e^{(6)} + \frac{\beta_1}{2H^{(0)}} \Delta F^{(1)} \left(H^{(0)} - \eta_c\right)^2 - \frac{C_E^* \left(\mathbf{v}_{\shortparallel}^{(1)}\right)^4 \Delta(\varrho\theta_e)_H^{(5)}}{H^{(0)^2} \left(\varphi^{(5)}\right)^{\frac{3}{2}}} \\ + \frac{C^*}{H^{(0)}} \left|\mathbf{v}_{\shortparallel}\right|^{(0)} \left(\theta_e^{(5)} - \theta_{e,0}^{(5)}\right) = 0, \end{aligned} \tag{127}$$

$$\frac{\partial q_t^{(6)}}{\partial T} + \mathbf{v}_{_{\shortparallel}}^{(0)}.\nabla_X q_t^{(6)} + \frac{\beta_1}{2H^{(0)}}\mathcal{P}^*\left(H^{(0)} - \eta_c\right)^{2\alpha_p} - \frac{C_E^*\left(\mathbf{v}_{_{\shortparallel}}^{(1)}\right)^4 \Delta(\varrho q_t)_H^{(5)}}{H^{(0)2}\left(\varphi^{(5)}\right)^{\frac{3}{2}}} \\ + C^*\left|\mathbf{v}_{_{\shortparallel}}\right|^{(0)}\left(q_t^{(5)} - q_{s,0}^{(5)}\right) = 0, \tag{128}$$

where $\Phi = \frac{H^{(0)}{}^2}{2}\left(-\theta_e{}^{(6)} + q_t{}^{(6)}\right) + \frac{\beta_1}{6}\left(\Gamma^* L_v^* - R^*\right)\eta_c^3$, the cloud base height η_c is given by (87) and $\varphi^{(5)} = -\theta_e{}^{(5)} + (1 + \Gamma^* L_v^* - R^*)q_t{}^{(5)}$.

7. Concluding Remarks

In this paper we have derived sets of equations that can be used to describe various regimes of the dynamics and the thermodynamics of the cloud topped boundary layer. Our derivation formally relates approximations one often makes in the representation of the thermodynamics properties of the fluid, to simplifications one would like to achieve in the dynamics, and as such helps justify bulk models frequently encountered in the literature. The equations we develop here have filtered out the fast time scale and small spatial scales and themselves identify a type of bulk model. In addition to the identification of a new reduced model for investigating the coupling of the fluid dynamics of the boundary layer on the mesoscale to the turbulent dynamics on the convective scale, a significant finding of this work is the demonstration of the strong link between perturbation velocity $\mathbf{v}_{_{\shortparallel}}^{(1)}$ and the thermodynamics perturbations. Traditional bulk models ignore the coupling between the shallow-water like dynamics of mesoscale motions and the fast turbulent dynamics that dominate the convective scales.

Further qualities of the reduced equations we derive include that:

- the evolution of $\mathbf{v}_{_{\shortparallel}}^{(1)}$ depends on the moist thermodynamics in addition to the depth of the boundary layer and surface fluxes of momentum;

- the entrainment rate depends directly on $\mathbf{v}_{\shortparallel}^{(1)}$, and through the evolution of this quantity on the accumulated effects of surface heat fluxes, the buoyancy jump across the inversion layer, radiative cooling and any drizzle effects;

- the velocity perturbations are also driven by a depth perturbation which depends on stratification, which is $H^{(1)}$ for the case of weak stratification on which we focus.

The mathematical and physical properties of the system of equations (125)–(128) will be explored in future papers.

Acknowledgments.

The authors would like to thank Prof Andrew Majda for constructive discussions. This work was supported by Deutsche Forschungsgemeinschaft (DFG) under Grant KL 611/14 and DFG Priority Research Program SPP 1167 "Quantitative Precipitation Forecast"; and by Leibniz-Gemeinschaft (WGL) through the PAKT Project "High-resolution Modelling of Clouds and Gravity Waves". B.S. thanks Freie Universität Berlin for financing an extended initial visit in 2007 through the Interdisciplinary Center "Efficient Mathematical Modelling in the Earth and Life Sciences"

APPENDIX

Thermodynamic Relations

a. Equivalent Potential Temperature, θ_e

The dimensional equivalent potential temperature is given by

$$\theta'_e = T' \left(\frac{p_{00}}{p'} \right)^{\frac{R_d}{c_{p_*}}} \Omega_e \exp \left(\frac{L_v q_v}{c_{p_*} T'} \right) \tag{A1}$$

where p is the pressure, $p_{00} = p_{\text{ref}} = 1000\text{hPa}$ is the reference pressure, L_V is latent heat of vaporization, T temperature, $q_t = q_v + q_l$ is the total water mixing ratio with q_v is the water vapor mixing ratio and q_l is the water content mixing ratio; ${c_p}^* = c_{p_d} + c_l q_t$ where c_{p_d} and c_l are specific heat capacities for dry air and water, respectively. Now we have

$$\frac{R_d}{c_{p_*}} = \frac{R_d}{c_{p_d} + c_l q_t} = \frac{R_d}{c_{p_d}} \left(1 + \frac{c_l}{c_{p_d}} q_t \right)^{-1} \tag{A2}$$

where R_d is the gas constants for dry air. We have

$$\frac{R_d}{c_{p_d}} = \frac{\gamma - 1}{\gamma} = \frac{2}{7} \approx \Gamma^* \varepsilon, \qquad \frac{c_l}{c_{p_d}} = \frac{4217}{1007} \approx c_p^* \varepsilon^{-1} \tag{A3}$$

neglecting any variations of these values with temperature and pressure. The $(.)^*$ superscripts indicate constants of order one. Thus

$$\frac{R_d}{c_{p_*}} = \Gamma^* \varepsilon (1 + c_p^* \varepsilon^{-1} q_t)^{-1} \tag{A4}$$

Also

$$\frac{R_v}{c_{p_*}} = \frac{R_v}{R_d} \frac{R_d}{c_{p_*}} = R^* \Gamma^* \varepsilon (1 + c_p^* \varepsilon^{-1} q_t)^{-1} \quad \text{with} \quad \frac{R_v}{R_d} = \frac{461.5}{287.0} \approx 1.61 \sim \varepsilon^0 \equiv R^* \tag{A5}$$

where R_v is the gas constant for water vapor.

The equation of state is given by

$$p' = \varrho' R_d T' (1 + q_v \frac{R_v}{R_d})(1 + q_t)^{-1} \tag{A6}$$

which implicitly defines expressions for the effective (two-phase) gas constant, R

$$R = R_d (1 + q_v \frac{R_v}{R_d})(1 + q_t)^{-1} \tag{A7}$$

The dimensionless temperature follows as

$$T = T'/T_{\text{ref}} = T' \left(\frac{R_d \varrho_{\text{ref}}}{p_{\text{ref}}} \right) = \frac{p}{\varrho} \frac{(1 + q_t)}{(1 + R^* q_v)} \tag{A8}$$

These relations can be used to define the dimensionless enthalpy ratio:

$$\frac{L_v q_v}{c_{p_*} T} = \frac{L_v q_v}{c_{p_*}} \frac{\varrho R_d}{p} \frac{(1 + q_v \frac{R_v}{R_d})}{(1 + q_t)} \equiv L_v^* \Gamma^* (1 + c_p^* \varepsilon^{-1} q_t)^{-1} \frac{\varrho q_v}{p} \frac{(1 + R^* q_v)}{(1 + q_t)} \tag{A9}$$

which introduces L_v^* as follows $\frac{L_v \varrho_{\text{ref}}}{p_{\text{ref}}} = 31.25 \sim \varepsilon^{-1} L_v^*$. These relations allow us to write Ω_e as follows

$$\begin{aligned} \Omega_e &= \left(1 + \frac{R_v}{R_d} q_v\right)^{\frac{R_d}{c_{p_*}}} \left(\frac{q_v}{q_{vs}}\right)^{-\frac{R_v q_t}{c_{p_*}}} \\ &= (1 + R^* q_v)^{\Gamma^* \varepsilon (1 + c_p^* \varepsilon^{-1} q_t)^{-1}} \left(\frac{q_v}{q_{vs}}\right)^{-R^* \Gamma^* \varepsilon q_v (1 + c_p^* \varepsilon^{-1} q_t)^{-1}} \end{aligned} \tag{A10}$$

All of which may be combined with the expression for θ_e (A1) to derive a dimensionless equation of state expressed in terms of distinguished limits for non-dimensional thermodynamic parameters:

$$\begin{aligned} \varrho \theta_e &= p^{[1 - \Gamma^* \varepsilon (1 + \varepsilon^{-1} c_p^* q_t)^{-1}]} (1 + q_t)(1 + R^* q_v)^{[-1 + \Gamma \varepsilon (1 + \varepsilon^{-1} c_p^* q_t)^{-1}]} \\ & \left(\frac{q_v}{q_{vs}}\right)^{-R^* \Gamma^* \varepsilon q_v (1 + c_p^* \varepsilon^{-1} q_t)^{-1}} \exp\left(\frac{L_v^* \Gamma^*}{(1 + c_p^* \varepsilon^{-1} q_t)} \frac{(1 + R^* q_v)}{(1 + q_t)} \frac{\varrho q_v}{p} \right) \end{aligned} \tag{A11}$$

b. θ_e Expansions

The expression for θ_e can be expanded in powers of δ as follows:

$$\theta_e^{(0)} = 1, \tag{A12}$$

$$\theta_e^{(1)} = 0, \tag{A13}$$

$$\theta_e^{(2)} = 0, \tag{A14}$$

$$\theta_e^{(3)} = T^{(3)} + L_v^*\Gamma^* q_v^{(3)}, \tag{A15}$$

$$\theta_e^{(4)} = T^{(4)} + L_v^*\Gamma^* q_v^{(4)} + \mathcal{Q}^{(4)}, \tag{A16}$$

$$\theta_e^{(5)} = T^{(5)} + L_v^*\Gamma^* q_v^{(5)} + \mathcal{Q}^{(5)}, \tag{A17}$$

$$\theta_e^{(6)} = T^{(6)} + L_v^*\Gamma^* q_v^{(6)} + \Gamma^* R^* q_v^{(4)} - L_v^*\Gamma^* c_p^* q_v^{(4)^2} + \mathcal{Q}^{(6)}, \tag{A18}$$

These can be further specified given an expansion for T in (A8) as

$$T^{(i)} = p^{(i)} - \varrho^{(i)} + q_t^{(i)} - R^* q_v^{(i)} \qquad \text{for} \quad i = 3, 4, 5, \tag{A19}$$

$$T^{(6)} = p^{(6)} - \varrho^{(6)} + q_t^{(6)} - R^* q_v^{(6)} - R^* q_v^{(3)} T^{(3)} - \left(T^{(3)} + R^* q_v^{(3)}\right) \varrho^{(3)} + q_t^{(3)} p^{(3)}, \tag{A20}$$

and $\mathcal{Q}^{(i)}$ depend on the background moisture as follows

$$\mathcal{Q}^{(4)} = -L_v^*\Gamma^* c_p^* q_t^{(4)} q_v^{(3)}, \tag{A21}$$

$$\mathcal{Q}^{(5)} = -L_v^*\Gamma^* c_p^* \left[q_t^{(4)} q_v^{(3)} + q_t^{(3)} q_v^{(4)} - c_p^* q_t^{(3)} q_v^{(3)^2}\right] + \Gamma^* R^* q_v^{(3)} \left[1 - \ln\left(\frac{q_v^{(3)}}{q_{vs}^{(3)}}\right)\right], \tag{A22}$$

$$\begin{aligned}\mathcal{Q}^{(6)} = & -L_v^*\Gamma^* c_p^* \left[q_t^{(5)} q_v^{(3)} + q_t^{(4)} q_v^{(4)} + q_t^{(3)} q_v^{(5)}\right] + \Gamma^* R^* \left[q_v^{(3)} \frac{q_{vs}^{(4)}}{q_{vs}^{(3)}} - q_v^{(4)} \ln\left(\frac{q_v^{(3)}}{q_{vs}^{(3)}}\right)\right] \\ & + L_v^*\Gamma^* c_p^{*2} q_t^{(3)} \left[2q_t^{(4)} q_v^{(3)} + q_t^{(3)} q_v^{(4)} - c_p^* q_v^{(3)} q_t^{(5)^2}\right] + \frac{1}{2}\left(L_v^*\Gamma^* q_v^{(3)}\right)^2 \\ & - \Gamma^* R^* c_p^* q_v^{(3)} q_t^{(3)} \left[1 - \ln\left(\frac{q_v^{(3)}}{q_{vs}^{(3)}}\right)\right].\end{aligned} \tag{A23}$$

c. Saturation Vapor Mixing Ratio, q_s

The saturated water vapor mixing ratio, q_{vs} is defined as

$$q_s(T) = \frac{p'_s(T)}{R^*(p' - p'_s(T))} \tag{A24}$$

where T is the temperature, p is the air pressure, and $p'_s(T)$ is the equilibrium saturation vapor pressure given by Clausius-Clapeyron formula:

$$\frac{dp'_s}{dT} = \frac{L_v}{R_v T^2} p'_s. \tag{A25}$$

Both R^* and L_v are given as before. Integrating the Clausius-Clapeyron equation assuming that L_v is independent of temperature yields the following approximation for p'_s :

$$p'_s = p'_{s,\mathrm{ref}} \exp\left(\frac{L}{R_v T_0}\frac{(T' - T_0)}{T'}\right) \tag{A26}$$

where $T_0 = 300\ K$ and $p'_{s,\mathrm{ref}} = 3500$ kg m^{-1} s^{-2}. Define $A = L/R_v T_0 = 18.05 \approx \varepsilon^{-1} A^*$ and make use of the distinguished limits (10) yields

$$p'_s = p'_{s,\mathrm{ref}} \exp\left(\frac{A^*}{\varepsilon}\left[1 - \frac{1}{T}\right]\right). \tag{A27}$$

where the dimensionless temperature is given by (A8). From this expression, and the additional distinguished limit $p'_{s,\mathrm{ref}}/p_{\mathrm{ref}} = 0.035 \approx \varepsilon^{\frac{3}{2}} e_s^*$, the following can be derived,

$$q_s = \frac{\delta^3 e_s^* \exp\left(\frac{A^*}{\delta^2}\left[1 - \frac{1}{T}\right]\right)}{R^* p - \delta^3 R^* e_s^* \exp\left(\frac{A^*}{\delta^2}\left[1 - \frac{1}{T}\right]\right)}, \tag{A28}$$

which concludes the derivation of (82) in the main text.

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List of Tables

TABLE 1. Dimensional magnitudes (with varying but convenient units) for reference quantities for $\varepsilon = 1/7$

α	ε^α	$\varepsilon^\alpha u_{\text{ref}}$	$\varepsilon^\alpha h_{\text{ref}}$	$\varepsilon^\alpha \theta_{\text{ref}}$	$\varepsilon^\alpha t_{\text{ref}}$
-3	345		3450 km		4 d
-5/2	130		1296 km		1.5 d
-2	49		490 km		14 hr
-3/2	19	185 m s^{-1}	185 km		5 hr
-1	7	70 m s^{-1}	70 km		2 hr
-1/2	2.65	26 m s^{-1}	26 km		44 min
0	1	10 m s^{-1}	10 km	300 K	17 min
1/2	0.38	3.78 m s^{-1}	3780 m	113 K	6 min
1	0.14	1.43 m s^{-1}	1429 m	43 K	2 min
3/2	0.054	0.54 m s^{-1}	540 m	16 K	54 s
2	0.020	0.20 m s^{-1}	204 m	6.1 K	20 s
5/2	0.0077	7.71 cm s^{-1}	77 m	2.3 K	8 s
3	0.0029	2.92 cm s^{-1}	29 m	0.87 K	3 s
7/2	0.0011	1.10 cm s^{-1}	11 m	0.33 K	1 s
4	0.0004	0.42 cm s^{-1}	4 m	0.12 K	0.4 s