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Far field velocity potential induced by a rapidly decaying vorticity distribution

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1. Introduction

The results to be obtained in this paper are applicable to a general problem regarding the far field behavior of the solution $A(x)$ of the vector Poisson equation in space,

$$\Delta A = -\Omega, \tag{1.1}$$

subjected to the boundary condition,

$$|A| \to 0 \quad \text{as} \quad r = |x| \to \infty. \tag{1.2}$$

Here x denotes the position vector with Cartesian coordinates x_i, $i = 1$, 2 and 3. In this paper, boldface symbols are used to denote vector quantities and the subscripts $i = 1$, 2 and 3 denote their components. Thus A_i and ω_i denote the three components of A and Ω respectively.

The inhomogeneous term, $\Omega(x)$, is assumed to be divergence free,

$$\nabla \cdot \Omega = 0, \tag{1.3}$$

and to decay exponentially in r, that is

$$|\Omega| = o(r^{-n}), \quad \text{for all } n \text{ as } r \to \infty. \tag{1.4}$$

Note that (1.4) is a realistic condition for vortical flows but is not necessary-for the results to be derived here. It suffices to require that (1.4) holds for n less than a sufficiently large integer instead of for all n.

The above problem originated from our analysis of a three dimensional unsteady incompressible flow induced by a vorticity field Ω. Therefore, we shall present our motivations and findings in the context of a fluid dynamics problem.

In the first subsection, we state the governing equations for a vortex dominated viscous flow, show the relevance of the above problem defined by (1.1) to (1.4) and then explain why we are interested in the far field behavior of A. We recount in Sec. 1.2 several known results regarding the far field behavior of A so as to provide the necessary background information for the current investigation and then point out in Sec. 1.3 several questions regarding the far field behavior. The answers to those questions, which are presented in the following sections, in turn constitute the new results mentioned in the Abstract.

1.1. *Governing equations for an incompressible viscous flow induced by an initial vorticity field*

The vorticity $\boldsymbol{\Omega}$ is related to the velocity $\boldsymbol{v}$ by

$$\boldsymbol{\Omega} = \nabla \times \boldsymbol{v}, \tag{1.5}$$

and hence has to be divergence free, (1.3). On the other hand the continuity equation,

$$\nabla \cdot \boldsymbol{v} = 0, \tag{1.6}$$

implies that the velocity, $\boldsymbol{v}$, can be expressed in terms of the curl of a vector potential, $\boldsymbol{A}$, i.e.,

$$\boldsymbol{v} = \nabla \times \boldsymbol{A}. \tag{1.7}$$

Since the addition of a vector function of the form ∇f to $\boldsymbol{A}$ does not change the velocity, we may suppose the function f to be chosen so that [1]

$$\nabla \cdot \boldsymbol{A} = 0. \tag{1.8}$$

The vector potential is then related to the vorticity $\boldsymbol{\Omega}$ by the vector Poisson equation (1.1). This equation is equivalent to (1.5), (1.7) and (1.8). We observe that (1.8) is consistent with (1.3).

The vorticity is assumed to decay exponentially in r, (1.4), consequently the induced flow field remains at rest at infinity, i.e.,

$$|\boldsymbol{v}| \to 0 \quad \text{as} \quad r \to \infty. \tag{1.9}$$

This in turn leads to the above boundary condition (1.2) for $\boldsymbol{A}$.

In the above, the dependence of the functions on time t has been suppressed because t can be treated as a parameter. A time derivative appears only in the vorticity evolution equation,

$$\boldsymbol{\Omega}_t + (\boldsymbol{v} \cdot \nabla)\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \nabla)\boldsymbol{v} = \nu\, \Delta\boldsymbol{\Omega}, \tag{1.10}$$

where ν denotes the kinematic viscosity. The above equation can be derived by applying the curl operator to the Navier–Stokes equation, thereby eliminating the gradient of pressure.

By specifying an initial vorticity distribution,

$$\boldsymbol{\Omega}(\boldsymbol{x}, \mathrm{t}) = \boldsymbol{\Upsilon}(\boldsymbol{x}) \quad \text{at} \quad t = 0, \tag{1.11}$$

where $\boldsymbol{\Upsilon}(\boldsymbol{x})$ fulfills the condition of exponential decay, (1.4), Eqs. (1.1), (1.7), (1.10), (1.2), (1.4) and (1.11) define an initial value problem in space for $\boldsymbol{A}(\boldsymbol{x}, t)$ and $\boldsymbol{\Omega}(\boldsymbol{x}, t)$. The solution $\boldsymbol{A}$ of the vector Poisson equation (1.1) subject to the boundary condition (1.2) then appears as a part of the initial value problem.

In general, the solution of this nonlinear initial value problem requires a numerical solution. This involves a discretization of the equations in a finite computational domain $\mathscr{D}$. Hence it is necessary to impose boundary conditions along $\partial\mathscr{D}$ which are consistent with the unbounded domain problem or to impose approximate ones and specify their degree of accuracy. A numerical scheme for the above problem was proposed in [2] and implemented in [3] and [4]. The scheme is efficient in the sense that the degree of accuracy of the approximate boundary conditions can be matched with that of the finite difference scheme to define the size of $\mathscr{D}$ and that for each time increment the number of computational steps for the evaluation of the boundary data is at most of the order of magnitude of that for the finite difference solution. In the scheme, the number for the former is $O(N)$ while the number for the latter is dominated by that needed to invert the Poisson equation and hence is $O(N \log N)$, where N is the number of grid points in $\mathscr{D}$. The approximate boundary conditions proposed in [2] were obtained by making use of the far field behavior of $\boldsymbol{A}$.

Another application of the far field behavior of $\boldsymbol{A}$ can be found in a recent study on aerodynamic noise induced by an unsteady vortical flow at low Mach number [5]. Matched asymptotic solutions of the unsteady compressible *N-S* equations are constructed such that the leading inner solution is identified as an incompressible vortical flow governed by the equations of the above initial value problem, Eqs. (1.1, 1.7, 1.10, 1.2, 1.4 and 1.11), while the leading outer solution, which obeys the simple wave equation, yields the acoustic field. From the matching conditions, the far field behavior of the inner solution, $\boldsymbol{A}$, provides the boundary data for the leading outer solution.

1.2. *Far field behavior of a vector potential—review of known results*

To study the far field behavior of the vector potential $\boldsymbol{A}$, we can deal with the problem defined by Eqs. (1.1) to (1.4) and suppress the dependence of the functions on t from hereon.

The solution of the Poisson equation, (1.1), subject to (1.2), is given by the Poisson integral

$$A(x) = \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{\Omega(x')}{|x - x'|} d^3x', \tag{1.12}$$

where d^3x' stands for $dx'_1\, dx'_2\, dx'_3$.

The far field behavior of A is defined by an expansion of the Poisson integral in powers of r^{-1}, see for example Ref. [6]. The result is,

$$A(x) = \sum_{n=0}^{m} A^{(n)}(x) + O(r^{-m-2}). \tag{1.13}$$

where

$$A^{(n)}(x) = \frac{1}{4\pi} Q^{(n)}(\theta, \phi) r^{-n-1} \tag{1.14}$$

and

$$Q^{(n)}(\theta, \phi, t) = \iiint_{-\infty}^{\infty} \Omega(x', t)(r')^n P_n(\mu)\, d^3x', \tag{1.15}$$

with $\mu = x \cdot x'/(rr') = \hat{x} \cdot \hat{x}'$. Here θ and ϕ denote the spherical angles of the unit vector $\hat{x}$ and P_n is a Legendre polynomial. The nth term $A^{(n)}$ in the series (1.13) will be called the vector potential of nth order because it fulfills the Laplace equation and is proportional r^{-n-1}.

We note that $(r')^n P_n(\mu)$ and $r^n P_n(\mu)$ are homogeneous polynomials of degree n in x'_i and x_i respectively. Consequently $r^n Q^{(n)}$ is a homogeneous polynomial in x_i of degree n and its coefficients are the nth moments of vorticity. The number of nth moments of the components of Ω is $3(n+2)(n+1)/2$. These moments exist for all n because Ω decays exponentially (1.4). Using (1.3) and (1.4), Truesdell [7], [8], showed that the nth coaxial moment along an axis parallel to a vector B should vanish:

$$I^{(n)}(b_1, b_2, b_3) = \langle [B \cdot x]^n B \cdot \Omega(x) \rangle = 0 \tag{1.16}$$

for $t \geq 0$ and for all b_i, which are the components of B. Here we use $\langle\ \rangle$ to denote the volume integral over the entire space in x, that is

$$\langle f \rangle = \iiint_{-\infty}^{\infty} f\, d^3x. \tag{1.17}$$

A simple proof of (1.16) follows from the application of integration by parts and far field conditions, (1.2) and (1.4),

$$\langle (x \cdot B)^n \Omega \cdot B \rangle = -\frac{1}{n+1} \langle (x \cdot B)^{n+1} \nabla \cdot \Omega \rangle = 0. \tag{1.18}$$

Since $I^{(n)}$ is a homogeneous polynomial in b_i of degree $n+1$, $I^{(n)}$ vanishes for all b_i if and only if all the $(n+3)(n+2)/2$ coefficients in the polynomial are equal to zero. By equating a coefficient in $I^{(n)}$ to zero, say that of $b_1^j b_2^k b_3^l$, we obtain the following consistency condition on nth moments of ω_1, ω_2 and ω_3,

$$j\langle x_1^{j-1} x_2^k x_3^l \omega_1\rangle + k\langle x_1^j x_2^{k-1} x_3^l \omega_2\rangle + l\langle x_1^j x_2^k x_3^{l-1} \omega_3\rangle = 0, \qquad (1.19)$$
$$\text{for} \quad j, k, l \geq 0 \quad \text{and} \quad j+k+l = n+1 \geq 1,$$

with the understanding that a moment is equal to zero whenever the exponent of one of the x_i's is negative. There are $(n+3)(n+2)/2$ consistency conditions, (1.19). In particular for $n = 0$, we have $\langle \omega_i \rangle = 0$ for $i = 1, 2, 3$ and hence $A^{(0)} = 0$. This fact is referred to in the Abstract in saying that the series representation of A, (1.13), begins with $n = 1$.

On account of these consistency conditions, (1.19), the number of nth moments of ω_i which can be specified or evaluated for the determination of $A^{(n)}$ in the series, (1.13), is J_n, where

$$J_n = 3(n+2)(n+1)/2 - (n+3)(n+2)/2 = n(n+2). \qquad (1.20)$$

It should be noted that by using the vorticity evolution equation, (1.10), Moreau [9], [10], showed that three linear combinations of the first moments and three linear combinations of second moments are time invariant and, therefore, are defined by the initial data. However, this result does not alter the above statement, leading to (1.20), on the number of nth moments to be specified.

1.3. *Motivations for the current analysis*

Using the consistency conditions, (1.19), on nth moments of vorticity, we identify J_n of those moments free to be specified. They form a set $\mathscr{C}^{(n)}$. The remaining nth moments are then related to those in $\mathscr{C}^{(n)}$ by the consistency conditions. The vector potential $A^{(n)}$ can then be expressed as a linear combination of J_n vector functions with the coefficients in $\mathscr{C}^{(n)}$. Each vector function is also a vector potential of nth order. An obvious question is

(i) *what are those J_n vector potentials of nth order associated with the nth moments in $\mathscr{C}^{(n)}$?*

We now note that the far field velocity, which is $\nabla \times A$, should be irrotational to all powers of r^{-1}, because $\boldsymbol{\Omega}$ decays exponentially, (1.4). This also follows directly from (1.14) which yields $\nabla \times (\nabla \times A^{(n)}) = -\Delta A^{(n)} = 0$. Consequently, the velocity can be expressed as a power series in r^{-1}, each term of which is a gradient of a scalar potential, $\Phi^{(n)}$. That is,

$$\boldsymbol{v} = \sum_{n=1,2,\ldots} \boldsymbol{v}^{(n)} = \sum_{n=1,2,\ldots} \nabla\Phi^{(n)} \tag{1.21}$$

with

$$\Phi^{(n)}(\boldsymbol{x}) = Y_n(\theta, \phi) r^{-n-1} \tag{1.22}$$

and

$$\boldsymbol{v}^{(n)} = \nabla \times \boldsymbol{A}^{(n)} = \nabla\Phi^{(n)}. \tag{1.23}$$

Here Y_n denotes Laplace spherical harmonics of dimension n, see for example Ref. [6]. The above relationships were employed in the study of the acoustic field induced by a vortical flow, [5].

Since Y_n can be represented by a linear combination of $2n + 1$ spherical harmonics of nth order, see [6], only $2n + 1$ terms will contribute to the far field velocity. Thus we may ask:

(ii) *what are the* $2n + 1$ *linear combinations of the J_n vector potentials of nth order, mentioned in* (i), *that do contribute to far field velocity,*
(iii) *what happened to the remaining* $J_n - (2n + 1) = n^2 - 1$ *linear combinations and*
(iv) *what are the* $2n + 1$ *linear combinations of nth moments of vorticity which define the coefficients of the* $2n + 1$ *spherical harmonics in the scalar potential* $\Phi^{(n)}$?

We also note that each component, $A_i^{(n)}$, of the nth order vector potential in the series (1.13) for large r should fulfill the Laplace equation. Each component $A_i^{(n)}$ can then be represented by r^{-n-1} times a sum of $2n + 1$ linearly independent spherical harmonics of nth order and the coefficients will be linear combinations of nth moments of ω_i. Consequently, we have

$$M_n \leq 3(2n + 1), \tag{1.24}$$

where M_n denotes the number of linearly independent vector potentials of nth order. This number M_n is certainly less than J_n for $n \geq 5$. Thus we encounter the question,

(v) *what is the number M_n of linearly independent vector potentials of nth order*?

It should be noted that we need only $2n + 1$ linear combinations of nth moments of vorticity to define the nth order far field velocity provided that the current vorticity distribution is specified. To determine the vorticity distribution from its preceding time step in a numerical algorithm, we have to solve the vorticity evolution equation (1.10) and the Poisson equation

(1.1) for $\boldsymbol{\Omega}$ and $\boldsymbol{A}$ simultaneously. We then need the far field behavior of $\boldsymbol{A}$, including the part which does not contribute to the velocity field, to define the appropriate boundary data for $\boldsymbol{A}$.

The first three questions will be answered in Sec. 2.1, the fourth in Sec. 2.2 and the fifth in Sec. 3. These answers are summarized in the Abstract. This first path of derivations goes from the motivations to the conjectures and then to their confirmations. Knowing what to prove we express, in Sec. 4, the moments of vorticity in terms of Cartesian tensors and then give independent proofs of our results by systematically using symbolic tensor operations. In the process of this second path of derivations we identify Truesdell's consistency conditions as symmetry constraints on the moment tensors of vorticity. Sec. 4 can be read independent of Sec. 2 and Sec. 3 and vice versa.

2. The far field vector potentials and the corresponding potential flows

In the first subsection, we shall introduce the Maxwell representation of spherical harmonics [6] for the nth order vector potential in the far field, $\boldsymbol{A}^{(n)}(\boldsymbol{x})$, apply the consistency conditions (1.19) to show that the corresponding flow is a potential flow and identify n^2-1 vector potentials in $\boldsymbol{A}^{(n)}$ which are curl free. In the second subsection, we shall derive explicit formulas for the corresponding scalar potential $\Phi^{(n)}$ and express the coefficients therein as linear combinations of nth moments of vorticity.

2.1. *Vector potentials of nth order*

It was mentioned in Sec. 1.3 that in the far field each component of the vector potential of nth order should fulfill the Laplace equation. We can then introduce the Maxwell representation [6] for each component,

$$A_l^{(n)}(x) = H_l^{(n)}(\xi, \eta, \zeta)r^{-1} \quad \text{for} \quad l = 1, 2, 3, \tag{2.1}$$

where

$$H_l^{(n)} = \sum_{i+j+k=n} C_{i,j,k,l}^{(n)} \xi^i \eta^j \zeta^k. \tag{2.2}$$

Each coefficient in $H_l^{(n)}$ is related to an nth moment of vorticity by,

$$C_{i,j,k,l}^{(n)} = \frac{(-1)^n \langle x_1^i x_2^j x_3^k \omega_l \rangle}{4\pi i! j! k!}. \tag{2.3}$$

Here $H_l^{(n)}$ is a homogeneous polynomial of nth degree and its variables ξ, η and ζ stand for partial differentiations with respect to x_1, x_2 and x_3

respectively. In particular, the gradient operator is

$$\nabla = \hat{\mathbf{1}}\xi + \hat{\mathbf{2}}\eta + \hat{\mathbf{3}}\zeta, \tag{2.4}$$

where $\hat{\boldsymbol{m}}$ stands for the unit vector along the mth axis with $m = 1, 2, 3$.

From (2.3), the $(n+3)(n+2)/2$ consistency conditions (1.19) become

$$C^{(n)}_{i-1,j,k,1} + C^{(n)}_{i,j-1,k,2} + C^{(n)}_{i,j,k-1,3} = 0, \tag{2.5}$$

for $i, j, k \geq 0$ and $n = i + j + k - 1$. From (2.2) and (2.4), we get

$$\nabla \cdot \boldsymbol{H}^{(n)} = \xi H^{(n)}_1 + \eta H^{(n)}_2 + \zeta H^{(n)}_3$$

$$= \sum_{i+j+k=n+1} [C^{(n)}_{i-1,j,k,1} + C^{(n)}_{i,j-1,k,2} + C^{(n)}_{i,j,k-1,3}]\xi^i \eta^j \zeta^k.$$

and then show that

(I) *the consistency conditions*, (1.19) *or* (2.5), *are equivalent to the condition that the vector polynomial* $\boldsymbol{H}^{(n)}$ *satisfies*

$$\nabla \cdot \boldsymbol{H}^{(n)} = \xi H^{(n)}_1 + \eta H^{(n)}_2 + \zeta H^{(n)}_3 = 0, \tag{2.6}$$

for all ξ, η *and* ζ.

It follows that $\boldsymbol{A}^{(n)}$ is divergence free,

$$\nabla \cdot \boldsymbol{A}^{(n)} = \nabla \cdot \boldsymbol{H}^{(n)} r^{-1} = 0.$$

Since $\boldsymbol{A}^{(n)}$ also fulfills the Laplace equation, we recover the result that

(II) *the corresponding velocity* $\boldsymbol{v}^{(n)}(\boldsymbol{x})$ *is irrotational and hence there exists a scalar potential* $\Phi^{(n)}(\boldsymbol{x})$, *such that*

$$\boldsymbol{v}^{(n)} = \nabla \times \boldsymbol{A}^{(n)} = \nabla \Phi^{(n)}.$$

This statement is noted in the Abstract and also in Sec. 1.3.

We note that in general a homogeneous vector polynomial $\boldsymbol{H}^{(n)}$ of degree n lies in a $3(n+2)(n+1)/2$ dimensional linear vector space spanned by $\xi^i \eta^j \zeta^k \hat{\boldsymbol{m}}$, $i, j, k \geq 0$, $i + j + k = n$ and $m = 1, 2, 3$. Because of the $(n+3)(n+2)/2$ consistency conditions, (1.19), (2.5) or (2.6), $\boldsymbol{H}^{(n)}$ has to be in a $J_n = n(n+2)$ dimensional subspace $\mathscr{H}^{(n)}$. We shall define $\mathscr{H}^{(n)}$ in Sec. 2.1.1 and then define in Sec. 2.1.2 an $n^2 - 1$ dimensional subspace $\mathscr{H}^{(n)}_0$ which satisfies $\nabla \times \boldsymbol{H}^{(n)} = 0$, and hence does not contribute to the far field velocity.

2.1.1. *The consistency conditions and the* J_n *divergence free vector polynomials*

Because of the consistency conditions on the nth moments, i.e., the relationships between the coefficients, (2.5), only $J_n = n(n+2)$ of the $\frac{3}{2}(n+2)(n+1)$

coefficients in $\boldsymbol{H}^{(n)}$ can be assigned. We now proceed to identify the set $\mathscr{C}^{(n)}$ of those J_n coefficients and the corresponding set $\mathscr{B}^{(n)}$ of J_n vector polynomials in $\boldsymbol{H}^{(n)}$.

For example, for $i = 1, \ldots, n$, we express the first coefficient in (2.5) in terms of the second and/or the third. The vector polynomials in $\boldsymbol{H}^{(n)}$ associated with the coefficients $C^{(n)}_{i,j-1,k,2}$ are

$$\boldsymbol{W}^{(n,3)}_{i,j} = [-\hat{\mathbf{1}}\eta + \hat{\mathbf{2}}\xi]\xi^{i-1}\eta^{j-1}\zeta^{k}, \tag{2.7}$$

for $j = 1, \ldots, n+1-i$ with $k = n+1-i-j$ while those associated with $C^{(n)}_{i,j,k-1,3}$ are

$$\boldsymbol{W}^{(n,2)}_{i,k} = [-\hat{\mathbf{1}}\zeta + \hat{\mathbf{3}}\xi]\xi^{i-1}\eta^{j}\zeta^{k-1}, \tag{2.8}$$

for $k = 1, \ldots, n+1-i$ with $j = n+1-i-k$.

For $i = 0$, the first coefficient in (2.5) vanishes and only one of the remaining two coefficients can be assigned, say the third. The vector functions in $\boldsymbol{H}^{(n)}$ associated with the third coefficients $C^{(n)}_{0,j,k-1,3}$ are

$$\boldsymbol{W}^{(n,1)}_{0,k} = [-\hat{\mathbf{2}}\zeta + \hat{\mathbf{3}}\eta]\eta^{j-1}\zeta^{k-1}, \tag{2.9}$$

for $k = 1, \ldots, n$ with $j = n+1-k$.

The above three equations define $n(n+1)$ vector polynomials $\boldsymbol{W}^{(n,3)}_{i,j}$ and $\boldsymbol{W}^{(n)}_{i,k}$ and n vector polynomials $\boldsymbol{W}^{(n,3)}_{0,k}$. Altogether there are J_n of them. They constitute the set $\mathscr{B}^{(n)}$, i.e.,

$$\mathscr{B}^{(n)} = \{\boldsymbol{W}^{(n,3)}_{i,j}, \boldsymbol{W}^{(n,2)}_{i,k}, \boldsymbol{W}^{(n,1)}_{0,k}\}. \tag{2.10}$$

The corresponding J_n coefficients referred to in the above three equations in turn form the set $\mathscr{C}^{(n)}$.

We note that the J_n vector polynomials in $\mathscr{B}^{(n)}$ are linearly independent, because they differ from each other either in the exponents of ξ, η and ζ or in the missing spatial component. Thus $\mathscr{B}^{(n)}$ serves as the basis for the vector space $\mathscr{H}^{(n)}$ and we show that,

(III) *the set $\mathscr{B}^{(n)}$ of those J_n vector polynomials spans a J_n dimensional vector space $\mathscr{H}^{(n)}$ which fulfills* (2.6).

It follows from statements (I) and (III) that,

(IV) *a homogeneous vector polynomial $\boldsymbol{H}^{(n)}$ of ξ, η and ζ of degree n fulfilling* (2.6) *has to be in $\mathscr{H}^{(n)}$.*

The vector potential of nth order, $\boldsymbol{A}^{(n)}(\boldsymbol{x})$, is therefore defined by a linear combination of the following J_n vector functions,

$$\boldsymbol{W}^{(n,3)}_{i,j}r^{-1}, \quad \boldsymbol{W}^{(n,2)}_{i,k}r^{-1} \quad \text{and} \quad \boldsymbol{W}^{(n,1)}_{0,k}r^{-1}. \tag{2.11}$$

This answers questions (i) raised in Sec. 1.3. It should be noted that those

J_n vector functions are not linearly independent for $n \geq 3$ because we have not made use of the fact that,

$$[\xi^2 + \eta^2 + \zeta^2]r^{-1} = \Delta r^{-1} = 0. \tag{2.12}$$

We shall address this question, (v) in Sec. 1.3, later in Sec. 3.

Now we shall identify the scalar potentials corresponding to those vector functions in (2.11). We note that all those vector functions have one component missing. For example, any one vector polynomial, $W^{(n)}$, of $W_{i,j}^{(n,1)}$, misses its first component, i.e., $W_1^{(n)} = 0$. Equation (2.6) then yields,

$$W_2^{(n)} = \zeta \tilde{W}_{2,3}, \quad W_3^{(n)} = -\eta \tilde{W}_{2,3} \quad \text{and} \quad W^{(n)} = [\hat{2}\zeta - \hat{3}\eta]\tilde{W}_{2,3}, \tag{2.13}$$

where $\tilde{W}_{2,3}$ is a homogeneous polynomial of degree $n-1$. Now we evaluate the velocity corresponding to $A^{(n)} = W^{(n)}r^{-1}$ and make use of (2.12). The result is

$$v^{(n)} = \nabla \xi \tilde{W}_{2,3} r^{-1}$$

and the corresponding scalar potential is

$$\Phi^{(n)} = \xi \tilde{W}_{2,3} r^{-1}. \tag{2.14}$$

Similarly, the scalar potential corresponding to a vector potential of $W_{i,k}^{(n,2)}$ having the second component missing, i.e., $W_2^{(n)} = 0$, is

$$\Phi^{(n)} = \eta \tilde{W}_{3,1} r^{-1}, \quad \text{with} \quad W_3^{(n)} = \xi \tilde{W}_{3,1} \quad \text{and} \quad W_1^{(n)} = \zeta \tilde{W}_{3,1} \tag{2.15}$$

and that for $W_3^{(n)} = 0$ is

$$\Phi^{(n)} = \zeta \tilde{W}_{1,2} r^{-1}, \quad \text{with} \quad W_1^{(n)} = \eta \tilde{W}_{1,2} \quad \text{and} \quad W_2^{(n)} = -\xi \tilde{W}_{1,2} \tag{2.16}$$

We can then apply (2.14), (2.15) or (2.16) to reconfirm that the velocity field corresponding to any one of the above J_n vector functions in (2.11) is a potential flow. For example, the scalar potential corresponding to $W_{i,j}^{(n,3)}$ is

$$\Phi_{i,j} = \xi^{i-1}\eta^{j-1}\zeta^{k+1}r^{-1}. \tag{2.17}$$

2.1.2. *Vector potential not contributing to the far field velocity*

To find $(n^2 - 1)$ linear combinations of vector polynomials in $\mathscr{B}^{(n)}$ which do not contribute to the far field velocity, we study the curl, i.e., $\nabla \times$, of the corresponding vector polynomials of degree $n-1$, i.e., those in $\mathscr{B}^{(n-1)}$. Let us denote the elements of the set $\mathscr{B}^{(n-1)}$ by $Z_m^{(n-1)}$ for $m = 1, \ldots, J_{n-1} = n^2 - 1$, and then denote $\nabla \times Z_m^{(n-1)}$ by T_m. From (2.4) we note that T_m is a homogeneous vector polynomial of ξ, η and ζ of degree n and

$$\nabla \cdot T_m = \nabla \cdot [\nabla \times Z_m^{(n-1)}] = 0. \tag{2.18}$$

It follows from statement (IV) that,

$$T_m = \nabla \times Z_m^{(n-1)} \in \mathscr{H}^{(n)} \tag{2.19}$$

On the other hand, statement (II) says that there exists a scalar potential $\Phi^{(n-1)}$ such that

$$T_m r^{-1} = \nabla \times Z_m^{(n-1)} r^{-1} = \nabla \Phi^{(n-1)}.$$

Consequently, we have

$$\nabla \times T_m r^{-1} = 0 \tag{2.20}$$

and conclude that:

(V) *the vector polynomials, T_m, $m = 1, \ldots, n^2 - 1$, span an $(n^2 - 1)$ dimensional subspace $\mathscr{H}_0^{(n)}$ of $\mathscr{H}^{(n)}$ and that*

(VI) *the vector potentials $T_m r^{-1}$ are curl free and hence do not contribute to the far field velocity.*

Thus question (iii) raised in Sec. 1.3 is answered.

We can also prove (V) directly by showing that T_m is a linear combination of elements of $\mathscr{B}^{(n)}$. For example, let $Z_m^{(n-1)}$ be $W_{i,j}^{(n-1,3)}$. We than have

$$T_m = \nabla \times W_{i,j}^{(n-1,3)} = -\nabla \xi^{i-1} \eta^{j-1} \zeta^{k+1} = -W_{i,j+1}^{(n,3)} - W_{i,k+1}^{(n,2)}.$$

Knowing the $n^2 - 1$ dimensional subspace $\mathscr{H}_0^{(n)}$ of $\mathscr{H}^{(n)}$, we can construct the subspace, $\mathscr{H}_c^{(n)}$ which is complementary to $\mathscr{H}_0^{(n)}$, i.e.,

$$\mathscr{H}^{(n)} = \mathscr{H}_0^{(n)} \oplus \mathscr{H}_c^{(n)}. \tag{2.21}$$

The dimension of $\mathscr{H}_c^{(n)}$ is $n(n+2) - (n^2 - 1) = 2n + 1$. In other words, we can determine $2n + 1$ linear combinations of vector polynomials in $\mathscr{B}^{(n)}$ that contribute to the far field velocity and answer question (ii) in Sec. 1.3. Also we can express the corresponding $2n + 1$ coefficients as linear combinations of nth moments of vorticity. However, it is easier to determine them directly from $A^{(n)}$ defined by (1.15). This is done in the next subsection.

2.2. *Relating the coefficients in the scalar potential of nth order to nth moments of vorticity*

We recall statement (II) which says that the far field velocity corresponding to the vector potential $A^{(n)}(x)$ can be identified as the gradient of a scalar potential $\Phi^{(n)}$. On the other hand, the latter can be expressed in terms of spherical harmonics, see [6],

$$\Phi^{(n)}(x) = Y_n(\theta, \phi) r^{-n-1}. \tag{2.22}$$

Here, Y_n is a linear combination of the $2n + 1$ spherical harmonics of nth order. Now we shall relate the $2n + 1$ coefficients in Y_n directly to the nth moments of vorticity by making use of Eq. (1.15). In particular, we note that the radial component of $\nabla \times A^{(n)}$ defines $\Phi_r^{(n)}(x)$, which in turn defines

$Y_n(\theta, \phi)$. The result is

$$\hat{\boldsymbol{x}} \cdot [\nabla \times \boldsymbol{A}^{(n)}(\boldsymbol{x})] = \frac{1}{4\pi r^{n+2}} \iiint_{-\infty}^{\infty} r'^{n-1} P_n'(\hat{\boldsymbol{x}}' \cdot \hat{\boldsymbol{x}}) \hat{\boldsymbol{x}} \cdot [\boldsymbol{x}' \times \boldsymbol{\Omega}(\boldsymbol{x}')]\, d^3\boldsymbol{x}'$$
$$= \Phi_r^{(n)}(r, \theta, \phi)$$
$$= -(n+1) Y_n(\theta, \phi) r^{-n-2}, \tag{2.23}$$

and then

$$Y_n(\theta, \phi) = \frac{-1}{4\pi(n+1)} \iiint_{-\infty}^{\infty} r'^{n-1} P_n'(\hat{\boldsymbol{x}}' \cdot \hat{\boldsymbol{x}})[\boldsymbol{x}' \times \boldsymbol{\Omega}] \cdot \hat{\boldsymbol{x}}\, d^3\boldsymbol{x}'. \tag{2.24}$$

where $P_n'(\mu)$ stands for the derivative of $P_n(\mu)$. Since the integrand is a homogeneous polynomial of x_i' and $\hat{x}_i$, $i = 1, 2, 3$, of degree n, the $2n+1$ coefficients in Y_n are linear combinations of nth moments of vorticity. Thus we have settled question (iv) in Sec. 1.3.

For example, there are three spherical harmonics in Y_1,

$$Y_1(\theta, \phi) = -\frac{\hat{\boldsymbol{x}} \cdot \langle \boldsymbol{x}' \times \boldsymbol{\Omega} \rangle}{8\pi} = -\frac{1}{8\pi} \sum_{i=1}^{3} \hat{x}_i \langle x_j' \omega_k - x_k' \omega_j \rangle \tag{2.25}$$

and five in Y_2,

$$Y_2(\theta, \phi) = -\frac{1}{8\pi} \sum_{l=1}^{3} \hat{\boldsymbol{x}} \cdot \langle \boldsymbol{x}' \times \boldsymbol{\Omega} x_l' \rangle \hat{x}_l$$
$$= \frac{1}{24\pi} \sum_{i=1}^{3} [9 F_i \hat{x}_j \hat{x}_k + 2\hat{x}_i^2 (G_j - G_k)], \tag{2.26}$$

with i, j, k in cyclic order. Here F_i and H_i are linear combinations of second moments of vorticity,

$$F_i = \langle \omega_i (x_j^2 - x_k^2) \rangle \quad \text{and} \quad G_i = \langle 2\omega_i x_j x_k - \omega_j x_k x_i - \omega_k x_i x_j \rangle. \tag{2.27}$$

We note that $G_1 + G_2 + G_3 \equiv 0$, so there are only five linearly independent combinations. The first two terms of the far field scalar potential, can be identified as three doublets and five quadrupoles respectively. They are

$$\Phi^{(1)}(\boldsymbol{x}) = \frac{1}{8\pi} \sum_{i=1}^{3} \langle x_j \omega_k - x_k \omega_j \rangle\, \partial_i r^{-1} \tag{2.28}$$

and

$$\Phi^{(2)}(\boldsymbol{x}) = \frac{1}{8\pi} \sum_{i=1}^{3} F_i\, \partial_j\, \partial_k r^{-1} + \frac{1}{36\pi} \sum_{i=1}^{3} [G_j - G_k]\, \partial_i^2 r^{-1}, \tag{2.29}$$

with i, j, k in cyclic order.

In this section, we began with Truesdell's $(n+3)(n+2)/2$ consistency conditions on nth moments of vorticity, found a set $\mathscr{C}^{(n)}$ of $J_n = n(n+2)$

moments which can be assigned and identified the corresponding set $\mathscr{B}^{(n)}$ of J_n vector polynomials and the J_n vector functions in $A^{(n)}$, see (2.11). We then employed the condition that in the far field each component of $A^{(n)}(x)$ is a potential solution to show that only $2n + 1$ linear combinations of those J_n vector functions in $A^{(n)}$ will contribute to the far field velocity.

In the next section, we shall carry out our investigations in the reverse order. We shall begin with the condition that each component of $A^{(n)}(x)$ is a potential solution and later employ the condition that each vector potential of nth order should be divergence free to deduce the answer to question (v) and then (iv) in Sec. 1.3. That is for $n \geq 2$ there are $4n$ linearly independent vector potentials of nth order, while only $2n + 1$ linear combinations of them contribute to the far field velocity.

3. Linearly independent vector potentials of nth order

It was pointed out in Sec. 1 and also in Sec. 2.1 that in the far field each component of $A^{(n)}$ is a potential solution. It can therefore be expressed in terms of spherical harmonics of n order,

$$A_l^{(n)}(x) = \sum_{k=0}^{2n} a_{k,l} y_{n,k}(\theta, \phi) r^{-n-1} \tag{3.1}$$

for $l = 1, 2, 3$. Here $y_{n,k}(\theta, \phi)$, $k = 0, \ldots, 2n$ denote the set $\mathscr{Y}^{(n)}$ of $2n + 1$ linearly independent spherical harmonics of nth order. They are [6]

$$\begin{aligned} y_{n,0} &= P_0(\cos\theta), \\ y_{n,2h-1} &= \cos h\phi P_n^h(\cos\theta), \\ y_{n,2h} &= \sin h\phi P_n^h(\cos\theta), \end{aligned} \tag{3.2}$$

for $h = 1, \ldots, n$. Each coefficient, $a_{k,l}$, can be related to a linear combination of nth moments of ω_l by (1.15). The vector potential becomes,

$$A^{(n)} = \sum_{l=1}^{3} \sum_{k=0}^{2n} a_{k,l}\, \hat{\mathbf{l}} y_{n,k}(\theta, \phi) r^{-n-1}. \tag{3.3}$$

We then conclude that,

(VII) *the vector potential* $A^{(n)}$ *lies in the vector space* $\mathscr{L}^{(n)}$ *spanned by the* $3(2n + 1)$ *linearly independent vector potentials of nth order,* $\hat{\mathbf{l}} y_{n,k} r^{-n-1}$, and

(VIII) $A^{(n)}(x)$ *is defined by* $3(2n + 1)$ *linear combinations of nth moments of vorticity, among* $3(n + 2)(n + 1)/2$ *of them.*

Now we shall make use of the fact that $A^{(n)}$ has to be divergence free, i.e.,

$$\nabla \cdot A^{(n)} = \sum_{k=0}^{2n} \sum_{l=1}^{3} a_{k,l}\, \partial_l [y_{n,k} r^{-n-1}] = 0. \tag{3.4}$$

We note that $\nabla \cdot A^{(n)}$ is a potential solution of $(n+1)$th order and hence can be expressed in terms of the spherical harmonics in $\mathscr{Y}^{(n+1)}$, i.e.,

$$\nabla \cdot A^{(n)} = \sum_{j=0}^{2n+2} b_j y_{n+1,j}(\theta, \phi) r^{-n-2}. \tag{3.5}$$

The divergence free condition, (3.4), requires that all the coefficients, b_j, equal to zero, which in turn yields $2n+3$ linear relationships on the $6n+3$ coefficients, $a_{k,l}$. The relationships are

$$\sum_{l=1}^{3} \sum_{k=0}^{2n} a_{k,l} \int_{-\pi}^{\pi} \int_{0}^{\pi} y_{n+1,j}(\theta, \phi)[\partial_l y_{n,k}(\theta, \phi) r^{-n-1}] \sin\theta \, d\theta \, d\phi = 0, \tag{3.6}$$

for $j = 0, \ldots, 2n+2$. Therefore, we can reduce the $6n+3$ coefficients, $a_{k,l}$, in $A^{(n)}$ to $4n$ and express $A^{(n)}$ in terms of $4n$ linearly independent vector potentials $A^{(n,i)}$, $i = 1, \ldots, 4n$. Each one of them is a linear combination of the elements of $\mathscr{Y}^{(n)}$ times r^{-n-1}. Since $4n \le n(n+2)$ for $n > 2$, we conclude that

(IX) *a divergence free vector potential of nth order, $A^{(n)}$, can be represented by a linear combination of M_n linearly independent vector potentials $A^{(n,i)}$ and the M_n coefficients are linear combinations of nth moments of vorticity, where $M_n = 3$ for $n = 1$ and $M_n = 4n$ for $n \ge 2$.*

This is the answer to question (v) in Sec. 1.3.

Another way of arriving at this is to note that the conditions (3.6) for $j = 0, \ldots, 2n+2$, exclude the vector potentials in a $2n+3$ dimensional subspace $\mathscr{L}_0^{(n)}$ of $\mathscr{L}^{(n)}$ and we have

$$\mathscr{L}^{(n)} = \mathscr{L}_0^{(n)} \oplus \mathscr{A}^{(n)} \tag{3.7}$$

where $\mathscr{A}^{(n)}$ denotes the complementary subspace of dimension $4n$. Let its basis be denoted by $\{A^{(n,i)}, i = 1, \ldots, 4n\}$. We then say that

(X) *a divergence free vector potential of nth order has to be in $\mathscr{A}^{(n)}$ for $n \ge 2$.*

Now we shall rederive the answer to question (iv) by the method of induction. We shall show that

(XI) *the far field velocity corresponding to $A^{(n)}$ depends only on $2n+1$ vector potentials $Z^{(n,j)}$, $j = 1, \ldots, 2n+1$ with the $2n+1$ coefficients defined by linear combinations of nth moments of vorticity. Each one of the $Z^{(n,j)}$'s is a linear combination of the $4n$ vector potentials $A^{(n,i)}$.*

This is equivalent to saying that

$$\mathscr{A}^{(n)} = \mathscr{A}_0^{(n)} \oplus \mathscr{A}_c^{(n)}, \tag{3.8}$$

where $\mathscr{A}_0^{(n)}$ is a $2n-1$ dimensional subspace which is curl free while a vector potential of nth order which contributes to the far field velocity is in the complementary subspace $\mathscr{A}_c^{(n)}$ of dimension $2n+1$. The dimension of the latter has to be $2n+1$ because a scalar potential of nth order has $2n+1$ spherical harmonics.

For $n = 1$, there are four linearly independent vector potentials $A^{(1,i)}$ $i = 1, \ldots, 4$. Since ∇r^{-1} is a divergence free vector potential of first order, and is curl free, it has to be in $\mathscr{A}_0^{(1)}$ and the dimension of $\mathscr{A}_c^{(1)}$ is $2n+1=3$. The above statement is proven for $n = 1$. Of course, we can arrive at this result immediately from statements (II) and (III) noting that for $n = 1$, $J_1 = n(n+2) = 2n+1$.

Now we assume that the above statement is true for $n-1$, that is, there are $2n-1$ vector potentials of $(n-1)$th order, $\boldsymbol{Z}^{(n-1,j)}$ $j = 1, \ldots, 2n-1$, which form the basis of $\mathscr{A}_c^{(n-1)}$. Note that $\nabla \times \boldsymbol{Z}^{(n-1,j)}$ represents a divergence free vector potential of nth order, hence it is in $\mathscr{A}^{(n)}$. Since it is also curl free, it has to be in $\mathscr{A}_0^{(n)}$ which is then spanned by the $2n-1$ vector potentials of nth order, $\nabla \times \boldsymbol{Z}^{(n-1,j)}$, $j = 1, \ldots, 2n-1$.

Thus the above statement is true for n and is thereby proven for all n.

It should be noted that the derivation presented in this section is much simpler than that in Sec. 2 but on the other hand it relaxes the connection with the explicit presentations. In the next section we shall present an alternative but systematic derivation of the results in Sec. 2 and Sec. 3.

4. Systematic derivation using tensor algebra

In this section we present a unified and systematic derivation of the answers to questions (i) to (v) raised in Sec. 1.3. All derivations in the preceding two sections use cartesian coordinates of vectors in the three dimensional Euclidian physical space $\mathscr{E}$. Those results may be rederived in a comprehensive way using the concept of affine tensors (see e.g. [12]). To this end we first introduce the relevant nomenclature. Let $\mathscr{T}^{(n)}$ denote the space of affine tensors of rank n of $\mathscr{E}$. Then, for example, the scalar function $\Phi(\boldsymbol{x})$ is in $\mathscr{T}^{(0)}$ and the vector functions $\boldsymbol{\Omega}(\boldsymbol{x})$, $\boldsymbol{A}(\boldsymbol{x})$, $\boldsymbol{v}(\boldsymbol{x})$ and the position vector $\boldsymbol{x}$ are in $\mathscr{T}^{(1)}$. The nth moments of vorticity $\langle x_1^i x_2^j x_3^k \omega_m \rangle$ are then canonical coordinates of the nth moment tensor

$$\langle (\boldsymbol{x} \circ)^n \boldsymbol{\Omega}(\boldsymbol{x}) \rangle \in \mathscr{T}^{(n+1)}. \tag{4.1}$$

The symbol '$\circ$' indicates the tensorial product and $(\boldsymbol{x} \circ)^n$ denotes the tensor of rank n: $(\boldsymbol{x} \circ \boldsymbol{x} \circ \cdots \circ \boldsymbol{x})$. Let $(\nabla \circ)^n$ stand for n-fold application of the gradient operator, then the multipoles $\partial_1^i \partial_2^j \partial_3^k(1/r)$ with $(i+j+k=n)$ are canonical coordinates of the nth gradient tensor

$$(\nabla\circ)^n\left(\frac{1}{r}\right)\in\mathscr{T}^{(n)}. \tag{4.2}$$

Using (4.1) and (4.2), we rewrite the far field expansion of the Poisson integral, (1.13–1.15), as

$$A=\sum_{n=0}^{\infty}A^{(n)}=\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\langle(x'\circ)^n\Omega'\rangle\odot^n(\nabla\circ)^n\left(\frac{1}{r}\right), \tag{4.3}$$

where Ω' stands for $\Omega(x')$ and the symbol '$\odot$' denotes the contraction of tensors from the left after tensorial multiplication. For instance the two-fold contraction, $\odot^2$, of tensors $(a\circ b\circ c)$ and $(z\circ y\circ x)$, both in $\mathscr{T}^{(3)}$, is

$$(a\circ b\circ c)\odot^2(z\circ y\circ x)=(a\cdot z)(b\cdot y)c\circ x. \tag{4.4}$$

From hereon we further say:

A tensor $T^{(n)}\in\mathscr{T}^{(n)}$ is symmetric with respect to its νth and μth argument, if in a canonical representation

$$T^{i_1\dots i_\mu\dots i_\nu\dots i_n}(\hat{i}_1\circ\dots\circ\hat{i}_\mu\circ\dots\circ\hat{i}_\nu\circ\dots\circ\hat{i}_n) \tag{4.5}$$
(Summation over double indices)

the coefficients do not alter upon interchange of the μth and νth index.

In Sec. 4.1, we address question (i). We present an alternative formulation of Truesdell's consistency conditions and show that they are equivalent to certain symmetry constraints on the nth moment tensor (4.1). These constraints confine the tensor to a $J_n=n(n+2)$-dimensional subspace $\hat{\mathscr{C}}^{(n)}$ of $\mathscr{T}^{(n+1)}$. This reflects the fact, pointed out in Sec. 1.2, that J_n is the number of nth moments free to be assigned. As a consequence $A^{(n)}$ is a linear combination of J_n vector potentials of nth order each of which corresponds to an element of a basis of $\hat{\mathscr{C}}^{(n)}$.

In Sec. 4.2 we use the symmetry constraints and the fact that $(1/r)$ satisfies the Laplace's equation to decompose $A^{(n)}$ as

$$A^{(n)}=\nabla\psi^{(n)}+\nabla\times B^{(n)}. \tag{4.6}$$

We express the vector function $B^{(n)}$ and the scalar function $\psi^{(n)}$ in terms of the nth moment tensor and of the $(n-1)$st gradient tensor, $(\nabla\circ)^{n-1}(1/r)$. The far field velocity $v^{(n)}=\nabla\times A^{(n)}$ then becomes $\nabla(\nabla\cdot B^{(n)})$ in anticipation of $\Delta B^{(n)}=0$. Thus the corresponding nth order scalar potential is related to $B^{(n)}$ by

$$\Phi^{(n)}=\nabla\cdot B^{(n)}. \tag{4.7}$$

It should be noted that it is not the Helmholtz decomposition theorem, [13], that leads to (4.6), because $\boldsymbol{A}^{(n)}$ is singular at $\boldsymbol{x} = 0$.

In Sec. 4.3 we show that $\psi^{(n)}$ and $\Phi^{(n)}$ are determined by two sets of linear combinations of nth moments of vorticity, $q_l^{(n)}$, $(l = 1, \ldots, 2n - 1)$ and $p_k^{(n)}$, $(k = 1, \ldots, 2n + 1)$ respectively, which are particular projections of the nth moment tensor onto subspaces $\mathscr{S}\mathscr{H}^{(n-1)}$ and $\mathscr{S}\mathscr{H}^{(n)}$ of $\mathscr{T}^{(n-1)}$ and $\mathscr{T}^{(n)}$. These subspaces are spanned by the $(n - 1)$st and nth gradient tensors (4.2) as $\boldsymbol{x}$ varies within $\mathscr{E}$. The second term $\nabla \times \boldsymbol{B}^{(n)}$ in (4.6) is also readily determined by the coefficients $p_k^{(n)}$. The construction of the $q_l^{(n)}$ and $p_k^{(n)}$ will provide the answers to questions (ii) to (v).

4.1. *Symmetry constraints on the moment of vorticity tensor*

We shall rederive Truesdell's consistency conditions (1.19) as symmetry constraints on the nth moment tensor $\langle(\boldsymbol{x} \circ)^n \boldsymbol{\Omega}\rangle$, which will restrict the tensor to a subspace $\hat{\mathscr{C}}^{(n)}$ of $\mathscr{T}^{(n+1)}$.

First we note that an nth moment tensor is symmetric with respect to its first n arguments, i.e.

$$\langle(\boldsymbol{x} \circ)^n \boldsymbol{\Omega}\rangle \in \mathscr{T}_S^{(n)} \otimes \mathscr{T}^{(1)}, \tag{4.8}$$

where $\mathscr{T}_S^{(n)}$ denotes the completely symmetric $\frac{1}{2}(n + 2)(n + 1)$-dimensional subspace of $\mathscr{T}^{(n)}$ and where $\otimes$ indicates the tensorial product of linear vector spaces.

Now we let $\boldsymbol{T}^{(n)}$ be any tensor of rank n in $\mathscr{T}^{(n)}$ and consider the volume integral

$$\begin{aligned}\langle\nabla' \cdot (\boldsymbol{T}^{(n)} \odot^n ((\boldsymbol{x}' \circ)^n \boldsymbol{\Omega}'))\rangle &= \boldsymbol{T}^{(n)} \odot^n \langle(\boldsymbol{x}' \circ)^n \nabla' \cdot \boldsymbol{\Omega}'\rangle \\ &\quad + \boldsymbol{T}^{(n)} \odot^n \sum_{\nu=0}^{n-1} \langle(\boldsymbol{x}' \circ)^{n-1-\nu} \boldsymbol{\Omega}' \circ (\boldsymbol{x}' \circ)^\nu\rangle.\end{aligned}$$

Here ∇' is the gradient operator with respect to $\boldsymbol{x}'$, and $\boldsymbol{\Omega}' \circ (\boldsymbol{x}' \circ)^\nu$ stands for $\boldsymbol{\Omega}'$ if $\nu = 0$ and for $\boldsymbol{\Omega}' \circ (\boldsymbol{x}' \circ)^{\nu - 1} \boldsymbol{x}'$ if $\nu \geq 1$. Using the divergence theorem and the far field behavior (1.4) of $\boldsymbol{\Omega}$ one can show that the left handed side of the above equation vanishes. The first term on the right handed side vanishes as well, because $\nabla \cdot \boldsymbol{\Omega} = 0$. The equation thus becomes

$$\boldsymbol{T}^{(n)} \odot^n \sum_{\nu=0}^{n-1} \langle(\boldsymbol{x}' \circ)^{n-1-\nu} \boldsymbol{\Omega}' \circ (\boldsymbol{x}' \circ)^\nu\rangle = 0, \tag{4.9}$$

for all $\boldsymbol{T}^{(n)} \in \mathscr{T}^{(n)}$. This relation only holds if the symmetric part of the tensor $\langle(\boldsymbol{x} \circ)^{n-1} \boldsymbol{\Omega}\rangle$ vanishes and, with $(n - 1)$ replaced by n, we obtain

$$\langle(\boldsymbol{x} \circ)^n \boldsymbol{\Omega}\rangle_S = \frac{1}{n+1} \sum_{\nu=0}^{n} \langle(\boldsymbol{x} \circ)^{n-\nu} \boldsymbol{\Omega} \circ (\boldsymbol{x} \circ)^\nu\rangle = 0, \tag{4.10}$$

or

$$\langle (\boldsymbol{x} \circ)^n \boldsymbol{\Omega} \rangle \notin \mathscr{T}_S^{(n+1)}.$$

Equation (4.10) is equivalent to the consistency conditions given in (1.19), but, as we shall see later, one can easily derive new identities from (4.9) by choosing tensors $\boldsymbol{T}^{(n)}$ different from $(\boldsymbol{B} \circ)^n$ which leads back to (1.16) and (1.19).

Since $T_S^{(n+1)}$ has dimension $\frac{1}{2}(n+2)(n+3)$ and is a subspace of $\mathscr{T}_S^{(n)} \otimes \mathscr{T}^{(1)}$ of dimension $\frac{3}{2}(n+1)(n+2)$, we conclude from Eqs. (4.8) and (4.10) that:

The n-th moment tensor $\langle (\boldsymbol{x}' \circ)^{(n)} \boldsymbol{\Omega}' \rangle$ belongs to the subspace

$$\hat{\mathscr{C}}^{(n)} := (\mathscr{T}_S^{(n)} \otimes \mathscr{T}^{(1)}) \backslash \mathscr{T}_S^{(n+1)} \tag{4.11}$$

of $\mathscr{T}^{(n+1)}$ with dimension $J_n = n(n+2)$.

Note that a basis of $\hat{\mathscr{C}}^{(n)}$ will constitute the set of linearly independent nth moments $\mathscr{C}^{(n)}$ mentioned in Sec. 1.3.

Subtracting from the nth moment tensor in $\boldsymbol{A}^{(n)}$ in (4.3) its symmetric part (4.10), we obtain

$$\boldsymbol{A}^{(n)} = d_n \langle (\boldsymbol{x}' \circ)^{n-1} [\boldsymbol{x}' \circ \boldsymbol{\Omega}' - \boldsymbol{\Omega}' \circ \boldsymbol{x}'] \rangle \odot^n (\nabla \circ)^n \left(\frac{1}{r}\right), \tag{4.12}$$

where $d_n = (-1)^n n / [4\pi (n+1)!]$. By carrying out the contractions on the right handed side and using the identity for the triple vector product

$$(\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = \boldsymbol{c} \odot (\boldsymbol{a} \circ \boldsymbol{b} - \boldsymbol{b} \circ \boldsymbol{a}), \tag{4.13}$$

with $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})$ replaced by $(\boldsymbol{x}', \boldsymbol{\Omega}', \nabla)$, we get

$$\begin{aligned} \boldsymbol{A}^{(n)} &= d_n \langle (\boldsymbol{x}' \cdot \nabla)^{n-1} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle \times \nabla \left(\frac{1}{r}\right) \\ &= d_n [\langle (\boldsymbol{x}' \circ)^{(n-1)} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle \times \nabla] \odot^{n-1} (\nabla \circ)^{n-1} \left(\frac{1}{r}\right). \end{aligned} \tag{4.14}$$

Note that ∇ is the gradient operator with respect to $\boldsymbol{x}$ and hence commutes with $\boldsymbol{x}'$ and $\boldsymbol{\Omega}'$.

We are now ready to answer question (i): The coefficient tensor $\langle (\boldsymbol{x}' \circ)^{(n-1)} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle$ is symmetric with respect to its $(n-1)$ leftmost arguments and therefore belongs to $\mathscr{T}_S^{(n-1)} \otimes \mathscr{T}^{(1)}$. Furthermore it has zero contraction $\langle (\boldsymbol{x}' \circ)^{n-2} \boldsymbol{x}' \cdot (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle \equiv 0$ of the rightmost with anyone of the other arguments. These contractions are in $\mathscr{T}_S^{(n-2)}$ of dimension $\frac{1}{2}n(n-1)$. Therefore, the coefficient tensor in $\boldsymbol{A}^{(n)}$ has only $\frac{3}{2}n(n+1) - \frac{1}{2}n(n-1) = J_n$ independent components free to be assigned. Due to the equivalence of the axial vector $(\boldsymbol{x} \times \boldsymbol{\Omega})$ and the skewsymmetric tensor of rank two ($x \circ \Omega - \Omega \circ x$) in three dimensions there is a one to one correspondence between a linearly independent set of these components of the coefficient tensor and a basis of $\hat{\mathscr{C}}^{(n)}$. Hence any basis of $\hat{\mathscr{C}}^{(n)}$, via (4.14), provides a set of J_n vector potentials of nth order. A particular such set is given by (2.10).

4.2. *The decomposition of $A^{(n)}$*

We shall decompose $A^{(n)}$ into a gradient and a curl contribution, according to (4.6). For $n = 1$ the decomposition is readily given by Eq. (4.14) which yields

$$A^{(1)} = \nabla \times B^{(1)} \quad \text{with} \quad B^{(1)} = \frac{\langle x' \times \Omega' \rangle}{8\pi r} \quad \text{and} \quad \psi^{(1)} = 0. \tag{4.15}$$

Equation (4.7) provides the corresponding scalar potential

$$\Phi^{(1)} = \frac{\langle x' \times \Omega' \rangle}{8\pi} \cdot \nabla\left(\frac{1}{r}\right). \tag{4.16}$$

We see that the vector potential for $n = 1$ is defined by the three components of $\langle x \times \Omega \rangle$ and thus we find $M_1 = 3$ (see question (v)).

For $n \geq 2$ we first evaluate the velocity of nth order. Using (4.13) and the fact that $\Delta(1/r) = 0$, we obtain from (4.12)

$$v^{(n)} = \nabla \times A^{(n)} = -d_n \nabla \langle (x' \cdot \nabla)^{n-1} (x' \times \Omega') \rangle \cdot \nabla\left(\frac{1}{r}\right) \tag{4.17}$$

and hence the corresponding scalar potential is

$$\begin{aligned} \Phi^{(n)} &= -d_n \langle (x' \cdot \nabla)^{n-1} (x' \times \Omega') \rangle \cdot \nabla\left(\frac{1}{r}\right) \\ &= -d_n \langle (x' \circ\,)^{(n-1)} (x' \times \Omega) \rangle \odot^n (\nabla \circ\,)^n \left(\frac{1}{r}\right) \\ &= -d_n \nabla \cdot \langle (x' \circ\,)^{(n-1)} (x' \times \Omega') \rangle_S \odot^{n-1} (\nabla \circ\,)^{n-1} \left(\frac{1}{r}\right). \end{aligned} \tag{4.18}$$

In the above equations we may replace the tensor $\langle (x' \circ\,)^{(n-1)} (x' \times \Omega') \rangle$ by its symmetric part because of the symmetry of the nth gradient tensor. Equation (4.7) now implies

$$B^{(n)} = -d_n \langle (x' \circ\,)^{(n-1)} (x' \times \Omega') \rangle_S \odot^{n-1} (\nabla \circ\,)^{n-1} \left(\frac{1}{r}\right). \tag{4.19}$$

This equation shows the special role played by the symmetric part of the coefficient tensor in the curl contribution to $A^{(n)}$ and suggests that the difference of $\boldsymbol{A}^{(n)}$ and $\nabla \times \boldsymbol{B}^{(n)}$ be curl free. This difference is

$$\frac{1}{d_n}[\boldsymbol{A}^{(n)} - \nabla \times \boldsymbol{B}^{(n)}] =$$

$$-\frac{n-1}{n} \nabla \times \langle (\boldsymbol{x}' \circ)^{(n-1)}(\boldsymbol{x}' \times \boldsymbol{\Omega}')\rangle \odot^{n-1} (\nabla \circ)^{n-1}\left(\frac{1}{r}\right)$$

$$-\frac{n-1}{n} \langle (\boldsymbol{x}' \cdot \nabla)^{n-2}((\boldsymbol{x}' \times \boldsymbol{\Omega}') \cdot \nabla)\boldsymbol{x}' \times \nabla \rangle \left(\frac{1}{r}\right). \tag{4.20}$$

We observe that the first term equals $(n-1)\boldsymbol{A}^{(n)}/nd_n$ and, by applying (4.13) twice and noting that $\Delta(1/r) = 0$, we find that the second term can be replaced by

$$-\frac{n-1}{n} \nabla(\langle (\boldsymbol{x}' \cdot \nabla)^{n-2}(\boldsymbol{x}' \times \boldsymbol{\Omega}') \times \boldsymbol{x}' \rangle \cdot \nabla r^{-1}) - \frac{n-1}{n}\frac{1}{d_n}\boldsymbol{A}^{(n)}.$$

Equation (4.20) then yields the decomposition

$$\boldsymbol{A}^{(n)} - \nabla \times \boldsymbol{B}^{(n)} = \nabla \psi^{(n)}$$

of $\boldsymbol{A}^{(n)}$ with

$$\psi^{(n)} = -\frac{n-1}{n} d_n \langle (\boldsymbol{x}' \cdot \nabla)^{n-2}(\boldsymbol{x}' \times \boldsymbol{\Omega}') \times \boldsymbol{x}' \rangle \cdot \nabla\left(\frac{1}{r}\right). \tag{4.21}$$

We will further simplify $\psi^{(n)}$ by using the following identity derived from the symmetry constraints (4.9) with $\boldsymbol{T}^{(n)} = \hat{\boldsymbol{\imath}} \circ \hat{\boldsymbol{\imath}} \circ (\nabla \circ)^{(n-2)}(1/r)$(*summation over* $\hat{\boldsymbol{\imath}}$) and with n replaced by $n+1$:

$$\langle |\boldsymbol{x}'|^2(\boldsymbol{x}' \cdot \nabla)^{n-1}(\boldsymbol{\Omega}' \cdot \nabla)\rangle \left(\frac{1}{r}\right) = \frac{-2}{n-1} \langle (\boldsymbol{x}' \cdot \nabla)^{n-1}(\boldsymbol{x}' \cdot \boldsymbol{\Omega}')\rangle \left(\frac{1}{r}\right).$$

With this relation and using again (4.13) we obtain from (4.21)

$$\psi^{(n)} = \frac{n+1}{n} d_n \langle (\boldsymbol{x}' \cdot \nabla)^{n-1}(\boldsymbol{x}' \cdot \boldsymbol{\Omega}')\rangle \left(\frac{1}{r}\right). \tag{4.22}$$

Thus the decomposition of $\boldsymbol{A}^{(n)}$ is accomplished.

4.3. *Linear combinations of nth moments in the vector potential $\boldsymbol{A}^{(n)}$ and the scalar potential $\Phi^{(n)}$*

Equation (4.18) says that only the contraction of the symmetric moments $\langle (\boldsymbol{x}' \circ)^{n-1}(\boldsymbol{x}' \times \boldsymbol{\Omega}')\rangle_S$ with the nth gradient tensor $(\nabla \circ)^n(1/r)$ can contribute to the nth order scalar potential $\Phi^{(n)}$. Now it is well known [6] that any multipole $\partial_1^i \, \partial_2^j \, \partial_3^k(1/r)$, with $(i + j + k = n)$, can be expressed as a linear combination of $(2n+1)$ spherical harmonics of nth order $Y_k^{(n)}(\boldsymbol{x})$, $(k = 1, \ldots, 2n+1)$. We infer that there is a representation of an nth gradient tensor as

$$(\nabla \circ)^n \left(\frac{1}{r}\right) = \sum_{k=1}^{2n+1} \boldsymbol{G}_k^{(n)} Y_k^{(n)}(\boldsymbol{x}), \tag{4.23}$$

where the tensors $\boldsymbol{G}_k^{(n)}$, which span a $(2n+1)$-dimensional subspace $\mathscr{SH}^{(n)}$ of $\mathscr{T}^{(n)}$, are defined once a set of nth spherical harmonics is chosen and vice versa. We consider a set of orthonormalized tensors $\boldsymbol{G}_k^{(n)}$ so that $\boldsymbol{G}_i^{(n)} \odot^n \boldsymbol{G}_j^{(n)} = \delta_{ij}$. Then the answer to question (iv) is:

The scalar potential $\Phi^{(n)}$ *is defined by a set* $\{p_k^{(n)}\}_{k=1}^{2n+1}$ *of* $2n+1$ *linear combinations of nth moments of vorticity as*

$$\Phi^{(n)}(\boldsymbol{x}) = -d_n \sum_{k=1}^{2n+1} p_k^{(n)} Y_k^{(n)}(\boldsymbol{x}), \tag{4.24}$$

where

$$p_k^{(n)} = \langle (\boldsymbol{x}' \circ)^{n-1} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle_S \odot^n \boldsymbol{G}_v^{(n)}. \tag{4.25}$$

The same argument holds for the scalar field $\psi^{(n)}$ in (4.22), except that the gradient tensor in (4.22) is of rank $(n-1)$. Consequently,

$\psi^{(n)}$ *is defined by a set of* $(2n-1)$ *independent projections* $q_l^{(n)}$ *as*

$$\psi^{(n)}(\boldsymbol{x}) = \frac{n+1}{n} d_n \sum_{l=1}^{2n-1} q_l^{(n)} Y_l^{(n-1)}(\boldsymbol{x}), \tag{4.26}$$

where

$$q_l^{(n)} = \langle (\boldsymbol{x}' \circ)^{n-1} (\boldsymbol{x}' \cdot \boldsymbol{\Omega}') \rangle \odot^{n-1} \boldsymbol{G}_l^{(n-1)}. \tag{4.27}$$

Our calculations in Sec. 2 are equivalent to choosing particular $\boldsymbol{G}_k^{(n)}$ and $\boldsymbol{G}_l^{(n-1)}$ and deriving explicit formulae for the projections in (4.25) and (4.27). It remains to show that the set $\{p_k^{(n)}\}_{k=1}^{2n+1}$ also completely determines $\nabla \times \boldsymbol{B}^{(n)}$, which is not obvious from equation (4.19). To prove this we will, in a first step, construct the complement

$$\mathscr{CSH}^{(n)} = \mathscr{T}_S^{(n)} \backslash \mathscr{SH}^{(n)} \tag{4.28}$$

of the subspace of $\mathscr{SH}^{(n)}$ in $\mathscr{T}_S^{(n)}$, thereby discovering a general representation of any tensor in this space. Then, given a basis $\{\boldsymbol{Z}_j^{(n)}\}$ of $\mathscr{CSH}^{(n)}$, the symmetric tensor appearing in $\boldsymbol{B}^{(n)}$ has the decomposition

$$\langle (\boldsymbol{x}' \circ)^{n-1} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle_S = \sum_{v=1}^{N_c} \beta_v^{(n)} \boldsymbol{Z}_v^{(n)} + \sum_{k=1}^{2n+1} p_k^{(n)} \boldsymbol{G}_k^{(n)}, \tag{4.29}$$

where N_c is the dimension of $\mathscr{CSH}^{(n)}$. Any of the $\boldsymbol{Z}_j^{(n)}$ will prove to yield a zero contribution to $\nabla \times \boldsymbol{B}^{(n)}$ when it replaces the moment tensor in (4.19).

Therefore the mere knowledge of the projections $\{p_k^{(n)}\}_{k=1}^{2n+1}$ is sufficient to compute the curl contribution to $\boldsymbol{A}^{(n)}$. We proceed by establishing the general form of members of $\mathscr{CSH}^{(n)}$.

Let $\mathscr{CSH}_*^{(n)}$ be the set

$$\{\boldsymbol{C}^{(n)} \mid \boldsymbol{C}^{(n)} = (\boldsymbol{C}_S^{(n-2)} \circ \hat{\boldsymbol{j}} \circ \hat{\boldsymbol{j}})_S, \quad \boldsymbol{C}_S^{(n-2)} \in \mathscr{T}_S^{(n-2)}\} \quad \text{(Summation over } \hat{\boldsymbol{j}}) \tag{4.30}$$

with $(\hat{\boldsymbol{j}} \circ \hat{\boldsymbol{j}})$ being the unit tensor of rank 2. Then we claim that

$$\mathscr{CSH}^{(n)} = \mathscr{CSH}_*^{(n)}. \tag{4.31}$$

If $C_S^{i_1, i_2, \ldots, i_{n-2}}$ are the coordinates of $\boldsymbol{C}_S^{(n-2)}$ with respect to the canonical basis $(\hat{\boldsymbol{i}}_1 \circ \cdots \circ \hat{\boldsymbol{i}}_{n-2})$ of $\mathscr{T}^{(n-2)}$, the symmetrization in (4.30) according to (4.10) reads

$$\boldsymbol{C}^{(n)} = C_S^{i_1, i_2, \ldots, i_{n-2}} \left[\frac{1}{n!} \sum_{\sigma \in \mathscr{P}^{(n)}} \sigma(\hat{\boldsymbol{i}}_1 \circ \hat{\boldsymbol{i}}_2 \cdots \circ \hat{\boldsymbol{j}} \circ \hat{\boldsymbol{j}}) \right], \tag{4.32}$$

where $\mathscr{P}^{(n)}$ denotes the set of all permutations of arguments in the tensorial product of n vectors. We note that the $\boldsymbol{C}^{(n)}$ span a vector space equivalent to $\mathscr{T}_S^{(n-2)}$, which has dimension $\frac{1}{2}n(n-1)$. In addition they are all orthogonal to the nth gradient tensor $(\nabla \circ)^n(1/r)$ under n-fold contraction, and hence $\mathscr{CSH}_*^{(n)}$ is orthogonal to $\mathscr{SH}^{(n)}$. To see the orthogonality one has to realize that the combination $(\hat{\boldsymbol{j}} \circ \hat{\boldsymbol{j}})$ in each term of the permutations in (4.32) produces the Laplacian operator when it meets two gradients in the contraction. It follows that the dimension of the direct sum $\mathscr{SH}^{(n)} \oplus \mathscr{CSH}_*^{(n)}$ is the sum of the dimensions of both spaces, namely $(2n+1) + \frac{1}{2}n(n+1) = \frac{1}{2}(n+1)(n+2)$. Since this is also the dimension of $\mathscr{T}_S^{(n)}$, the claim (4.31) is confirmed.

We now show that none of the projections $\beta_\nu^{(n)}$ of $\langle(\boldsymbol{x}' \circ)^{(n-1)}(\boldsymbol{x}' \times \boldsymbol{\Omega}')\rangle_S$ into $\mathscr{CSH}^{(n)}$ can influence $\nabla \times \boldsymbol{B}^{(n)}$. Any basis of $\mathscr{T}_S^{(n-2)}$, via (4.30), yields a basis $\{\boldsymbol{Z}_\nu^{(n)}\}$ of $\mathscr{CSH}^{(n)}$, and when an element of this basis replaces the moment tensor in (4.19) the result for $n > 2$ after application of the curl operator is

$$[(\boldsymbol{Z}_\nu^{(n-2)} \circ \hat{\boldsymbol{j}} \circ \hat{\boldsymbol{j}})_S \times \nabla] \odot^{n-1} (\nabla \circ)^{n-1} \left(\frac{1}{r}\right)$$
$$= -a_n \nabla^2 (\nabla \times \boldsymbol{z}_\nu^{(n-2)}) + b_n (\hat{\boldsymbol{j}} \cdot \nabla) \hat{\boldsymbol{j}} \times \nabla(\nabla \cdot \boldsymbol{z}_\nu^{(n-2)}), \tag{4.33}$$

where $a_n = (n-2)/n$, $b_n = 2/n$ and

$$\boldsymbol{z}_\nu^{(n-2)}(\boldsymbol{x}) = [\boldsymbol{Z}_\nu^{(n-2)} \odot^{n-3} (\nabla \circ)^{n-3}] \left(\frac{1}{r}\right). \tag{4.34}$$

The first term on the right handed side of (4.33) drops out due to the Laplacian acting on $(1/r)$ and the second term covers the zero operator

$(\hat{j} \cdot \nabla)\hat{j} \times \nabla = (\nabla \times \nabla) \equiv 0$. For $n = 2$ the expression on the left handed side of (4.33) vanishes as well. This finishes our argument and shows that the coefficients $p_k^{(n)}$ given in (4.25) not only determine completely the nth scalar potential $\Phi^{(n)}$, but also fix the curl contribution $\nabla \times \boldsymbol{B}^{(n)}$ to the nth vector potential.

The results of the Sec. 4.2 and 4.3 now yield the answers to questions (ii), (iii) and (v):

Question (ii): Given a basis $\{\boldsymbol{G}_k^{(n)}\}$ of $\mathscr{S}\mathscr{H}^{(n)}$ one obtains $(2n+1)$ independent vector potentials which contribute to $\boldsymbol{v}^{(n)}$ by

$$\boldsymbol{A}_k^{(n)} = -d_n \nabla \times \left(\boldsymbol{G}_k^{(n)} \odot^{n-1} (\nabla \circ)^{n-1} \left(\frac{1}{r} \right) \right). \tag{4.35}$$

Question (iii): The coefficients of these vector potentials in $\boldsymbol{A}^{(n)}$ and the $p_k^{(n)}$ given in (4.25). Section 4.2 shows that only the symmetric part of $\langle (\boldsymbol{x}' \circ)^{n-1} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle$ can contribute to $\nabla \times \boldsymbol{B}^{(n)}$, while the nonsymmetric components yield the gradient $\nabla \psi^{(n)}$. The number of moments excluded in this way is $J_n - \dim(\mathscr{T}_S^{(n)}) = \frac{1}{2}n(n+1) - 1$. The $N_c = \frac{1}{2}n(n-1)$ components $\beta_v^{(n)} \boldsymbol{Z}_v^{(n)}$ of $\langle (\boldsymbol{x}' \circ)^{n-1} (\boldsymbol{x}' \times \boldsymbol{\Omega}') \rangle_S$ in $\mathscr{C}\mathscr{S}\mathscr{H}^{(n)}$ neither influence the velocity field. Together these are the $(n^2 - 1)$ combinations of moments which are irrelevant for $\boldsymbol{v}^{(n)}$.

Questions (iv), (v): We have shown explicitly that $M_1 = 3$ at the beginning of the present subsection. For $n > 1$ the projections $q_j^{(n)}$ and $p_k^{(n)}$ are $M_n = 4n$ independent combinations of moments governing $\boldsymbol{A}^{(n)}$.

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Abstract

We study the velocity field induced by a vorticity distribution decaying rapidly in the distance r from the origin. In the far field, the vector potential for the velocity field can be represented by a series $\sum A^{(n)}$, with $A^{(n)}$ proportional to r^{-n-1}, for $n = 1, 2, \ldots$. We show that $A^{(n)}$ can be expressed as a linear combination of M_n linearly independent vector functions. The number M_n is equal to 3 for $n = 1$ and $4n$ for $n \geq 2$ and the coefficient of a vector function is defined by a linear combination of $\frac{3}{2}(n+2)(n+1)$ nth moments of vorticity. We then show that only $2n + 1$ linear combinations of those M_n vector functions contribute to the far field velocity which is irrotational. The corresponding scalar potential $\Phi^{(n)}$ is then represented by a linear combination of $2n + 1$ spherical harmonics of nth order whose coefficients are again linear combinations of nth moments of vorticity.

Zusammenfassung

Die vorliegende Arbeit beschreibt das Geschwindigkeitsfeld fernab einer Wirbelverteilung, welche mit dem Abstand r vom Ursprung eines geeigneten Bezugssystems hinreichend schnell abklingt. Die Geschwindigkeit besitzt ein Vektorpotential, dessen Fernfeldverhalten einer Reihenentwicklung $\sum A^{(n)}$, genügt. Dabei ist $A^{(n)}$ proportional zu r^{-n-1} für $n = 1, 2, \ldots$. Wir entwickeln eine explizite Darstellung von $A^{(n)}$ als Linearkombination von M_n linear unabhängigen Vektorfunktionen. Die auftretenden Koeffizienten sind ihrerseits Kombinationen n-ter Momente der Wirbelverteilung. Die Zahl M_1 ist gleich 3 und es ist $M_n = 4n$ für $n \geq 2$, während die Gesamtzahl der $n$ten Momente $\frac{3}{2}(n+1)(n+2)$ beträgt. Weiterhin zeigen wir, daß nur $2n + 1$ dieser Vektorfunktionen auch zum drehungsfreien Fernfeld der Geschwindigkeit n-ter Ordnung beitragen können und identifizieren die zugehörigen Kombinationen von Wirbelmomenten. Dieselben Kombinationen liefern dann auch die Koeffizienten in einer Entwicklung des *skalaren* Fernfeldpotentials nach Kugelfunktionen.

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