ZAMM · Z. angew. Math. Mech. **80** (2000) 11-12, 765–785

KLEIN, R.

Asymptotic Analyses for Atmospheric Flows and the Construction of Asymptotically Adaptive Numerical Methods

Prandtl's boundary layer theory may be considered one of the origins of systematic scale analysis and asymptotics in fluid mechanics. Due to the vast scale differences in atmospheric flows such analyses have a particularly strong tradition in theoretical meteorology. Simplified asymptotic limit equations, derived through scale analysis, yield a deep insight into the dynamics of the atmosphere. Due to limited capacities of even the fastest computers, the use of such simplified equations has traditionally been a necessary precondition for successful approaches to numerical weather forecasting and climate modelling.

In the face of the continuing increase of available compute power there is now a strong tendency to relax as many simplifying scaling assumptions as possible and to go back to more complete and more complex balance equations in atmosphere flow computations. However, the simplified equations obtained through scaling analyses are generally associated with singular asymptotic limits of the full governing equations, and this has important consequences for the numerical integration of the latter. In these singular limit regimes dominant balances of a few terms in the governing equations lead to degeneracies and singular changes of the mathematical structure of the equations. Numerical models based on comprehensive equation systems must simultaneously represent these dominant balances and the subtle, but important, deviations from them. These requirements are partly in contradiction, and this can lead to severe restrictions of the accuracy and/or efficiency of numerical models.

The present paper makes a case for a somewhat unconventional use of the results of scale analyses and multiple scales asymptotics. It demonstrates how, through the judicious implementation of asymptotic results, numerical discretizations of the full governing equations can be designed so that they operate with uniform accuracy and efficiency even when a singular limit regime is approached.

Keywords: Multiple Scales Asymptotics, Atmospheric Flows, Numerical Methods, Asymptotic Adaptivity

1 Introduction

Prandtl's boundary layer theory may be considered one of the origins of systematic scale analysis and asymptotics in fluid mechanics, (see, e.g., [34, 41, 36]). Due to the vast scale differences in atmospheric flows, such analyses have a particularly strong tradition in theoretical meteorology. Classical textbooks and research monographs (see, e.g., [17, 32, 44]) explain how scaling arguments, in combination with order-of-magnitude-estimates of observed data, may be used to deduce simplified equation systems that capture the essence of atmospheric flows, while suppressing unnecessary details. These systems often allow a more comprehensive understanding of many observed phenomena, because they contain only those physical interactions that are truly relevant.

These limit equations generally emerge because of a singularity of the governing equations that is associated with the particular flow regime considered. This aspect is important in the context of numerical modelling based on the full governing equations. Flows that are generally within or close to the regime of validity of the asymptotic limit equations will be governed essentially by the dominant balances revealed by the asymptotics. Yet, a flow solver designed for the full governing equations will generally not be able to properly represent these and, at the same time, capture the remaining higher order effects. A prominent example, which is also relevant to atmospheric flow modelling, is the numerical simulation of low Mach number flows using fully compressible flow solvers. It has been shown, e.g., in [42, 38, 18], that standard compressible flow solvers become inaccurate or even fail completely for Mach numbers smaller than about $\mathrm{M} = 10^{-2}$.

In the remainder of this paper we will recount in section 2 a version of the full three-dimensional compressible flow equations for flow in an atmospheric layer on the rotating earth in non-dimensional form. In section 3 we discuss qualitatively three singular asymptotic limit regimes that are relevant for different atmospheric flow applications, namely the quasi-geostrophic regime for synoptic scale flows ($\approx$ 1000km), and the pseudo-incompressible and anelastic regimes for mesoscale flows on scales less than or equal to the pressure scale height (i.e., $\leq$ 10km). Section 4 then provides some of results from (multiple scales) asymptotics of atmospheric flows. The scalings for this analysis are chosen so as to match the relevant numerically resolved scales in modern numerical weather forecast systems or regional climate models. Various issues regarding the numerical modelling of such flows on the basis of the fully three-dimensional compressible flow equations will be explained in section 5. A strategy for the design of suitable numerical schemes, which has been adopted in the author's earlier work on low Mach number flows, will be summarized briefly. Section 6 provides conclusions and an outlook to future activities.

2 Non-dimensional compressible flow equations in a rotating frame

The discussions in this paper will be based on the compressible non-homentropic three dimensional flow equations in a rotating frame of reference (see, e.g., [17, 32, 44]):

$$\begin{aligned}
\mathrm{Sr}\,\rho_t &+ \nabla\cdot(\rho\vec{v}) &&= 0\,,\\
\mathrm{Sr}\,\vec{v}_t &+ \vec{v}\cdot\nabla\vec{v} + \frac{1}{\mathrm{Ro}}\,\vec{\Omega}\times\vec{v} + \frac{1}{\mathrm{M}^2}\,\frac{1}{\rho}\nabla p + \frac{1}{\mathrm{Fr}^2}\,\vec{k} &&= \vec{D}_v\,,\\
\mathrm{Sr}\,p_t &+ \vec{v}\cdot\nabla p + \gamma p\nabla\cdot\vec{v} &&= D_p\,.
\end{aligned} \tag{1}$$

Let $\rho_{\mathrm{ref}}, v_{\mathrm{ref}}, p_{\mathrm{ref}}$ denote characteristic values of density, flow velocity, and pressure, let $\ell_{\mathrm{ref}}, t_{\mathrm{ref}}$ be characteristic length and time scales for the considered flow, and let $\Omega = \frac{1}{\mathrm{day}}$ be the earth rotation frequency, and g the modulus of the gravitational acceleration. Then the characteristic non-dimensional numbers Fr, M, Ro, and Sr are defined and labelled as

$$
\begin{aligned}
\mathrm{Fr} &= \frac{v_{\mathrm{ref}}}{\sqrt{g\ell_{\mathrm{ref}}}} && \text{Froude Number}\\
\mathrm{M} &= \frac{v_{\mathrm{ref}}}{\sqrt{p_{\mathrm{ref}}/\rho_{\mathrm{ref}}}} && \text{Mach Number}\\
\mathrm{Ro} &= \frac{v_{\mathrm{ref}}}{2\Omega\ell_{\mathrm{ref}}} && \text{Rossby Number}\\
\mathrm{Sr} &= \frac{\ell_{\mathrm{ref}}/t_{\mathrm{ref}}}{v_{\mathrm{ref}}} && \text{Strouhal Number}
\end{aligned}
$$

Furthermore, in (1), $\rho, \vec{v}, p$ are the non-dimensional density, velocity and pressure, t denotes time, and $\vec{\Omega}, \vec{k}$ are unit vectors pointing in the directions of the earth rotation axis and of the gravitational acceleration, respectively. $\vec{D}_v$ and D_p are associated with the effects of viscous (turbulent) friction, dissipation, heat conduction, and radiation on the balances of momentum and energy. D_p may also include effects of latent heat release, even though in this case the equation system would have to be extended to include the transport of the water phases. $\gamma = c_p/c_v$ is the ratio of the specific heat capacities at constant pressure and at constant volume, respectively.

In meteorological terminology, the homogeneous parts of (1) describe merely the "dynamics" of the atmosphere. In order to account for the so-called subgrid scale effects, suitable expressions representing turbulent transport, radiation effects, moisture transport, evaporation, condensation, precipitation, and possibly many other processes must be added to the "dynamic" part of the model. Such expressions are introduced in numerical atmospheric flow models, because it is generally not possible to resolve the atmospheric dynamics down to the very smallest flow scales by any realistic computational grid. Therefore, the effects of "unresolved scales" on the resolved ones must be parameterized, and the according terms are termed "the physics" of the model. They are subsumed here in $\vec{D}_v$ and D_p.

In the following we concentrate on the dynamics only and we discuss various singular flow regimes that emerge when one or more of the non-dimensional characteristic numbers defined above become very small or very large.

3 Examples of singular limit regimes in atmosphere flows

3.1 Quasi-geostrophic flow

One prominent example of a singular flow regime is the quasi-geostrophic limit, [17, 32, 44, 30], derived comprehensively by Pedlosky [32]. Following his exposition, we consider

1. a shallow atmosphere, with a vertical extent $h_{\mathrm{ref}} \ll \ell_{\mathrm{ref}}$,

2. horizontal scales small compared with the earth radius, so the flow occurs approximately in a tangent plane,
3. horizontal scales over which the vertical component of the earth rotation vector changes appreciably, so that $\vec{k} \cdot \vec{\Omega} = 1 + \beta y$, where y is a cartesian coordinate pointing in the meridional direction, and β is a constant,
4. small vertical velocities w so that $w/v_{\text{ref}} \ll O(h_{\text{ref}}/\ell_{\text{ref}})$,
5. low Mach, Froude, and Rossby numbers so that

$$\mathrm{M} \to 0,\ \mathrm{Fr} \to 0,\ \mathrm{Ro} \to 0, \qquad \text{with} \qquad \frac{\mathrm{M}}{\mathrm{Fr}} = 1,\ \text{and}\ \frac{\mathrm{M}^2}{\mathrm{Ro}} \to 0 \quad \text{as} \quad \mathrm{M} \to 0\,, \tag{2}$$

6. and unsteady flows with $\mathrm{Sr} = O(1)$ as $\mathrm{M}, \mathrm{Fr}, \mathrm{Ro} \to 0$.

At the leading order, the vertical momentum balance yields hydrostatics, and the horizontal momentum balance shows that the leading order pressure and density depend on the vertical coordinate z only. The Coriolis effect induces a perturbation of order

$$\delta = \frac{\mathrm{M}^2}{\mathrm{Ro}} \ll 1\,, \tag{3}$$

so that density and pressure may be expanded as

$$\begin{aligned} p &= \overline{p}(z) + \delta\, \overline{p}(z)\ p'(\vec{x}, z, t) + o(\delta), \\ \rho &= \overline{\rho}(z) \left(1 + \delta\ \rho'(\vec{x}, z, t) + o(\delta)\right), \end{aligned} \qquad \text{where} \qquad \frac{d\overline{p}}{dz} = -\overline{\rho} = -\overline{\theta}(z)\overline{p}^{\frac{1}{\gamma}}\,. \tag{4}$$

Here $\vec{x}$ denotes the horizontal coordinates, and $\overline{\theta}(z)$ is the leading order potential temperature distribution which must be considered an input to the problem formulation. The velocity field is expanded as

$$\vec{v} = \vec{u}_{\text{qg}} + \mathrm{Ro}\ (\vec{u}^{(1)} + w^{(1)}\vec{k}) + o(\mathrm{Ro})\,, \tag{5}$$

where vectors $\vec{u}$ denote the horizontal velocity components and $w\vec{k}$ the vertical one.

Asymptotic expansions then yield a closed set of equations for the quasi-geostrophic horizontal velocity $\vec{u}_{\text{qg}}$, the leading pressure perturbation p' and the potential temperature perturbation $\theta' = -\rho' + \dfrac{\overline{\rho}}{\gamma\overline{p}}(z)\ p'$:

Horizontal momentum balance

$$\vec{k} \times \vec{u}_{\text{qg}} + \nabla_{\parallel} p' = 0 \tag{6}$$

Vertical momentum balance / hydrostatics

$$\theta' = \frac{\partial p'}{\partial z} \tag{7}$$

Transport equation for the vertical vorticity $\zeta_{\text{qg}} = \vec{k} \cdot (\nabla \times \vec{u}_{\text{qg}})$

$$\left(\frac{\partial}{\partial t} + \vec{u}_{\text{qg}} \cdot \nabla_{\parallel}\right) \left(\zeta_{\text{qg}} + \beta y + \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho}}{S}\theta'\right)\right) = \frac{1}{\overline{\rho}} \frac{\partial}{\partial z} \left(\frac{\overline{\rho} D_p'}{\gamma\overline{p}S}\right) \tag{8}$$

In these equations D_p' is the suitably scaled source term from the original pressure evolution equation in (1), S is the static stability parameter defined by

$$S = \frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{dz} , \tag{9}$$

and $\nabla_\|$ abbreviates the horizontal components of the gradient operator.

We notice that (6)–(9) is the first closed set of equations obtained in this asymptotic regime of low Mach, Froude, and Rossby numbers. While (6) and (7) are the *leading order* horizontal and vertical momentum equations the vorticity transport equation (8) is a result of higher order balances. Thus, flow simulations based on more comprehensive equation systems, such as the three-dimensional compressible flow equations, must use highly accurate numerical schemes that can reliably represent such higher order balances without explicitly extracting them through asymptotics.

Furthermore, the fact that the velocity field is related to a scalar function p' as in (6) introduces a compatibility constraint for the horizontal velocity field. One cannot assign an arbitrary two-dimensional initial velocity field, but must ensure that $\vec{u}_{\rm qg}$ is divergence free to begin with. In addition, a meaningful computation must generally start with an almost balanced flow, so that unbalanced fast wave components do not pollute the solution.

(Note, however, that Embid and Majda [13, 14] reveal a flow regime in which the long time average effects of fast gravity waves do not influence the underlying slow time scale balanced mean flow. Thus, the often cost intensive procedure of finding balanced initial data may sometimes be omitted.)

3.2 The incompressible, pseudo-incompressible, and anelastic approximations

High frequency acoustic waves are generally considered unimportant for the purposes of weather prediction or climate modelling. Here "high frequency" is meant to denote acoustic modes with wavelengths on the order of 10 km or less and characteristic frequencies higher than 1 $\min^{-1}$. A plausibility argument states that sizeable elastic perturbations cannot establish in the atmosphere, because acoustic waves rapidly redistribute the associated energy and lead to an equilibration void of any acoustic modes. This intuitive explanation may be quantified through an asymptotic analysis for vanishing Mach number,[36, 28], that has been backed up by rigorous justifications, e.g., in [11, 22, 37]. The resulting set of zero Mach number, variable density flow equations without gravity and rotation can be written as

$$\begin{aligned}
\rho_t \quad + \quad \vec{v}\cdot\nabla\rho \quad &= \quad -\rho\nabla\cdot\vec{v}\,,\\
\vec{v}_t \quad + \quad \vec{v}\cdot\nabla\vec{v} + \frac{1}{\rho}\nabla p' \quad &= \quad \vec{D}_v\,,\\
\nabla\cdot\vec{v} \quad &= \quad \frac{D_p}{\gamma\overline{p}_\infty}\,.
\end{aligned} \tag{10}$$

Here p' is a higher order perturbation pressure satisfying $p = \overline{p} + \mathrm{M}^2 p'$, with $\overline{p} \equiv$ const.. (Notice that for combustion in closed systems the background pressure $\overline{p}$ may be time dependent, but will always be spacially homogeneous.)

The (elliptic) divergence constraint $(10)_3$ replaces the total energy or pressure evolution equation. It effectively suppresses acoustic modes and is therefore extremely important for numerical flow simulations: Fast acoustic wave propagation is eliminated and thus will neither lead to undesired time step restrictions, nor to unphysical oscillations that may occur when straight-forward discretizations are used to represent hyperbolic equation systems (see, e.g., [27]). The constraint does induce a complication, in that it establishes an elliptic, Poisson-type equation for the pressure perturbation p', namely

$$\nabla \cdot (\frac{1}{\rho} \nabla p') = -\nabla \cdot (\vec{v} \cdot \nabla \vec{v} - \vec{D}_v) - \frac{\partial}{\partial t} \left(\frac{D_p}{\gamma \overline{p}} \right) , \tag{11}$$

which

- is not trivially solved numerically, and
- has no connection with the thermodynamic equation of state which for compressible flows would establish a relation between the density and pressure fields.

In practice, these complications are often considered less severe than the time step constraints associated with a proper representation of anyway unimportant acoustic waves.

Durran [10] describes two similar velocity divergence constraints that are used frequently in the formulation of numerical atmosphere flow models when the horizontal scales considered are comparable to the vertical ones (see also [17, 44, 12]). On such scales one may describe, e.g., the transport of pollutants in the atmosphere after they have exited a smoke stack (relevant length scales: ≈ 100 m – 1 km), or one may simulate the formation and evolution of convective clouds on scales comparable to the atmospheric scale height of about 10 km.

The first constraint,

$$\nabla \cdot (\overline{\rho} \vec{v}) = 0 \tag{12}$$

leads to the "anelastic approximation", [3, 31, 9]. The second yields the "pseudo-incompressible approximation", [9], and reads

$$\nabla \cdot (\overline{\rho} \overline{\theta} \vec{v}) = \frac{D_p}{\gamma \overline{p}^{\frac{\gamma - 1}{\gamma}}} . \tag{13}$$

Here

$$\theta = \frac{p^{1/\gamma}}{\rho} \tag{14}$$

is the standard "potential temperature" except for a constant factor. The potential temperature is a function of the thermodynamic entropy, (see, e.g., [17], section 3.7.3). The overbar denotes horizontal averaging at constant height, $z =$ const..

If either one of these constraints is adopted in a numerical model, the desired effect of suppression of acoustic modes is obtained. Yet, these constraints are not equivalent and which one best suits a given situation must be carefully evaluated. Estimates of the respective regimes of validity of the constraints in (12) and (13) as well as a discussion of consequences for numerical discretizations will be given in section 4.1 below. Here we merely re-iterate an interesting observation from [44, 5]:

If the anelastic and the pseudo-incompressible approximations hold simultaneously, then either

- the vertical flow velocity w satisfies the "diagnostic relation"

$$w = \frac{D_p}{\gamma \overline{p}} \left(\frac{1}{\overline{\theta}} \frac{\partial \overline{\theta}}{\partial z} \right)^{-1}, \tag{15}$$

or

- the potential temperature has an expansion $\theta = \theta_\infty + \mathrm{M}^2 \theta^{(2)}(\vec{x}, z, t)$, where $\theta_\infty \equiv \text{const.}$

Notice that the second option has been assumed by Ogura and Phillips [31] in their derivation of the anelastic approximation. (See also sections 45, 46 of [44].)

4 Asymptotic analysis

4.1 The anelastic and pseudo-incompressible divergence constraints

Here we consider atmospheric motions on length scales of up to the pressure scale height, i.e.,

$$\ell_{\mathrm{ref}} \leq h_{\mathrm{sc}} \approx 10\mathrm{km}\,,$$

and we assume an isotropic scaling of characteristic lengths, i.e., no shallowness is assumed. Also we are interested in convective time scales, so that we choose

$$t_{\mathrm{ref}} = \frac{\ell_{\mathrm{ref}}}{v_{\mathrm{ref}}} \quad \text{and} \quad \mathrm{Sr} \equiv 1\,.$$

The Rossby number may be estimated as

$$\mathrm{Ro} = \frac{v_{\mathrm{ref}}}{2\Omega \ell_{\mathrm{ref}}} > \frac{v_{\mathrm{ref}}}{2\Omega h_{\mathrm{sc}}} \approx 5\,,$$

so that $1/\mathrm{Ro} \leq O(1)$ and the Coriolis effects are non-singular.

To assess the relative order of magnitude of the Mach and Froude numbers we first observe that the Froude number based on the pressure scale height actually *equals* the Mach number. The pressure scale height may be assessed using the approximate hydrostatic balance $\partial p/\partial z \approx -\rho g$, and the scale height is defined as the vertical distance over which the pressure changes appreciably, i.e., by an amount comparable to the reference pressure at the ground. Therefore

$$h_{\mathrm{sc}} \left|\frac{\partial p}{\partial z}\right| \approx h_{\mathrm{sc}} \rho_{\mathrm{ref}} g := p_{\mathrm{ref}}\,, \qquad \text{or} \qquad h_{\mathrm{sc}} g \approx \frac{p_{\mathrm{ref}}}{\rho_{\mathrm{ref}}}\,.$$

Thus, from the definitions of the characteristic numbers we have

$$\mathrm{Fr} \geq \mathrm{M} \qquad (\text{with} \quad \mathrm{Fr} = \mathrm{M} \quad \text{for} \quad \ell_{\mathrm{ref}} = h_{\mathrm{sc}})\,.$$

The pseudo-incompressible approximation. It is generally observed that pressures in the troposphere at any given location do not change appreciably in comparison with the background pressure of about 1 bar. Thus pressure variations are always small and we may write

$$p = \overline{p}(\vec{x}, z) + O(\varepsilon_p) \qquad \text{where} \qquad \varepsilon_p \ll 1\,, \tag{16}$$

and $\overline{p}(\vec{x}, z)$ is a time independent local mean pressure. Also, high frequency pressure fluctuations on time scales much shorter than the convective time scales are either absent or of such low amplitudes that they do not appreciably affect the pressure time derivative. As a consequence we have

$$\frac{\partial p}{\partial t} = O(\varepsilon_{p,t}) \ll 1\,. \tag{17}$$

With these estimates, the pressure evolution equation $(1)_3$ yields

$$\vec{v} \cdot \nabla p + \gamma p \nabla \cdot \vec{v} \;=\; \gamma p^{\frac{\gamma-1}{\gamma}} \;\nabla \cdot (p^{\frac{1}{\gamma}} \vec{v}) \;=\; D_p + O(\varepsilon_{p,t})\,. \tag{18}$$

If the pressure perturbations of order $O(\varepsilon_p)$ allow the additional estimate

$$\nabla p = \nabla \overline{p} + O(\varepsilon_{p,x}) \qquad \text{with} \qquad \varepsilon_{p,x} \ll 1\,, \tag{19}$$

we may neglect errors of order $O(\varepsilon_p, \varepsilon_{p,t}, \varepsilon_{p,x})$ to find the "generalized pseudo-incompressible constraint"

$$\nabla \cdot (\overline{p}^{\frac{1}{\gamma}} \vec{v}) = \frac{D_p}{\gamma \overline{p}^{\frac{\gamma-1}{\gamma}}}\,. \tag{20}$$

The only approximations that are needed to obtain this result concern the smallness of the overall pressure variations and of the pressure time and space derivatives. Thus this divergence constraint is a direct consequence of a degeneration of the pressure evolution, and not one of mass conservation (see also [23]). In contrast, the original pseudo-incompressible divergence constraint from (13) necessitates the additional requirement $\overline{\rho}\overline{\theta} = \overline{p}^{\frac{1}{\gamma}}$. Unless one *defines* $\overline{\theta}$ and $\overline{\rho}$ through

$$\overline{\theta} = \langle \theta \rangle \qquad \text{and} \qquad \overline{\rho} = \langle \frac{1}{\rho} \rangle^{-1} \tag{21}$$

with a suitable averaging operator$\langle \cdot \rangle$, this requirement will be satisfied only when fluctuations of potential temperature and density are small. This would not be the case, e.g., for hot exhaust gases exiting a smoke stack.

The low Mach number limit for small scales. When $\ell_{\text{ref}} \ll h_{\text{sc}}$, we have $\text{Fr} \gg \text{M}$ and an appropriate asymptotic pressure expansion reads

$$p = \overline{p} + \varepsilon_{\text{Fr}} p^{(\text{Fr})}(\vec{x}, z, t) + \text{M}^2 p^{(2)}(\vec{x}, z, t) + o(\varepsilon_{\text{Fr}}, \text{M}^2) \tag{22}$$

where

$$\varepsilon_{\text{Fr}} = \frac{\text{M}^2}{\text{Fr}^2} \ll 1\,. \tag{23}$$

The momentum equation at order $O(\text{M}^{-2})$ reads

$$\nabla \overline{p} \equiv 0\,, \tag{24}$$

so that $\overline{p}$ is a constant. For this case the generalized pseudo-incompressible constraint from (20) reduces to the standard divergence condition for non-adiabatic zero Mach number flow

$$\nabla \cdot \vec{v} = \frac{D_p}{\gamma \overline{p}}\,. \tag{25}$$

This constraint is common, e.g., in the theory of low Mach number combustion, [28].

When $\text{Fr} = O(1)$ we have $\varepsilon_{\text{Fr}} = \text{M}^2$ and the pressure perturbations $p^{(\text{Fr})}$ and $p^{(2)}$ need not be distinguished. In this regime, which describes flow fields on the 10 m scale, the leading order system of governing equations is that presented in (10), provided the gravitation term is included in the momentum source $\vec{D}_v$. There are interesting intermediate regimes for which the condition in (23) holds, and for which there is a non-trivial distinguished limit $\text{Fr} = \text{M}^\alpha$ with $0 < \alpha < 1$. The discussion of these intermediate regimes is beyond the scope of this paper.

Mesoscale flow on scales comparable to the pressure scale height. Of particular interest in meteorology are "deep convection events" characterized by vertical motions over distances as large as the pressure scale height. As pointed out earlier, this flow regime is peculiar in that it involves a distinguished limit of the Froude and Mach numbers, namely $\text{Fr}/\text{M} = O(1)$ as $\text{M} \to 0$. The standard low Mach number pressure expansion

$$p = \overline{p}(\vec{x}, z) + \text{M}^2 p^{(2)}(\vec{x}, z, t) + o(\text{M}^2) \tag{26}$$

together with perturbation expansions for the density and velocity

$$\vec{v} = \vec{v}^{(0)}(\vec{x}, z, t) + \text{M}\vec{v}^{(1)}(\vec{x}, z, t) + \text{M}^2\vec{v}^{(2)}(\vec{x}, z, t) + o(\text{M}^2) \tag{27}$$

$$\rho = \rho^{(0)}(\vec{x}, z, t) + \text{M}\rho^{(1)}(\vec{x}, z, t) + \text{M}^2\rho^{(2)}(\vec{x}, z, t) + o(\text{M}^2) \tag{28}$$

yield the following asymptotic limit equations. First, the leading order vertical momentum equation establishes hydrostatics for $\overline{p}, \rho^{(0)}$, i.e.,

$$\frac{\partial \overline{p}}{\partial z} = -\rho^{(0)} \tag{29}$$

The horizontal momentum balance at order $O(\mathrm{M}^{-2})$ yields

$$\nabla_{\|}\,\overline{p} \equiv 0 \qquad \text{or} \qquad \overline{p} = \overline{p}(z)\,. \tag{30}$$

Here $\nabla_{\|}$ abbreviates the horizontal components of the gradient operator. From (29) and the (assumed) time independence of $\overline{p}$ we conclude that

$$\rho^{(0)} \equiv \overline{\rho}(z)\,. \tag{31}$$

With this result we obtain the anelastic divergence constraint (12) as the leading order continuity equation, i.e.,

$$\nabla \cdot (\overline{\rho}\vec{v}^{(0)}) = 0\,. \tag{32}$$

We realize furthermore that all the conditions needed to justify the pseudo-incompressible divergence constraint from (20), namely $p = \overline{p}(\vec{x}, z) + \varepsilon_p$, $\partial p/\partial t = O(\varepsilon_{p,t})$, and $\nabla p = \nabla\overline{p} + O(\varepsilon_{p,x})$ are satisfied, with $\varepsilon_p = \varepsilon_{p,t} = \varepsilon_{p,x} = \mathrm{M}^2$. Thus, the anelastic and the pseudo-incompressible divergence constraints hold *simultaneously*! Using (29), (30), and (31) we conclude, after some straight-forward manipulations, that

$$w^{(0)} \left(\frac{1}{p^{\frac{1}{\gamma}}} \frac{dp^{\frac{1}{\gamma}}}{dz} - \frac{1}{\overline{\rho}} \frac{d\overline{\rho}}{dz} \right) = w^{(0)}\,\frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = \frac{D_p^{(0)}}{\gamma\overline{p}}\,, \tag{33}$$

where $w^{(0)}$ is the leading order vertical velocity and $\overline{\theta} = \overline{p}^{\frac{1}{\gamma}}/\overline{\rho}$. At this point we have to distinguish different regimes concerning the sign and order of magnitude of the dry adiabatic stability parameter

$$\frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = S =: \mathrm{M}^2 N^2\,, \tag{34}$$

where N is the — non-dimensional — "buoyancy frequency" or "Brunt-Väisälä" frequency. This quantity is a measure of the oscillation frequency of particles that are vertically displaced in a stably stratified atmosphere with $d\overline{\theta}/dz > 0$, (see, e.g., [17, 32, 44, 12]). For many, if not most, practical purposes one may assume stable stratification with $N^2 > 0$ in the troposphere and stratosphere. Pedlosky [32] provides an order of magnitude estimate (see his eq. (6.4.13)), which in the present notation amounts to

$$\mathrm{M}^2 N^2 = \frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{dz} = O(10^{-1})\,. \tag{35}$$

With $\mathrm{M} \approx 0.03$, $v_{\mathrm{ref}} = 10\mathrm{m/s}$, $\ell_{\mathrm{ref}} = h_{\mathrm{sc}} = 10^4\mathrm{m}, t_{\mathrm{ref}} = \ell_{\mathrm{ref}}/v_{\mathrm{ref}}$ this amounts to a dimensional frequency of $N/t_{\mathrm{ref}} = 10^{-2}s^{-1}$, (see also [32], Fig. 6.4.2, and [17], page 52).

Stable Stratification: Assuming now that the stability parameter is non-zero and positive, the simultaneous realization of the anelastic and pseudo-incompressible divergence constraints from (33) leads us to a *diagnostic*, algebraic relation for the leading order vertical velocity,

$$w^{(0)} = \frac{D_p^{(0)}}{\gamma\overline{p}} \left(\frac{1}{\overline{\theta}} \frac{d\overline{\theta}}{dz} \right)^{-1} \qquad \text{if} \qquad \frac{d\overline{\theta}}{dz} > 0\,, \tag{36}$$

as pointed out in section 1. Here is a summary of the relevant asymptotic limit equations for the horizontal velocity $\vec{u}$, the perturbation pressure $p^{(2)}$, and the vertical velocity w in this regime.

$$\begin{aligned} \vec{u}_t \quad + \quad \vec{u}\cdot\nabla_\parallel\,\vec{u} + w\vec{u}_z + \frac{1}{\overline{\rho}}\nabla_\parallel\, p^{(2)} \quad &= \quad \vec{D}_u\,, \\ \nabla_\parallel\cdot\vec{u} \quad &= \quad -\frac{1}{\overline{\rho}}\frac{\partial\overline{\rho}w}{\partial z}\,, \\ w \quad &= \quad \frac{D_p^{(0)}}{\gamma\overline{p}}\left(\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz}\right)^{-1}\,. \end{aligned} \tag{37}$$

The energy source term D_p may depend on $w, \vec{u}, \overline{\rho}, \overline{p}$, and on additional variables, such as humidity, for which one would have to include additional inhomogeneous scalar transport equations. Since the purpose of the present derivations is to point out the structure of the low Mach / low Froude number singularities rather than being comprehensive, we omit detailled specifications of D_p here.

The equations in (37) are strongly degenerate, [44], in that the vertical momentum balance is no longer part of the *prognostic* set of evolution equations. In this regime, vertical motion can take place only if there is sufficiently strong heat input or heat loss through sources such as latent heat release of absorption of radiation. In the absence of heat exchange there is no vertical motion and the airflow occurs in vertically stacked, decoupled layers. One consequence of the algebraic constraint in (36), which is consistent with the discussions in [39], is that it generally forces airflow *around* topographical obstacles rather than allowing flow *over* them.

Nearly Neutral Stratification: The desire to model events of "deep convection" in well-mixed atmospheric boundary layers has led Ogura and Phillips [31] to introduce the assumption of near neutral stability with

$$\frac{1}{\overline{\theta}}\frac{d\overline{\theta}}{dz} = O(\mathrm{M}^2)\,. \tag{38}$$

With the dimensional parameters as in the previous paragraph, this corresponds to $N/t_{\mathrm{ref}} \approx 10^{-3}s^{-1}$, and thus to a much weaker stratification than before. Zeytounian ([44], chapter 45) provides a compact derivation of the relevant governing equations for the full three-dimensional velocity field, $\vec{v}$, and the second order perturbation of the potential temperature, $\theta^{(2)}$. In our notation these equations read

$$\begin{aligned} \vec{v}_t \quad + \quad \vec{v}\cdot\nabla\vec{v} + \frac{1}{\overline{\rho}}\nabla p^{(2)} \quad &= \quad \vec{D}_v + \theta^{(2)}\vec{g}\,, \\ \nabla\cdot(\overline{\rho}\vec{v}) \quad &= \quad 0\,, \\ \theta_t^{(2)} \quad + \quad \vec{v}\cdot\nabla\theta^{(2)} \quad &= \quad \frac{D_p^{(2)}}{\gamma\overline{p}}\,. \end{aligned} \tag{39}$$

Importantly, in this regime the energy input represented by the source term D_p must be small of order $O(\mathrm{M}^2)$, i.e.,

$$D_p = \mathrm{M}^2 D_p^{(2)} + o(\mathrm{M}^2)\,. \tag{40}$$

The leading order pressure and density stratifications now correspond to a neutrally stable atmosphere and are explicitly given by

$$\overline{p}(z) = \left(1 - \frac{\gamma - 1}{\gamma}\, z\right)^{\frac{\gamma}{\gamma-1}}, \qquad \overline{\rho}(z) = \overline{p}(z)^{\frac{1}{\gamma}}\,. \tag{41}$$

Zeytounian [44] discusses two-dimensional steady state solutions of the above equations. He reduces the equations to a single second order elliptic equation for a stream function ψ. This special case provides an excellent basis for the design of test problems for comprehensive solvers of the three dimensional anelastic weak stratification equations from (39).

4.2 Long wave length dynamics on short convective time scales

Thus far we have considered asymptotic flow regimes that were characterized by single length and time scales. We did account for shallow atmospheres with vertical extent small compared to the horizontal scales, but *in each direction* only a single characteristic scale has been accounted for. Furthermore, the reference time scale has been the characteristic time scale of convection, i.e., $t_{\mathrm{ref}} = \ell_{\mathrm{ref}}/v_{\mathrm{ref}}$. Meteorological applications, however, generally feature a multitude of scales.

On the one hand, there are the "unresolved" or "subgrid scales", which are smaller than what can be resolved by the computational grids of atmospheric flow models. The overall effects of these subgrid scales on the resolved ones is a very subtle and important issue. The complexity of the problem is even higher than that of classical turbulence modelling. Many of the simplifying assumptions that are often used in this latter area, such as homogeneous isotropic turbulence characteristics, are not valid in the atmosphere due to anisotropic scalings, the influence of gravity and stratification, etc.. In addition, there are microscale processes, e.g., in the context of cloud formation, that can only be described by multi-phase fluid dynamics. We will not address the problems of "parametrization of subgrid scale effects" here.

Another multiple scales aspect arises from the continuous increase of modern computational capacities. It is now possible to simultaneously resolve a multitude of length scales within a single computation. For example, the current generation of weather forecast codes uses a horizontal grid spacing as small as $\sim$ 3km, while spanning overall horizontal distances of thousands of kilometers. Some of the consequences of the simultaneous presence of multiple length scales can be explored using asymptotic multiple scales expansions, and this is the topic of the present section. The following discussions will be based on the detailed derivations in [5].

During previous discussions we have seen already that the mathematical structure of the effective dynamical equations changes drastically with the considered length scales. When considering multiple length scales in a single solution, a subtle issue regarding the underlying characteristic time scales arises in addition: At low flow Mach numbers the characteristic time scale for convection is always much longer than that of acoustic wave propagation for a given length scale. If, however, the largest present scale, $\ell_{\max}$, exceeds the smallest one, $\ell_{\min}$, by a factor of 1/M or more, then the characteristic time of convection on the $\ell_{\min}$ scale becomes comparable to the acoustic time scale for $\ell_{\max}$. It is then not at all clear that the "filtering of acoustic modes", as implied, e.g., in the quasi-geostrophic, anelastic, and pseudo-incompressible approximations, should affect all flow scales, or whether it should be restricted to the smallest scales only. In fact, there are very long wavelength combined acoustic — gravitational wave modes, the so-called Lamb waves (see [17], p. 504–506), which are inherently compressible and must be accounted for together with the quasi-incompressible small scale dynamics.

Botta et al. [5] consider flows with a smallest scale comparable to the pressure scale height $h_{\rm sc}$ and chose reference length and times for non-dimensionalization according to

$$\ell_{\rm ref} = h_{\rm sc} \approx 10^4 {\rm m}\,, \qquad t_{\rm ref} = \frac{\ell_{\rm ref}}{v_{\rm ref}} \approx 10^3 {\rm s} \tag{42}$$

(i.e., the reference flow velocity is $v_{\rm ref} \approx 10{\rm m/s}$). However, they are interested at the same time in much larger horizontal scales

$$\ell_{\rm ac} = \frac{1}{\rm M} \ell_{\rm ref} \approx 300{\rm km} \tag{43}$$

for which one finds the acoustic time scale to match the convection time scale for $\ell_{\rm ref}$

$$\frac{\ell_{\rm ac}}{c_{\rm ref}} = \frac{{\rm M}\ell_{\rm ac}}{v_{\rm ref}} = \frac{\ell_{\rm ref}}{v_{\rm ref}} = t_{\rm ref}\,. \tag{44}$$

As discussed earlier, the Froude and Rossby numbers based on the reference scales may be assessed through the distinguished limits

$$\mathrm{Ro} = O(1)\,, \qquad \frac{\mathrm{Fr}}{\rm M} = O(1) \qquad \text{as} \qquad \mathrm{M} \to 0\,. \tag{45}$$

Notice that the effective Rossby number for the larger "acoustic" scales becomes

$$\mathrm{Ro}_{\rm ac} = \frac{v_{\rm ref}}{\Omega \ell_{\rm ac}} = O(\mathrm{M}) \qquad \text{as} \qquad \mathrm{M} \to 0\,, \tag{46}$$

so that we may expect the effects of rotation to achieve increasing importance at the larger length scales.

Following the low Mach number single time – multiple space scale analysis for weakly compressible flows from [23], Botta et al. then choose an asymptotic ansatz for solutions of (1) of the form

$$\mathcal{U}(\vec{x}, z, t; \mathrm{M}) = \mathcal{U}^{(0)}(\vec{x}, \mathrm{M}\vec{x}, z, t) + \mathrm{M}\, \mathcal{U}^{(1)}(\vec{x}, \mathrm{M}\vec{x}, z, t) + \mathrm{M}^2\, \mathcal{U}^{(2)}(\vec{x}, \mathrm{M}\vec{x}, z, t) + o(\mathrm{M}^2)\,, \tag{47}$$

where $\mathcal{U} = (\rho, p, \vec{u}, w)^t$ denotes the solution vector. Without going into details we will summarize here some of the key results of the subsequent analysis. For future reference we note that horizontal spatial gradients for the above solution ansatz translate as follows

$$\nabla_{\parallel} \mathcal{U} = \sum_{\nu} \mathrm{M}^{\nu}\, (\nabla_{\vec{x}} + \mathrm{M}\nabla_{\vec{\xi}})\, \mathcal{U}^{\nu}\,, \tag{48}$$

where $\vec{\xi} = \mathrm{M}\vec{x}$.

Small scale dynamics: On the small scales comparable to $h_{\rm sc}$ it is found that

$$\nabla_{\vec{x}}\, p^{(0)} = \nabla_{\vec{\xi}} p^{(0)} = \nabla_{\vec{x}}\, p^{(1)} \equiv 0 \qquad \text{so that} \qquad p^{(0)} = \overline{p}(z,t)\,, \quad p^{(1)} = p^{(1)}(\vec{\xi}, z, t)\,. \tag{49}$$

Consistent with observations, temporal variations of the background pressure $\overline{p}$ are suppressed, so that $\overline{p} = \overline{p}(z)$.

Next one recovers the pseudo-incompressible flow equations for stratified flows from (37), except for one modification concerning the influence of long wavelength pressure gradients. The horizontal momentum equation now reads

$$\vec{u}_t + \vec{u}\cdot\nabla_{\vec{x}}\,\vec{u} + w\vec{u}_z + \frac{1}{\overline{\rho}}\nabla_{\vec{x}}\, p^{(2)} = \vec{D}_u^{\rm Ro} - \frac{1}{\overline{\rho}}\nabla_{\vec{\xi}} p^{(1)}\,, \tag{50}$$

where we have included the Coriolis terms in $\vec{D}_u^{\rm Ro}$ to abbreviate the notation. One may expect to find the fully three-dimensional anelastic approximation from (39), plus the long wave pressure gradient term as in (50), if variations of the potential temperature are assumed to be of order $O({\rm M}^2)$ only. This case was not discussed in [5].

The only new effect on the small scales is thus the pressure gradient term $\frac{1}{\overline{\rho}}\nabla_{\vec{\xi}} p^{(1)}$, which is responsible for a bulk acceleration of the fluid. Notice that the $\vec{x}$-dependence of the second order pressure $p^{(2)}$ is determined by a small scale divergence constraint analogous to $(37)_2$.

Large scale dynamics: In a standard fashion, the relevant equation for the dynamics on the $\frac{1}{\rm M}\ell_{\rm ref}$-scale are obtained by horizontally averaging over the small scale coordinate $\vec{x}$. The $\vec{x}$-averaging operator will be abbreviated by $\langle\cdot\rangle$. The large scale dynamics involves three variables, namely the long wave pressure mode $p^{(1)}$, the averaged vertical component of vorticity $\overline{\zeta} = \vec{k}\cdot\nabla_{\vec{\xi}}\times\langle\vec{u}^{(0)}\rangle$, and the averaged first order vertical velocity $\overline{w}^{(1)} = \langle w^{(1)}\rangle$. The long wavelength dynamical equations for the considered regime read

$$\begin{aligned}
\Big(\frac{\partial^2}{\partial t^2} - \overline{c}^2\nabla_{\vec{\xi}}^{\;2}\Big)\, p^{(1)} &= \Big(1-\overline{c}^2\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial t}\,(\overline{\rho}\,\overline{w}^{(1)}) - \overline{c}^2\,\overline{\zeta} + Q_{p^{(1)}}\,,\\
N^2\overline{c}^2\,(\overline{\rho}\,\overline{w}^{(1)}) &= -\Big(1+\overline{c}^2\frac{\partial}{\partial z}\Big)\frac{\partial}{\partial t}\,p^{(1)} + Q_{\overline{\rho}\overline{w}^{(1)}}\,,\\
\frac{\partial}{\partial t}\,\overline{\zeta} &= \frac{1}{\overline{c}^2}\frac{\partial}{\partial t}\,p^{(1)} + \frac{\partial}{\partial z}\,(\overline{\rho}\,\overline{w}^{(1)}) + Q_{\overline{\zeta}}\,.
\end{aligned} \tag{51}$$

Here Q_i are inhomogeneous terms that combine the effects of small scale averages of nonlinear terms, and of the momentum and energy inhomogeneities $\vec{D}_v, D_p$ from (1). The system in (51) supports internal gravity waves, long wave acoustics, and the Lamb wave. It represents the link between the multiple length scale asymptotics from [5] and classical analyses for small perturbations from a state of rest.

We notice also that the in the stationary limit ($\partial/\partial t \equiv 0$) the first equation in (51) reduces to the pressure Poisson equation from the quasi-geostrophic theory obtained by applying the horizontal divergence $\nabla_{\parallel}\cdot(\cdot)$ to the geostrophic balance equation (6). Thus we verify that the Coriolis effects achieve dominant importance for quasi-steady flow on the larger $\vec{\xi}$-scale as announced earlier.

5 Numerical issues

5.1 Discussion of the singular limit regimes

In the previous sections we have provided a summary of some important singular limit regimes for atmospheric flows. Simplified asymptotic limit equations have been discussed that can be derived systematically from the fully three-dimensional compressible flow equations by considering various distinguished limits connecting the Mach, Froude, and Rossby numbers. (We have focused on convective time scales, so that the Strouhal number has been set to unity throughout). Based on these considerations we will now point out a number of difficulties that must be addressed when near-singular solutions to the original compressible flow equations are to be obtained via numerical integration.

Low Mach number small scale flows: Consider first the regimes of pseudo-incompressible and anelastic flows from section 3.2 and 4.1. It has been pointed out that the stratification of potential temperature plays an crucial role for the flow dynamics. Under near neutral conditions, where $\theta = \theta_\infty + \mathrm{M}^2\theta^{(2)}$, there is fully three-dimensional dynamics, whereas the vertical motion is strongly blocked by buoyancy forces if the stratification is stronger. Furthermore, in this regime there is an asymptotic decoupling between the thermodynamic component of the pressure, the background pressure $\overline{p}$, and the pressure gradients in the momentum equation, which are represented by the second order pressure $p^{(2)}$.

Suppose now that a fully compressible flow solver using pressure, density, and velocity as the dependent variables were used to compute a weakly stratified flow. Let us suppose further that a second order accurate discretization is used and that the numerical resolution is of the order of $h = \ell_{\mathrm{ref}}/\Delta x \approx 10^{-2}$, where Δx characterizes the mesh size of the computational grid. Local truncation errors within the pressure and density evolution equations will then be of order $\delta\rho, \delta p = O(h^2) \sim 10^{-4}$. After a limited number of time steps the potential temperature $\theta = p^{\frac{1}{\gamma}}/\rho$ will have accumulated errors of order $O(10^{-3})$, which is comparable to the square of the Mach number. Thus, unless additional special measures are taken, the buoyancy forces in the vertical momentum equation will be dominated by the net effect of truncation errors, and an accurate prediction of vertical velocities cannot be expected.

Suppose next that the same flow solver is used to compute a flow field with stable stratification as described by equations (37). The diagnostic constraint for the leading order vertical velocity states that, upon heat addition, a mass element will move vertically within the stratification so as to maintain a position of neutral buoyancy. This motion is a result of an intimate coupling that simultaneously involves the pressure, density, and vertical momentum equations. Unless specially designed to respect this balance, a fully compressible flow solver will respond to heat addition by generating unphysical acoustic waves with amplitudes of the order of the numerical discretization error.

Buoyancy imbalances induced by truncation errors can in addition excite unphysical internal gravity waves.

Another deterioration of accuracy with decreasing Mach number is associated with the asymptotic pressure decomposition. Compressible flow solvers which feature a single pressure variable without explicitly implementing the asymptotic scalings have been observed to fail for sufficiently small Mach number in [6, 42, 38, 18]. This failure can be traced to back to cancellation of significant digits in the discretization of the pressure gradient. There is a large volume of literature addressing the issue of low Mach number flow numerics. Besides the references already cited here is an additional, unfortunately still incomplete, list for further reading, [1, 2, 4, 6, 16, 21, 23, 25, 26, 29, 40]. (See also section 5.2 below.)

In addition to these accuracy issues, compressible flow solvers often become inefficient at low Mach numbers, because they are subject to stability related time step restrictions associated with the acoustic wave speed. The regime of anelastic, weakly stratified flow, (39), allows the construction of more efficient schemes with time steps restricted by the magnitude of the convection velocity rather than the sound speed. The complexity of an associated computational model is comparable to that of a three-dimensional incompressible flow solver. If one is interested furthermore in flows with adiabatically stable stratification, (37), then the vertical velocity is given diagnostically and one is left with two-dimensional divergence constraints for the horizontal velocity in each horizontal layer of the computational grid. Imposing a set of independent horizontal divergence constraints should be much more efficient than a single three-dimensional one, and a considerable further gain in computational speed can potentially be achieved.

Multiple scales: Large (synoptic) scale flows in the atmosphere may be approximated quite well by the quasi-geostrophic theory summarized in section 3.1. However, the quasi-geostrophic equations are highly filtered in that they suppress most of the wave phenomena in the atmosphere, such as long wave acoustic gravity waves and internal gravity waves. Furthermore, modern computational capacities allow the simultaneous numerical representation of such long wavelength phenomena and flow features on length scales of a few kilometres. We have seen that the anelastic and pseudo-incompressible flow limits, which are relevant for these smaller scales, are described by equation systems of a very different nature. It is a highly non-trivial challenge to construct numerical schemes which represent the almost balanced large scale flows, at the same time resolve the much richer small scale dynamics, and provide high accuracy on both. In the next section we briefly summarize a strategy for the construction of multiple scales numerics that combines ideas from asymptotic analysis with numerical multi-grid techniques.

5.2 Asymptotically adaptive numerical methods

Here we consider low Mach number flows without gravity and rotation that are characterized by a single time scale, but multiple space scale analogous to what has been presented in section 4.2. In [23] the author derives simplified asymptotic limit equations that describe interaction of the long wave acoustic and short wave quasi-incompressible flow phenomena. Using a solution ansatz analogous to that in (47) the author obtains the following leading order set of equations for inviscid, adiabatic flows:

Small scale quasi-incompressible flow

$$\begin{aligned}
\rho^{(0)}_t \quad &+ \quad \nabla_{\vec{x}} \cdot (\rho^{(0)} \vec{v}^{(0)}) &&= \quad 0 \\
(\rho^{(0)} \vec{v}^{(0)})_t \quad &+ \quad \nabla_{\vec{x}} \cdot (\rho^{(0)} \vec{v}^{(0)} \circ \vec{v}^{(0)}) + \nabla_{\vec{x}}\, p^{(2)} &&= \quad -\nabla_{\vec{\xi}}\, p^{(1)} \\
& \quad \nabla_{\vec{x}} \cdot \vec{v}^{(0)} &&= \quad 0
\end{aligned} \tag{52}$$

Long wave acoustic modes

$$\begin{aligned}
\overline{\rho^{(0)}}_t \quad & &&= \quad 0 \\
\overline{(\rho^{(0)} \vec{v}^{(0)})}_t \quad &+ \quad \nabla_{\vec{\xi}}\, p^{(1)} &&= \quad 0 \\
p^{(1)}_t \quad &+ \quad \gamma P_0 \nabla_{\vec{\xi}} \cdot \overline{\vec{v}^{(0)}} &&= \quad 0
\end{aligned} \tag{53}$$

Here P_0 is the leading order pressure, which is constant in this regime. As in section (47), the first order pressure does not have small scale variation, so that $\nabla_{\vec{x}}\, p^{(1)} \equiv 0$ and $p^{(1)} = p^{(1)}(\vec{\xi}, t)$.

Solution techniques for the full compressible flow equations from (1) are suggested in [15, 16, 23, 24, 33] which are designed to operate efficiently and accurately in the present asymptotic regime of weakly compressible low Mach number flows. The idea is to first obtain efficient and accurate estimates of the key physical effects in the flow by using appropriate, optimized numerical methods to integrate the asymptotic limit equations over one time step. Then, ideas similar to those behind projection methods for incompressible flows, [19, 7, 8], are used to compose a final solution that actually obeys the full governing equations instead of merely the asymptotic limit equations. Here is a qualitative description of the key steps need to address the flow regime described by (52), (53).

In step 1 one obtains an explicit estimate of the effects of nonlinear convection using an upwind based higher order discretization. The time step restriction for this procedure is given by a stability constraint based on the maximum flow velocity (instead of on the sound speed).

Step 2 consists of (i) a filtering technique that extracts the current long wave components of the pressure, momentum, and velocity fields, $p^{(1)}, \overline{\vec{v}}, \overline{\rho\vec{v}}$. These data are then used to solve integrate (53) over one time step using a discretization that overcomes the sound speed based stability constraints. Here one may use an implicit discretization on the original mesh. Alternatively, one may consider the long wave – short wave decomposition as the basis of a physics-induced multi-grid technique and solve (53) by an explicit method on a coarse grid, [33].

The final step determines the small scale, second order pressure $p^{(2)}$, which is responsible for guaranteeing that the discrete flow field locally obeys a divergence constraint as in $(52)_3$. This step corresponds to a "projection step" in classical incompressible flow solvers.

Notice that steps 2 and 3 yield independent evaluations of discrete analogues of the first and second order pressures $p^{(1)}, p^{(2)}$ from the asymptotic analysis. In the numerical scheme, these pressure contributions are computed on the basis discretizations that are non-singular as the Mach number vanishes. Thus, the above-mentioned cancellation of significant digits is avoided. Only after completion of the time step are the pressure contributions re-composed to yield the full thermodynamic pressure.

A subtle issue in this approach is the fact that the numerical discretization makes explicit use of the current value of the Mach number. It is thus necessary to implement intelligent filtering techniques that extract the long wave and short wave solution components and, simultaneously, simultaneously yield the instantaneous relevant value of the Mach number. This issue has been addressed in [24].

Depending on the flow regimes that a computational code is mostly used for, the detailed components for steps 1 to 3 may vary. The original idea in [23] was to extend a higher order shock capturing technique for fully compressible flows to the regime of small and even zero Mach number, and to include long wave acoustic effects. The desired extension to zero Mach number, variable density, multi-dimensional flows has been achieved in [35], code versions for flows including long wave acoustics have been presented in [15, 16].

If one is interested mostly in flows at very low Mach number, then the overhead associated with full-fledged shock capturing techniques are unnecessary and simplifications can be introduced. Thus, Roller et al. [33] start from SIMPLE-type incompressible flow solvers, (see, e.g., [21]), and extend these technologies to the weakly compressible flow regime using the above techniques. Another alternative strategy has been proposed in by Worlikar et al. [43] in order to simulate the flow in a thermo-acoustic device. Their scheme is based on a vorticity stream function formulation for the small scale quasi-incompressible flow, and uses the modified Godunov-Type scheme from [23] for long wave acoustics.

6 Conclusions and outlook

The "traditional" aim of scale analysis in meteorology is to obtain simplified asymptotic limit equations that are easier to solve than the more comprehensive fully compressible flow equations. There is a large number of such simplified model equations, and each has its particular advantages and shortcomings in practice. In recent years the rapid increase of compute power has stimulated strong efforts to relax the various simplifying scaling assumptions associated with asymptotic limit equations and to tackle the numerical solution of the full governing equations directly. It has become clear, however, that a lack of computing power has not been the only obstacle in attempts at solving these equations in earlier decades. Additional numerical issues arise, which are intimately related to the fact that scale analysis and asymptotic modelling have been successful in the first place. The derivation of simplified asymptotic models usually takes advantage of a singular degeneration of the governing equations in the respective limit regime, and it is these singularities that can induce severe numerical stiffnesses, and dynamic range problems.

This paper advocates a so far relatively unconventional use of asymptotic scale analysis in the context of atmosphere flow modelling (see, however, [20]). We propose to construct new classes of "asymptotically adaptive numerical methods", which do solve the full three dimensional compressible flow equations, but use the results of asymptotic scale analysis in the design of the discretizations. Such a scheme would assess a small number of nondimensional characteristic numbers "on the fly" during a computation. These characteristic numbers are chosen so as to indicate whether the current flow state is or is not within the vicinity of a singular limit regime. As a singular limit is approached, the discretizations automatically adapt and they merge into a scheme for the asymptotic limit equations when the limit is actually achieved.

The strategy has been explained for the case of low Mach number multi-dimensional flows, for which it has been shown to be successful in a series of publications.

Acknowledgement. The author gratefully acknowledges the continous constructive co-operation with Dr. Ann Almgren, Lawrence Berkeley National Laboratory, USA, Dr. Omar M. Knio, Johns Hopkins University, Baltimore, MD, USA, and Dr. Nicola Botta, Potsdam Institute for Climate Impact Research, Germany. This work has been sponsored partly by the Deutsche Forschungsgemeinschaft under Grant KL 611/6-1,2.

References

1 ABARBANEL, S.; DUTH, P.; GOTTLIEB, D.: Splitting method for low Mach number Euler and Navier-Stokes equations. Computers and Fluids, **17** (1989), 1–12.

2 ALMGREN, A.; BELL, J.; COLELLA, P.; HOWELL, L.; WELCOME, M.: A conservative adaptive projection method for the variable density incompressible navier-stokes equations. LBNL Preprint, (1996) 39075 UC-405.

3 BATCHELOR, G. K.: The condition for dynamical similarity of motions of a frictionless perfect gas atmosphere. Quarterly Journal of Royal Meteorological Society, **79** (1953), 224–235.

4 BIJL, H.; WESSELING, P. A.: Unified method for computing incompressible and compressible flows in boundary-fitted coordinates. Journal of Computational Physics, **141** (1998), 153173.

5 Botta, N.; Klein, R.; Almgren, A.: Dry atmosphere asymptotics. Techn. rep., Potsdam Institute for Climate Impact Research, Potsdam, Germany, 1999.

6 Casulli, V.; Greenspan, D.: Pressure method for the numerical solution or transient compressible fluid flow. Int. J. Num. Meth. Fluids, **4** (1984), 1001–1012.

7 Chorin, A. J.: A numerical method for solving incompressible flow problems. J. Comput. Phys., **2** (1967), 12–26.

8 Chorin, A. J.: Numerical solution of the Navier-Stokes equations. Math. Comp., **22** (1968), 745–762.

9 Durran, D. R.: Improving the anelastic approximation. Journal of the Atmospheric Sciences, **46** (1989), 1453–1461.

10 Durran, D. R.: Numerical Methods for Wave Equations in Geophysical Fluid Dynamics. Springer, 1999.

11 Ebin, D. G.: Motion of Slightly Compressible Fluids in a Bounded Domain. i. Comm. Pure Appl. Math., **35** (1982), 451–485.

12 Emanuel, K. A.: Atmospheric Convection. Oxford University Press, 1994.

13 Embid, P.; Majda, A. J.: Averaging over fast gravity waves for geophysical flows with arbitrary potential vorticity. Communications in Partial Differential Equations, **21** (1996), 619–658.

14 Embid, P.; Majda, A. J.: Averaging over fast gravity waves for geophysical flows with unbalanced initial data. Theoretical and Computational Fluid Dynamics, **11** (1998), 155–169.

15 Geratz, K. J.: Erweiterung eines Godunov-Typ-Verfahrens für mehrdimensionale kompressible Strömungen auf die Fälle kleiner und verschwindender Machzahl. PhD thesis, RWTH Aachen, 1998.

16 Geratz, K.; Klein, R.; Munz, C.-D.; Roller, S.: Multiple Pressure Variable (MPV) Approach for Low Mach Number Flows Based on Asymptotic Analysis. In: Hirschel, E. H. (ed.): Flow simulation with high-performance computers II. DFG priority research programme results. Notes on Numerical Fluid Mechanics, 52. Vieweg Verlag, Braunschweig, 1996.

17 Gill, A. E. (ed.): Atmosphere-Ocean Dynamics. International Geophysical Series, 30. Academic Press, London, 1982.

18 Guillard, H.; Viozat, C.: On the behoviour of upwind schemes in the low mach number limit. Computers and Fluids, **28** (1999), 63–86.

19 Harlow, F. H.; Welch, J. E.: Numerical calculation of time dependent viscous incompressible flow. Phys. Fluids, **8** (1965), 2182–2189.

20 Kaper, G.; Garbey, M. (eds.): Asymptotic analysis and the numerical solution of partial differential equations. Lecture Notes in Pure and Applied Mathematics, 130. Marcel Dekker, New York, 1991.

21 Karki, K. C.; Patankar, S. V.: Pressure based calculation procedure for viscous flows at all speeds in arbitrary configurations. AIAA J., **27** (1989), 1167–1174.

22 Klainerman, S.; Majda, A. J.: Compressible and Incompressible fluids. Comm. Pure Appl. Math., **35** (1982), 629–.

23 Klein, R.: Semi-implicit extension of a godunov-type scheme based on low mach number asymptotics i: One-dimensional flow. Journal of Computational Physics, **121** (1995), 213–237.

24 Klein, R.; Botta, N.; Schneider, T.; Munz, C.; Roller, S.; Meister, A.; Hoffmann, L.; Sonar, T.: Asymptotic adaptive methods for multi-scale problems in fluid mechanics. Journal of Engineering Mathematics, – (2000), —.

submitted.

25 Lai, M.; Bell, J. B.; Colella, P.: A projection method for combustion in the zero Mach number limit. AIAA Paper, **3369** (1993).

26 LeVeque, R. J.: A large time step generalization of Godunov's method for systems of conservation laws. SIAM J. Num. Anal., **22** (1985), 1051–1073.

27 LeVeque, R. J.: Numerical Methods for Conservation Laws. Birkhäuser Verlag, Zürich, Schweiz, 1992. IBSN 0-521-43009-7.

28 Majda, A.; Sethian, J.: The derivation and numerical solution of the equations for zero mach number combustion. Combustion Science and Technology, **42** (1985), 185–205.

29 Merkle, C. L.; Choi, Y.-H.: Computation of low speed flow with heat addition. AIAA J., **25** (1987), 831–838.

30 Muraki, D. J.; Snyder, C.; Rotunno, R.: The next-order corrections to quasigeostrophic theory. Journal of the Atmospheric Sciences, **56** (1998), 1547–1560.

31 Ogura, Y.; Phillips, N. A.: Scale analysis of deep moist convection and some related numerical calculations. Journal of Atmosphere Science, **19** (1962), 173–179.

32 Pedlosky, J. (ed.): Geophysical Fluid Dynamics. Springer, 2 edition, 1987.

33 Roller, S.; Munz, C.-D.; Geratz, K.; Klein, R.: The extension of incompressible flow solvers to the weakly compressible regime. Theoretical and Numerical Fluid Dynamics, (1999), submitted for publication.

34 Schlichting, H. (ed.): Boundary Layer Theory. MacGraw-Hill, 1968.

35 Schneider, T.; Botta, N.; Klein, R.; Geratz, K. J.: Extension of finite volume compressible flow solvers to multi-dimensional, variable density zero mach number flows. Journal of Computational Physics, **155** (1999), 248–286.

36 Schneider, W. (ed.): Mathematische Methoden in der Strömungsmechanik. Vieweg, 1978.

37 Schochet, S.: Fast singular limits of hyperbolic pdes. Journal of Differential Equations, **114** (1994), 476–512.

38 Sesterhenn, J.; Müller, B.; Thomann, H.: Computation of compressible low mach number flow. Computational Fluid Dynamics, **2** (1992), 829–833.

39 Smith, R. B.: Why can't stably stratified air rise over high ground. In: Blumen, W. (ed.): Atmospheric processes over complex terrain. Meteorological Monographs, 23, 1990, pp. 105–107.

40 Turkel, E.: Preconditioned Methods for Solving the Incompressible and Low Speed Compressible Equations. Journal of Computational Physics, **72** (1987), 277 – 298.

41 van Dyke, M. (ed.): Perturbation Methods in Fluid Mechanics, Annotated Ed. Parabolic Press, 1975.

42 Volpe, G.: On the use and accuracy of compressible flow codes at low mach numbers. AIAA Paper 91-1662, 1991.

43 Worlikar, A.; Knio, O.; Klein, R.: Numerical simulation of a thermo-acoustic refrigerator: Ii stratified flow around the stack. Journal of Computational Physics, **144** (1998), 299–324.

44 Zeytounian, R. K. (ed.): Meteorological Fluid Dynamics. Lecture Notes in Physics, m5. Springer, 1991.

Address: Prof. Rupert Klein, Potsdam Institut für Klimafolgenforschung, Telegrafenberg C4, 14412 Potsdam, Germany, email: `rupert.klein@pik-potsdam.de`