Nonuniqueness of generalised weak solutions to the primitive and Prandtl equations

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Abstract

We develop a convex integration scheme for constructing nonunique weak solutions to the hydrostatic Euler equations (also known as the inviscid primitive equations of oceanic and atmospheric dynamics) in both two and three dimensions. We also develop such a scheme for the construction of nonunique weak solutions to the three-dimensional viscous primitive equations, as well as the two-dimensional Prandtl equations.

While in [D.W. Boutros, S. Markfelder and E.S. Titi, arXiv:2208.08334 (2022)] the classical notion of weak solution to the hydrostatic Euler equations was generalised, we introduce here a further generalisation. For such generalised weak solutions we show the existence and nonuniqueness for a large class of initial data. Moreover, we construct infinitely many examples of generalised weak solutions which do not conserve energy. The barotropic and baroclinic modes of solutions to the hydrostatic Euler equations (which are the average and the fluctuation of the horizontal velocity in the z-coordinate, respectively) that are constructed have different regularities.

Keywords: Convex integration, primitive equations of oceanic and atmospheric dynamics, Prandtl equations, Onsager's conjecture, energy dissipation, hydrostatic Euler equations, hydrostatic Navier-Stokes equations, weak solutions, barotropic mode, baroclinic mode, nonuniqueness of weak solutions

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1 Introduction

1.1 Problems considered in this paper and context

In this work, we consider the following general equation (with $(d+1)$-dimensional spatial domain for $d = 1, 2$)

$$\partial_t u - \nu_h^* \Delta_h u - \nu_v^* \partial_{zz} u + u \cdot \nabla_h u + w \partial_z u + \nabla_h p = 0, \tag{1.1}$$

$$\partial_z p = 0, \tag{1.2}$$

$$\nabla_h \cdot u + \partial_z w = 0, \tag{1.3}$$

where the horizontal velocity field $u : \mathbb{T}^{d+1} \times (0,T) \to \mathbb{R}^d$, the vertical velocity field $w : \mathbb{T}^{d+1} \times (0,T) \to \mathbb{R}$, and the pressure $p : \mathbb{T}^{d+1} \to \mathbb{R}$ are unknown, and the horizontal and vertical viscosity parameters $\nu_h^*, \nu_v^* \geq 0$ are given constants. The d-dimensional horizontal gradient is denoted by ∇_h and the d-dimensional horizontal Laplacian by Δ_h. Since the pressure p is only determined up to an additive constant, we may require that p is mean-free.

In this paper we are interested in the following cases:

- Taking $d = 2$ and $\nu_h^* = \nu_v^* = 0$ gives the three-dimensional hydrostatic Euler equations of an incompressible fluid (also known as the inviscid primitive equations of oceanic and atmospheric dynamics). In this paper, the terms inviscid primitive equations and hydrostatic Euler equations will be used interchangeably.

- Taking $d = 2$ and $\nu_h^*, \nu_v^* > 0$ leads to the three-dimensional viscous primitive equations. We remark that the cases with anistropic viscosities ($\nu_h^* > 0$ and $\nu_v^* = 0$, or $\nu_h^* = 0$ and $\nu_v^* > 0$) have also been studied.
- Taking $d = 1$ and $\nu_h^* = \nu_v^* = 0$ yields the two-dimensional inviscid primitive equations (or hydrostatic Euler equations).
- Taking $d = 1$, $\nu_h^* = 0$ and $\nu_v^* > 0$ yields the two-dimensional Prandtl equations.

In this paper, we will develop a convex integration scheme for system (1.1)-(1.3) for the cases mentioned above. In particular, we will work with a generalised notion of weak solution. While classical weak solutions have sufficient Lebesgue integrability for the nonlinearity to make sense as an $L^1(\mathbb{T}^{d+1} \times (0,T))$ function, another notion of weak solution was introduced in [7] where the nonlinearity is interpreted as a paraproduct. The generalised weak solutions introduced in this paper treat the nonlinearity in an even more general way, see section 1.3.3 below.

In all the cases of (1.1)-(1.3) that we are interested in, we will show the existence of such generalised weak solutions (for a dense set of initial data in the relevant spaces). In addition, we will show that such weak solutions are nonunique.

If $\nu_h^* = \nu_v^* = 0$, we recall that classical spatially analytic solutions of (1.1)-(1.3) (see [40, 50, 51]) conserve the energy, i.e. the spatial $L^2(\mathbb{T}^{d+1})$ norm of u. In [7] an analogue of Onsager's conjecture was studied for the three-dimensional hydrostatic Euler equations and it was found that there exist several sufficient regularity criteria for weak solutions which guarantee the conservation of energy. In particular, there exist several notions of weak solutions for these equations, each of which have their own version of the analogue of the Onsager conjecture.

In this work, we will construct generalised weak solutions to these equations, which do not conserve energy and do not satisfy the regularity criteria mentioned above. In other words, in this paper we prove a first result towards the aim of resolving the dissipation part of the analogue of the Onsager conjecture for the inviscid primitive equations (hydrostatic Euler), while the conservation part of the analogue of the Onsager conjecture has been studied in [7], as was mentioned before.

1.2 Literature overview

In this section we will provide an overview of some of the literature that is related to this work. As both the primitive and Prandtl equations as well as the Onsager conjecture have been the subject matter of many works in recent years, this overview is by no means comprehensive and is by necessity incomplete in reviewing all the relevant work.

Onsager's conjecture was originally posed in [75] for the incompressible Euler equations. The conjecture states that if a weak solution lies in $L^3((0,T); C^{0,\alpha}(\mathbb{T}^3))$ for $\alpha > \frac{1}{3}$ it must conserve energy. If $\alpha < \frac{1}{3}$ energy might not be conserved.

In [37] a proof of a slightly weaker result than the first half of the conjecture was given. A full proof of the first half was then given in [30]. In [35] a different proof was presented, which relied on an equation of local energy balance and a defect measure. In [5, 6] (see also [4]) the problem was considered in the presence of physical boundaries and the first half of the conjecture was proved in this case.

The existence of non-energy conserving solutions of the Euler equations of an incompressible fluid was first shown in [84, 85]. To prove the existence of dissipative weak solutions of the Euler equations (and to prove the second half of Onsager's conjecture), techniques from convex integration were used. They were introduced for the first time in the context of incompressible fluid mechanics in [32, 34].

The second half of the conjecture was then proven in [47], after gradual success in the papers [12, 33] (and see references therein). The proof in [47] relied on the Mikado flows that were developed in [31]. In the work [13] dissipative Hölder continuous solutions of the Euler equations up to $\frac{1}{3}$ were constructed.

Subsequently, an intermittent version of convex integration was developed. This was first used in [15] to prove the nonuniqueness of very weak (not Leray-Hopf) solutions to the Navier-Stokes equations. In [11] this result was extended to show the existence of nonunique weak solutions with a bound on the singular set. In [14, 72] an intermittent scheme was constructed to prove the existence of non-energy conserving weak solutions of the Euler equations with Sobolev regularity. In [64] the method of [15] was generalised to the hyperviscous Navier-Stokes equations to show the sharpness of the Lions exponent.

After the works [67–69] where a spatially intermittent convex integration scheme was developed for the transport equation, temporal intermittency was introduced to the scheme in [26, 28] to prove the nonuniqueness of weak solutions to the transport equation. This scheme was then adapted to the Navier-Stokes equations in [27] to prove the sharpness of the Prodi-Serrin criteria, and in [25] to show that L^2 is the critical space for nonuniqueness for the 2D Navier-Stokes equations.

The primitive equations of oceanic and atmospheric dynamics were introduced in [81]. They were studied mathematically for the first time in [60–62], in which the global existence of weak solutions was proved. The short time existence of strong solutions was then obtained in [43]. The global well-posedness of the viscous primitive equations was proven in [23], see also [48]. In [53, 54] different boundary conditions were considered, and in [45] global well-posedness was established using a semigroup method.

Subsequently, the cases with only horizontal viscosity (as well as only horizontal diffusivity) were studied in [19–21]. The case with only vertical diffusivity and full viscosity was looked at in [18, 24]. The case with only horizontal diffusivity and full viscosity was investigated in [17]. The small aspect ratio was rigorously justified in a weak sense in [2] (see also [10]). It was subsequently proven in a strong sense with full viscosity in [56] and with only horizontal viscosity in [57] with error estimates in terms of the small aspect ratio .

The case with only vertical viscosity was studied in [80], in which linear ill-posedness was proven. The ill-posedness can be counteracted by adding a linear damping term, see [22] for more details. By considering the case of initial data with Gevrey regularity with certain convexity conditions, in [39] local well-posedness was established. The local well-posedness for analytic data was proven in [50, 51] (without rotation) and [40] (with rotation). By considering small data which are analytic in the horizontal variables, the paper [77] established global well-posedness for the case without rotation and Dirichlet boundary conditions. Finally, [59] considered the case of impermeable and stress-free boundary conditions.

The linear and nonlinear ill-posedness of the inviscid primitive equations in all Sobolev spaces was proven in [44, 80]. The ill-posedness results in Sobolev spaces suggest that the natural space for showing local well-posedness of the inviscid primitive equations is the space of analytic functions, which was proved in [40, 50, 51]. In [40] the role of fast rotation in prolonging the life-space of solutions was investigated.

In [16] it was shown that smooth solutions of the inviscid primitive equations can form a singularity in finite time, see also [90]. In [29] the existence and nonuniqueness of weak solutions with L^∞ data was proven. In [7] several sufficient criteria for energy conservation were proven. In the inviscid setting there have also been works studying the case of initial data with a monotonicity assumption, see [9, 49, 66].

The Prandtl equations for the boundary layer were derived by Prandtl in [78]. In [73, 74] the local well-posedness of the equations was shown under a monotonicity assumption. In [82] the local well-posedness for analytic data was proven, while in [36] the blow-up of solutions for certain classes of C^∞ data was proven. Further local well-posedness results were proved in [46, 49, 52, 76, 77]. In [42] it was shown that the equations are nonlinearly unstable.

The linear ill-posedness of the Prandtl equations in all Sobolev spaces was shown in [38] (for further work see [63] and references therein). In the three-dimensional case a convex integration scheme was developed in [65]. The analytic local well-posedness has been improved to Gevrey function spaces, see [58] and references therein.

1.3 Definitions and main results

1.3.1 Baroclinic and barotropic modes

Now we introduce the notion of barotropic and baroclinic modes, which is an important decomposition of the solutions which has been explored extensively in the investigation of the primitive equations. In the construction of the convex integration scheme for the primitive equations we will not use this decomposition explicitly. However, it is an important idea underlying the scheme.

We will illustrate this concept for the equations in the inviscid case, the viscous case is similar and can be found in [23]. The 3D inviscid primitive equations are given by

$$\partial_t u + u \cdot \nabla_h u + w \partial_z u + \nabla_h p = 0, \tag{1.4}$$

$$\partial_z p = 0, \tag{1.5}$$

$$\nabla_h \cdot u + \partial_z w = 0, \tag{1.6}$$

where $u : \mathbb{T}^3 \times (0,T) \to \mathbb{R}^2$ is the horizontal velocity field, $w : \mathbb{T}^3 \times (0,T) \to \mathbb{R}$ the vertical velocity field and $p : \mathbb{T}^3 \times (0,T) \to \mathbb{R}$ the pressure.

The barotropic mode $\overline{u}$ of a velocity field u is defined as follows

$$\overline{u}(x_1, x_2, t) := \int_{\mathbb{T}} u(x_1, x_2, z, t) \, \mathrm{d}z. \tag{1.7}$$

The baroclinic mode $\widetilde{u}$ is defined as the fluctuation

$$\widetilde{u} := u - \overline{u}. \tag{1.8}$$

The primitive equations (1.4)-(1.6) can then be written formally as a coupled system of evolution equations for the barotropic and baroclinic modes $\overline{u}$ and $\widetilde{u}$, which are

$$\partial_t \overline{u} + (\overline{u} \cdot \nabla_h)\overline{u} + \overline{\left[(\widetilde{u} \cdot \nabla_h)\widetilde{u} + (\nabla_h \cdot \widetilde{u})\widetilde{u}\right]} + \nabla_h p = 0, \tag{1.9}$$

$$\partial_t \widetilde{u} + (\widetilde{u} \cdot \nabla_h)\widetilde{u} + w\partial_z \widetilde{u} + (\widetilde{u} \cdot \nabla_h)\overline{u} + (\overline{u} \cdot \nabla_h)\widetilde{u} - \overline{\left[(\widetilde{u} \cdot \nabla_h)\widetilde{u} + (\nabla_h \cdot \widetilde{u})\widetilde{u}\right]} = 0. \tag{1.10}$$

Moreover, we have the following incompressibility conditions

$$\nabla_h \cdot \overline{u} = \nabla_h \cdot \widetilde{u} + \partial_z w = 0, \tag{1.11}$$

which formally follow from equation (1.6) and the periodicity of the functions.

In the convex integration scheme, we will add separate barotropic and baroclinic perturbations. This leads to different regularities of the barotropic and baroclinic modes of the solution and allows us to control different parts of the error.

The following estimates on the baroclinic and barotropic modes are standard

$$\|\overline{u}\|_{L^p} \lesssim \|u\|_{L^p}, \qquad \|\widetilde{u}\|_{L^p} \lesssim \|u\|_{L^p}.$$

1.3.2 Notation

Throughout the paper we will use the following notation.

- The components of the spatial variable are given by $x = (x_1, z)$ if $d = 1$, and $x = (x_1, x_2, z)$ if $d = 2$. For $d = 1$, x_1 represents the horizontal direction, for $d = 2$ the horizontal position is given by (x_1, x_2). In both cases, z is the vertical direction.
- The horizontal velocity field is called u, the vertical velocity is denoted by w and the full velocity by $\mathbf{u} = (u, w)$. They are d-, 1- and $(d+1)$-dimensional, respectively.
- We use the symbol ∇_h for the horizontal gradient (which equals ∂_{x_1} if $d = 1$), and ∇ for the full ($(d+1)$-dimensional) gradient.
- For an integrability parameter $1 \leq p \leq \infty$, the Hölder conjugate is denoted by p', i.e. $\frac{1}{p} + \frac{1}{p'} = 1$.
- Let $1 < p \leq \infty$. In section 1, $p-$ denotes any parameter $1 \leq p- < p$. In the other sections we have to be a bit more precise. In particular there is a need to quantify the '$-$' in $p-$. More precisely there will be a $\delta > 0$ and we set $p- := \frac{1}{\frac{1}{p}+\delta}$. Here we tacitly assume that δ is sufficiently small, such that $p- \geq 1$.
- For $1 \leq p, q \leq \infty$ and $s \in \mathbb{R}$ the Besov space $B^s_{p,q}(\mathbb{T}^3)$ is defined in appendix A.1. Let us emphasise here that $B^s_{2,2}(\mathbb{T}^3) = H^s(\mathbb{T}^3)$, see Remark A.2.

- Throughout this paper, we will omit the domain of a space-time norm if it is $\mathbb{T}^{d+1} \times [0, T]$, e.g. we write $\|\cdot\|_{L^p(H^s)} = \|\cdot\|_{L^p((0,T);H^s(\mathbb{T}^{d+1}))}$.
- In view of section 1.3.1 we define the barotropic and baroclinic part of any quantity $a = a(x)$ by

$$\overline{a} = \int_{\mathbb{T}} a(x)\,\mathrm{d}z, \qquad \widetilde{a} = a - \overline{a}.$$

1.3.3 Generalised weak solutions

In [7] two new types of weak solutions to the hydrostatic Euler equations (1.4)-(1.6) were introduced. In the present paper we will consider a slightly different notion of weak solution, which we will refer to as a *generalised weak solution*. This notion of solution is inspired by the notion of a type III weak solution, as introduced in [7].

Before we state the results for the different cases of the system (1.1)-(1.3), we will be more specific regarding the notion of weak solution used in this paper. The weak solutions of (1.1)-(1.3) we consider are defined as follows: We assume that $u \in L^2(\mathbb{T}^3 \times (0,T))$, $w \in \mathcal{D}'(\mathbb{T}^3 \times (0,T))$ and $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ for some suitably large $s \in \mathbb{R}$. System (1.1)-(1.3) must then be satisfied in the sense of distributions, where the vertical advection term

$$\int_0^T \langle uw, \partial_z \phi \rangle_{B^{-s}_{1,\infty} \times B^{s}_{\infty,1}}\,\mathrm{d}t,$$

is interpreted as a duality bracket between the term uw and the test function $\phi \in \mathcal{D}(\mathbb{T}^3 \times (0,T))$.

If u and w happen to have sufficient regularity, for example when $u \in L^2((0,T); H^{s+\delta}(\mathbb{T}^3))$ and $w \in L^2((0,T); H^{-s}(\mathbb{T}^3))$ (for some small $\delta > 0$), then by applying the paradifferential calculus (see appendix A) we know that $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$. This is a stronger notion of solution compared to the notion of a generalised weak solution that we introduced above, as u is required to have (positive) Sobolev regularity and w has to possess some regularity (i.e., it is more than just a distribution).

In the next few subsections, we will give precise definitions of the notion of weak solution we will use, and we will state the theorems we will prove for the different cases of the system (1.1)-(1.3). But generally speaking, we will split the nonlinearity uw into the barotropic-vertical and baroclinic-vertical interactions, i.e., the terms $\overline{u}w$ and $\widetilde{u}w$.

The baroclinic mode $\overline{u}$ of the constructed solutions will have sufficient regularity such that $\overline{u}w$ can be interpreted as a paraproduct. The terms $\widetilde{u}$ and w do not have sufficient regularity to apply the paradifferential calculus. However, as part of the convex integration scheme we will obtain separate estimates on $\widetilde{u}w$ in order to show that it lies in $L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ for some suitable s. Therefore the weak solutions we obtain are partly 'generalised' (as for the baroclinic-vertical part of the nonlinearity) and partly 'paradifferential' (for the barotropic-vertical part of the nonlinearity).

1.3.4 Results for the 3D inviscid primitive equations

We first introduce the notion of weak solution for the 3D inviscid primitive equations (1.4)-(1.6).

Definition 1.1. A triple (u, w, p) is called a weak solution of the hydrostatic Euler equations (1.4)-(1.6) if $u \in L^2(\mathbb{T}^3 \times (0,T))$, $w \in \mathcal{D}'(\mathbb{T}^3 \times (0,T))$ and $p \in L^1(\mathbb{T}^3 \times (0,T))$ such that $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ (where $s > 0$ is referred to as the regularity parameter) and the equations are satisfied in the sense of distributions, i.e.

$$\int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} u \otimes u : \nabla_h \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \\ + \int_0^T \langle uw, \partial_z \phi_1 \rangle_{B^{-s}_{1,\infty} \times B^{s}_{\infty,1}} \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} p \nabla_h \cdot \phi_1 \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.12}$$

$$\int_0^T \int_{\mathbb{T}^3} p \partial_z \phi_2 \,\mathrm{d}x\,\mathrm{d}t = 0, \tag{1.13}$$

$$\int_0^T \langle \mathbf{u}, \nabla \phi_3 \rangle \,\mathrm{d}t = 0, \tag{1.14}$$

for all test functions ϕ_1, ϕ_2 and ϕ_3 in $\mathcal{D}(\mathbb{T}^3 \times (0,T))$.

Remark 1.2. We emphasise that this definition of weak solutions to (1.4)-(1.6) is more general than the notion of weak solution introduced in [7]. While in [7] the velocity field of a weak solution has sufficient regularity to automatically guarantee that $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ (by using the paradifferential calculus), in Definition 1.1 we do not have sufficient (separate) regularity requirements on u and w such that the product uw is well-defined. Hence $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ is a separate independent requirement of Definition 1.1.

In this paper we will prove the following result.

Theorem 1.3. *Let $T > 0$ and suppose there exist smooth solutions of the hydrostatic Euler equations* (1.4)-(1.6) (u_1, w_1, p_1) *on* $[0, T/2]$ *and* (u_2, w_2, p_2) *on* $[T/2, T]$. *Moreover, let* $1 \leq q_1, q_2, q_3 \leq \infty$ *and*[1] $0 < s_1, s_3$ *be parameters satisfying*

$$q_2 > 2, \quad q_3 \leq q_1, \quad s_1 > s_3, \quad \frac{2}{q_1} > s_1 + 1. \tag{1.15}$$

Then there exists a weak solution (u, w, p) in the sense of Definition 1.1 with regularity parameter $s = 1$ and with the following properties:

1. *The solution satisfies that*

$$(u, w, p)(\cdot, t) = \begin{cases} (u_1, w_1, p_1)(\cdot, t) & \text{if } t \in [0, T/4), \\ (u_2, w_2, p_2)(\cdot, t) & \text{if } t \in (3T/4, T]. \end{cases} \tag{1.16}$$

[1] Note that s_2 does not appear in this paper.

2. *We have that*

$$\overline{u} \in L^2(\mathbb{T}^3 \times (0,T)) \cap L^{q_1}((0,T); H^{s_1}(\mathbb{T}^3)),$$
$$\widetilde{u} \in L^{q_2-}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3-}((0,T); H^{s_3}(\mathbb{T}^3)),$$
$$w \in L^{q_2'}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3'}((0,T); H^{-s_3}(\mathbb{T}^3)),$$

where $\overline{u}$ *and* $\widetilde{u}$ *denote the barotropic and baroclinic modes of* u *respectively.*

Remark 1.4. Alternatively one can construct a weak solution with the properties stated in Theorem 1.3 where the only difference is that the endpoint time integrability is attained for $\widetilde{u}$ rather than w. In other words

$$\widetilde{u} \in L^{q_2}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3}((0,T); H^{s_3}(\mathbb{T}^3)),$$
$$w \in L^{q_2'-}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3'-}((0,T); H^{-s_3}(\mathbb{T}^3)),$$

see Remarks 2.6 and 5.6 below. To this end however, we have to require that $q_3 < q_1$ (strictly) in (1.15).

Remark 1.5. By proceeding as in section 7 below, we can achieve in addition that $\overline{u}, \widetilde{u} \in L^1((0,T); W^{1,1}(\mathbb{T}^3))$. To this end, however, we have to require the constraints (1.20) rather than (1.15), see also Theorem 1.10 and Remark 1.13, below.

Remark 1.6. Again we would like to remark that the solutions constructed in Theorem 1.3 are partially 'generalised' (see section 1.3.3) and partially 'paradifferential' as in [7]. In particular, they have been inspired by the type III weak solutions that were introduced in [7].
More precisely, from the regularities of $\overline{u}$ and w stated in Theorem 1.3 it follows that $\overline{u}w \in L^1((0,T); B^{-1}_{1,\infty}(\mathbb{T}^3))$ (see the proof of Theorem 1.3 in sections 4-6 for details). The term $\widetilde{u}w$ is estimated directly in $L^1((0,T); B^{-1}_{1,\infty}(\mathbb{T}^3))$ as part of the convex integration scheme, as one cannot obtain the regularity of the product $\widetilde{u}w$ simply from the regularities of $\widetilde{u}$ and w (as they are insufficient to apply the paradifferential calculus directly).
The specific form of the perturbations allows for a direct estimate, as was done for example in [28]. Therefore the interpretation of the term $\overline{u}w$ can be seen as 'paradifferential', while the interpretation of the term $\widetilde{u}w$ is in the sense of a 'generalised weak solution' (as in Definition 1.1).

Remark 1.7. In addition, we would like to emphasise that in the presence of physical boundaries the primitive equations are often studied with no-normal flow boundary conditions on the top and bottom of the channel, i.e. $w|_{z=0,1}= 0$. However, in the convex integration scheme developed in this paper we will work on the three-dimensional torus rather than the channel. Note that solutions in the torus can be understood as solutions in the channel with an in-flow out-flow boundary condition, i.e.

$$w(x_1, x_2, 0, t) = w(x_1, x_2, 1, t) = w_B(x_1, x_2, t),$$

for a flow w_B. In our case w_B will be constructed as part of the convex integration scheme. In other words we will not solve the boundary value problem for given w_B and in particular, not for the case of the impermeability boundary condition $w_B = 0$.
We also remark that the constructed flow w_B belongs to the space $L^{q_2'}((0,T); L^2(\mathbb{T}^2)) \cap L^{q_3'}((0,T); H^{-s_3}(\mathbb{T}^2))$, where the parameters q_2', q_3' and s_3 are the same as in Theorem 1.3.

Theorem 1.3 allows to show the nonuniqueness and existence of solutions which do not conserve energy:

Corollary 1.8. *For any analytic initial data there exist infinitely many global-in-time weak solutions* (u, w, p) *of the hydrostatic Euler equations* (1.4)-(1.6) *(in the sense of Definition 1.1 which satisfy the regularity properties of Theorem 1.3) and they do not conserve energy.*

Proof. We take the smooth local-in-time solution for the given choice of analytic data (whose existence can be proven using the methods from [40, 51]) as the first solution (u_1, w_1, p_1) on $[0, T/2]$, and the zero solution on $[T/2, T]$ as (u_2, w_2, p_2). Then Theorem 1.3 yields a weak solution, which we may extend by zero for $t > T$. If the initial data are non-zero, we can conclude that the energy is not conserved as it is positive on $[0, T/4)$ and zero on $(3T/4, \infty)$.

Another global-in-time weak solution can be constructed similarly with replacing T by $T/4$. This solution has positive energy on $[0, T/16)$ while the energy is zero on $(3T/16, \infty)$. Consequently the two solutions cannot coincide. Repeating this argument leads to infinitely many global-in-time weak solutions with the same initial data, which are smooth and unique for a small initial interval of time, but which do not conserve energy.

For zero initial data, we observe that Theorem 1.3 allows one to 'connect' any analytic initial data with any analytic data in finite time. Hence we may connect the zero initial data to arbitrary analytic data with positive energy at $t = \widetilde{T}$. On the time interval $[\widetilde{T}, \infty)$ we then proceed as above. □

1.3.5 Results for the 3D viscous primitive equations

We now consider the viscous primitive equations, which are given by

$$\partial_t u - \nu_h^* \Delta_h u - \nu_v^* \partial_{zz} u + u \cdot \nabla_h u + w \partial_z u + \nabla_h p = 0, \tag{1.17}$$

$$\partial_z p = 0, \tag{1.18}$$

$$\nabla_h \cdot u + \partial_z w = 0, \tag{1.19}$$

where ν_h^* and ν_v^* are the horizontal and vertical viscosities. As before, $u : \mathbb{T}^3 \times (0, T) \to \mathbb{R}^2$ is the horizontal velocity field, $w : \mathbb{T}^3 \times (0, T) \to \mathbb{R}$ the vertical velocity and $p : \mathbb{T}^3 \times (0, T) \to \mathbb{R}$ the pressure. We have the following notion of weak solution for these equations.

Definition 1.9. A triple (u, w, p) is called a weak solution of the viscous primitive equations (1.17)-(1.19) if $u \in L^2(\mathbb{T}^3 \times (0, T)) \cap L^1((0, T); W^{1,1}(\mathbb{T}^3))$, $w \in \mathcal{D}'(\mathbb{T}^3 \times (0, T))$ and $p \in L^1(\mathbb{T}^3 \times (0, T))$ such that $uw \in L^1((0, T); B^{-s}_{1,\infty}(\mathbb{T}^3))$ (where $s > 0$ is referred to as the regularity parameter) and the equations are satisfied in the sense of distributions, i.e.

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \phi_1 \,\mathrm{d}x\,\mathrm{d}t - \nu_h^* \int_0^T \int_{\mathbb{T}^3} \nabla_h u : \nabla_h \phi_1 \,\mathrm{d}x\,\mathrm{d}t - \nu_v^* \int_0^T \int_{\mathbb{T}^3} \partial_z u \cdot \partial_z \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \\
+ \int_0^T \int_{\mathbb{T}^3} u \otimes u : \nabla_h \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle uw, \partial_z \phi_1 \rangle_{B^{-s}_{1,\infty} \times B^s_{\infty,1}} \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} p \nabla_h \cdot \phi_1 \,\mathrm{d}x\,\mathrm{d}t = 0, \\
\int_0^T \int_{\mathbb{T}^3} p \partial_z \phi_2 \,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

$$\int_0^T \langle \mathbf{u}, \nabla \phi_3 \rangle \, \mathrm{d}t = 0,$$

for all test functions ϕ_1, ϕ_2 and ϕ_3 in $\mathcal{D}(\mathbb{T}^3 \times (0,T))$.

In this paper we will prove the following result.

Theorem 1.10. *Let $T > 0$ and suppose there exist smooth solutions of the viscous primitive equations (1.17)-(1.19) (u_1, w_1, p_1) on $[0, T/2]$ and (u_2, w_2, p_2) on $[T/2, T]$. Moreover, let $1 \leq q_1, q_2, q_3 \leq \infty$ and $0 < s_1, s_3$ be parameters satisfying the following relations*

$$q_2 > 2, \quad q_3 < q_1, \quad s_1 > s_3, \quad \frac{2}{q_1} > s_1 + 1, \quad s_3 > \frac{1}{2\left(1 - \frac{1}{q_2}\right)} \left(\frac{1}{q_3} - \frac{1}{q_2} \right). \tag{1.20}$$

Then there exists a weak solution (u, w, p) in the sense of Definition 1.9 with regularity parameter $s = 1$ and with the following properties:

1. *The solution satisfies that*

$$(u, w, p)(\cdot, t) = \begin{cases} (u_1, w_1, p_1)(\cdot, t) & \text{if } t \in [0, T/4), \\ (u_2, w_2, p_2)(\cdot, t) & \text{if } t \in (3T/4, T]. \end{cases}$$

2. *We have that*

$$\begin{aligned} \overline{u} &\in L^2(\mathbb{T}^3 \times (0,T)) \cap L^{q_1}((0,T); H^{s_1}(\mathbb{T}^3)) \cap L^1((0,T); W^{1,1}(\mathbb{T}^3)), \\ \widetilde{u} &\in L^{q_2 -}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3 -}((0,T); H^{s_3}(\mathbb{T}^3)) \cap L^1((0,T); W^{1,1}(\mathbb{T}^3)), \\ w &\in L^{q_2' -}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3' -}((0,T); H^{-s_3}(\mathbb{T}^3)). \end{aligned}$$

Remark 1.11. Similar to Theorem 1.3 one can even obtain endpoint time integrability for w. With the modification described in Remarks 1.4, 2.6 and 5.6 one can alternatively establish endpoint time integrability for $\widetilde{u}$.

Remark 1.12. The reader should notice that there exist parameters $1 \leq q_1, q_2, q_3 \leq \infty$ and $0 < s_1, s_3$ satisfying (1.20). Indeed for every $q_3 < 3/2$, we have

$$\frac{1}{q_3} > \frac{1}{4q_3} + \frac{1}{2}.$$

Hence there exists q_1 with

$$\frac{1}{q_3} > \frac{1}{q_1} > \frac{1}{4q_3} + \frac{1}{2}.$$

Thus $q_3 < q_1$ and for $q_2 > 2$ sufficiently large, the estimate

$$\frac{2}{q_1} - 1 > \frac{1}{2\left(1 - \frac{1}{q_2}\right)} \left(\frac{1}{q_3} - \frac{1}{q_2} \right)$$

holds since the right-hand side converges to $\frac{1}{2q_3}$ for $q_2 \to \infty$. This allows to choose s_1 and s_3 such that

$$\frac{2}{q_1} - 1 > s_1 > s_3 > \frac{1}{2\left(1 - \frac{1}{q_2}\right)} \left(\frac{1}{q_3} - \frac{1}{q_2} \right),$$

so all constraints in (1.20) are satisfied.

Remark 1.13. We would like to emphasise that Theorem 1.10 holds for any choice of viscosities $\nu_h^*, \nu_v^* \in \mathbb{R}$, in particular even for the inviscid case $\nu_h^* = \nu_v^* = 0$.

Remark 1.14. Finally, we emphasise that the solutions constructed in Theorem 1.10 are not of Leray-Hopf type, as they do not have a finite rate of mean energy dissipation (i.e. the horizontal velocity field does not belong to the space $L^2((0,T);H^1(\mathbb{T}^3))$).

We now obtain the global existence of weak solutions as a corollary.

Corollary 1.15. *For $\nu_h^*, \nu_v^* > 0$ and any initial data $u_0 \in H^1(\mathbb{T}^3)$ there exists infinitely many global-in-time weak solutions (u,w,p) of the viscous primitive equations* (1.17)-(1.19) *(in the sense of Definition 1.9) which satisfy the regularity properties of Theorem 1.10.*

Proof. The proof works exactly as the proof of Corollary 1.8 where the corresponding local (even global) well-posedness result can be achieved by using the methods from [23]. □

Remark 1.16. The proof of nonuniqueness of global weak solutions works equally well in the three cases of full, horizontal or vertical viscosity, which were studied in the works [19–21]. Moreover, in the case of full viscosity the result can also be adapted to classes of initial data belonging to different function spaces, by relying on the well-posedness results from [41].

1.3.6 Results for the 2D hydrostatic Euler equations

It is also possible to develop a convex integration scheme for the two-dimensional hydrostatic Euler equations. They are given by

$$\partial_t u + u\partial_{x_1} u + w\partial_z u + \partial_{x_1} p = 0, \tag{1.21}$$

$$\partial_z p = 0, \tag{1.22}$$

$$\partial_{x_1} u + \partial_z w = 0, \tag{1.23}$$

where $u : \mathbb{T}^2 \times (0,T) \to \mathbb{R}$ is the horizontal velocity, $w : \mathbb{T}^2 \times (0,T) \to \mathbb{R}$ is the vertical velocity and $p : \mathbb{T}^2 \times (0,T) \to \mathbb{R}$ is the pressure. We first state the definition of weak solution to these equations.

Definition 1.17. A triple (u,w,p) is called a weak solution of the two-dimensional hydrostatic Euler equations (1.21)-(1.23) if $u \in L^2(\mathbb{T}^2 \times (0,T))$, $w \in \mathcal{D}'(\mathbb{T}^2 \times (0,T))$ and $p \in L^1(\mathbb{T}^2 \times (0,T))$ such that $uw \in L^1((0,T); B^{-s}_{1,\infty}(\mathbb{T}^2))$ (where $s > 0$ is the regularity parameter) and the equations are satisfied in the sense of distributions, i.e.,

$$\begin{aligned}
\int_0^T \int_{\mathbb{T}^2} u\partial_t \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^2} u^2 \partial_{x_1}\phi_1 \,\mathrm{d}x\,\mathrm{d}t + \\
+ \int_0^T \langle uw, \partial_z \phi_1 \rangle_{B^{-s}_{1,\infty} \times B^{s}_{\infty,1}} \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^2} p\partial_{x_1}\phi_1 \,\mathrm{d}x\,\mathrm{d}t &= 0, \\
\int_0^T \int_{\mathbb{T}^2} p\partial_z \phi_2 \,\mathrm{d}x\,\mathrm{d}t &= 0, \\
\int_0^T \langle \mathbf{u}, \nabla \phi_3 \rangle \,\mathrm{d}t &= 0,
\end{aligned}$$

for all test functions ϕ_1, ϕ_2 and ϕ_3 in $\mathcal{D}(\mathbb{T}^2 \times (0,T))$.

In particular, we will prove the following theorem.

Theorem 1.18. *Let $T > 0$ and suppose there exist smooth solutions of the two-dimensional hydrostatic Euler equations* (1.21)-(1.23) (u_1, w_1, p_1) *on* $[0, T/2]$ *and* (u_2, w_2, p_2) *on* $[T/2, T]$. *Moreover, let* $1 \leq q_2, q_3 \leq \infty$ *and* $0 < s_3$ *be parameters satisfying*[2]

$$\frac{3}{2} q_2 \left(\frac{1}{q_3} - \frac{1}{q_2} \right) > s_3 > \frac{1}{1 - \frac{2}{q_2}} \left(\frac{1}{q_3} - \frac{1}{q_2} \right) > 0, \quad 1 \geq s_3. \tag{1.24}$$

Then there exists a weak solution (u, w, p) *in the sense of Definition 1.17 with regularity parameter* $s = 1$ *and with the following properties:*

1. *The solution satisfies that*

$$(u, w, p)(\cdot, t) = \begin{cases} (u_1, w_1, p_1)(\cdot, t) & \text{if } t \in [0, T/4), \\ (u_2, w_2, p_2)(\cdot, t) & \text{if } t \in (3T/4, T]. \end{cases}$$

2. *We have that*

$$\begin{aligned} u &\in L^{q_2 -}((0, T); L^2(\mathbb{T}^2)) \cap L^{q_3 -}((0, T); H^{s_3}(\mathbb{T}^2)), \\ w &\in L^{q_2' -}((0, T); L^2(\mathbb{T}^2)) \cap L^{q_3' -}((0, T); H^{-s_3}(\mathbb{T}^2)). \end{aligned}$$

Remark 1.19. It might seem slightly odd to label the parameters by q_2, q_3 and s_3 (rather than q_1 etc.). The reason we chose to do so is because it will allow for easy comparisons with the three-dimensional scheme from Theorem 1.3. We emphasise that there are no equivalent parameters to q_1 and s_1 in the two-dimensional version of the scheme.

Remark 1.20. Remark 1.11 is also true in the context of the two-dimensional hydrostatic Euler equations (1.21)-(1.23), see Remark 8.7 below.

Remark 1.21. By proceeding as in section 9 we can achieve in addition that $\overline{u}, \widetilde{u} \in L^1((0, T); W^{1,1}(\mathbb{T}^3))$. In contrast to the three-dimensional case (cf. Remark 1.5) in two dimensions there is no need to require stronger constraints for the parameters, see also Theorem 1.23 and Remark 1.25.

We observe that it is possible to establish a two-dimensional analogue of Corollary 1.8 using the local well-posedness result from [50, 51] for analytic data in the channel. This yields existence of infinitely many global weak solutions for suitable initial data.

1.3.7 Results for the two-dimensional Prandtl equations

Now we turn to studying the two-dimensional Prandtl equations, which are given by

$$\partial_t u - \nu_v^* \partial_{zz} u + u \partial_{x_1} u + w \partial_z u + \partial_{x_1} p = 0, \tag{1.25}$$

$$\partial_z p = 0, \tag{1.26}$$

$$\partial_{x_1} u + \partial_z w = 0, \tag{1.27}$$

[2] Note that (1.24) implies $q_3 < q_2$ and $q_2 > 2$.

where $u:\mathbb{T}^2\times(0,T)\to\mathbb{R}$ is the horizontal velocity, $w:\mathbb{T}^2\times(0,T)\to\mathbb{R}$ is the vertical velocity and $p:\mathbb{T}^2\times(0,T)\to\mathbb{R}$ the pressure.

We observe that these equations differ from the two-dimensional hydrostatic Euler equations (1.21)-(1.23) by the vertical viscosity term $\nu_v^*\partial_{zz}u$. We introduce the following notion of weak solution to the Prandtl equations (1.25)-(1.27).

Definition 1.22. A triple (u,w,p) is called a weak solution of the two-dimensional Prandtl equations (1.25)-(1.27) if $u\in L^2(\mathbb{T}^2\times(0,T))\cap L^1((0,T);W^{1,1}(\mathbb{T}^2))$, $w\in\mathcal{D}'(\mathbb{T}^2\times(0,T))$ and $p\in L^1(\mathbb{T}^2\times(0,T))$ such that $uw\in L^1((0,T);B^{-s}_{1,\infty}(\mathbb{T}^2))$ (where $s>0$ is the regularity parameter) and the equations are satisfied in the sense of distributions, i.e.,

$$\begin{aligned}\int_0^T\int_{\mathbb{T}^2} u\partial_t\phi_1\,\mathrm{d}x\,\mathrm{d}t-\nu_v^*\int_0^T\int_{\mathbb{T}^2}\partial_z u\partial_z\phi_1\,\mathrm{d}x\,\mathrm{d}t+\int_0^T\int_{\mathbb{T}^2}u^2\partial_{x_1}\phi_1\,\mathrm{d}x\,\mathrm{d}t+&\\ +\int_0^T\langle uw,\partial_z\phi_1\rangle_{B^{-s}_{1,\infty}\times B^{s}_{\infty,1}}\,\mathrm{d}t+\int_0^T\int_{\mathbb{T}^2}p\partial_{x_1}\phi_1\,\mathrm{d}x\,\mathrm{d}t&=0,\\ \int_0^T\int_{\mathbb{T}^2}p\partial_z\phi_2\,\mathrm{d}x\,\mathrm{d}t&=0,\\ \int_0^T\langle\mathbf{u},\nabla\phi_3\rangle\,\mathrm{d}t&=0,\end{aligned}$$

for all test functions ϕ_1, ϕ_2 and ϕ_3 in $\mathcal{D}(\mathbb{T}^2\times(0,T))$.

We will prove the following result.

Theorem 1.23. *Let $T>0$ and suppose there exist smooth solutions of the two-dimensional Prandtl equations (1.25)-(1.27) (u_1,w_1,p_1) on $[0,T/2]$ and (u_2,w_2,p_2) on $[T/2,T]$. Moreover, let $1\le q_2,q_3\le\infty$ and $0<s_3$ be parameters satisfying*

$$\frac{3}{2}q_2\left(\frac{1}{q_3}-\frac{1}{q_2}\right)>s_3>\frac{1}{1-\frac{2}{q_2}}\left(\frac{1}{q_3}-\frac{1}{q_2}\right)>0,\quad 1\ge s_3. \tag{1.28}$$

Then there exists a weak solution (u,w,p) in the sense of Definition 1.22 with regularity parameter $s=1$ and with the following properties:

1. *The solution satisfies that*

$$(u,w,p)(\cdot,t)=\begin{cases}(u_1,w_1,p_1)(\cdot,t) & \text{if } t\in[0,T/4),\\ (u_2,w_2,p_2)(\cdot,t) & \text{if } t\in(3T/4,T].\end{cases}$$

2. *We have that*

$$\begin{aligned}u&\in L^{q_2-}((0,T);L^2(\mathbb{T}^2))\cap L^{q_3-}((0,T);H^{s_3}(\mathbb{T}^2))\cap L^1((0,T);W^{1,1}(\mathbb{T}^2)),\\ w&\in L^{q_2'-}((0,T);L^2(\mathbb{T}^2))\cap L^{q_3'-}((0,T);H^{-s_3}(\mathbb{T}^2)).\end{aligned}$$

Remark 1.24. Remark 1.11 is also true in the context of the Prandtl equations (1.25)-(1.27).

Remark 1.25. Similar to Remark 1.13, Theorem 1.23 holds for any $\nu_v^* \in \mathbb{R}$, in particular even for the inviscid case $\nu_v^* = 0$.

We note that it is possible to establish an analogue of Corollary 1.8 where one has to use the local well-posedness result from [77, p. 6] (see also [89, p. 7186]) for analytic data in the strip/channel. A straightforward adaption of the proof of Corollary 1.8 yields the existence of infinitely many global weak solutions for suitable initial data.

1.4 Further remarks and outline of the paper

Now that we have presented the results for the four cases of system (1.1)-(1.3) that we consider in this paper, we would like to make some further remarks on these results. Some conclusions that can be drawn are:

1. There exist weak solutions of the inviscid primitive equations (1.4)-(1.6) that do not conserve energy. Compared to the solutions constructed in [29], the solutions that we construct in this paper have Sobolev regularity. Moreover, they are related to the notion of type III weak solutions, as introduced in [7], as the barotropic-vertical part of the nonlinearity is interpreted as a paraproduct.

2. In addition, the scheme is able to construct solutions where the baroclinic and barotropic modes have different regularities. This is expected, as the loss of derivative in the advective term only occurs in the baroclinic equation. The barotropic mode must have higher Sobolev regularity than the baroclinic mode in the scheme as otherwise the paraproduct between the vertical velocity and the barotropic mode will not make sense.

3. As far as we can tell, this is the first proof of nonuniqueness of weak solutions for the viscous primitive equations. It shows that although the system is globally well-posed (as shown in [23]), at low regularity the system has nonunique weak solutions. This is true even if one has sufficiently regular Sobolev data for which global well-posedness holds in the class of strong solutions.

4. To the best of our knowledge, this is the first convex integration scheme for the two-dimensional Prandtl equations (in the three-dimensional case there is the work [65]), as well as the two-dimensional hydrostatic Euler equations.

There are a few new features of the scheme that we wish to highlight:

- We have introduced a splitting of the Reynolds stress tensor into a barotropic and baroclinic part. We add perturbations to separately deal with both these parts of the error. We then ensure that the interactions between the two perturbations are controlled.

- The splitting of the Reynolds stress tensor requires us to construct and use horizontal and vertical inverse divergence operators, as the barotropic part depends only on the horizontal variables, while the baroclinic part is mean-free with respect to the z-variable.

- Having two parts of the perturbation allows us to use different scalings of the temporal intermittency functions for the barotropic and baroclinic parts of the perturbation. This makes it possible to ensure that the different perturbations have different regularities, such that the interactions between the different parts can be controlled. In particular, this is crucial to control the terms $\overline{u}_p \otimes \widetilde{u}_p$ and $w_p \overline{u}_p$ (the barotropic-baroclinic and vertical-baroclinic parts of the nonlinearity).

Now we present an outline of the paper. In sections 2-6 we will develop the convex integration scheme for the 3D inviscid primitive equations, in order to prove Theorem 1.3. In section 2 we state the core inductive proposition of the convex integration scheme and prove Theorem 1.3 using this proposition. In section 3 we discuss several preliminaries. In particular, we introduce the inverse divergence operators, the spatial building blocks for the convex integration, as well as the temporal intermittency functions. In addition, we will discuss the choice of the frequency parameters.

In section 4 we introduce the perturbation that will be used in each iteration of the convex integration scheme, and compute the new Reynolds stress tensor after adding the perturbation. We will prove the estimates on the perturbation required for Proposition 2.4 in section 5. The estimates on the Reynolds stress tensor will be proven in section 6.

In sections 7-9 we will develop convex integration schemes to study the other cases of equations (1.1)-(1.3) that we are interested in this paper. These schemes differ from the scheme presented in sections 3-6 in some aspects, while other parts are similar. Therefore for the sake of brevity, in sections 7-9 we will focus on the parts that differ from the convex integration scheme for the 3D inviscid primitive equations.

In section 7 we provide an extension of the convex integration scheme to the viscous primitive equations with full viscosity and prove Theorem 1.10. The cases with anisotropic viscosities can be studied in a similar manner. In section 8 we investigate the two-dimensional hydrostatic Euler equations and prove Theorem 1.18. Finally, in section 9 we consider the (two-dimensional) Prandtl equations and provide the proof for Theorem 1.23.

In appendix A we give a short introduction to Littlewood-Paley theory, Besov spaces and paradifferential calculus, in order to make the paper self-contained. In appendix B we state the improved Hölder inequality, which was introduced in [68, Lemma 2.1], and we prove an oscillatory paraproduct estimate based on this inequality. Moreover, we provide another proof of Lemma 5.3 as an alternative to the proof given in section 5.1.2. This Lemma states a new inequality needed to control the interaction between the vertical velocity and the baroclinic mode, which turns out to be a critical part of the scheme.

2 The inductive proposition

The following underdetermined system of equations is called the hydrostatic Euler-Reynolds system

$$\partial_t u + u \cdot \nabla_h u + w \partial_z u + \nabla_h p = \nabla_h \cdot R_h + \partial_z R_v, \tag{2.1}$$

$$\partial_z p = 0, \tag{2.2}$$

$$\nabla_h \cdot u + \partial_z w = 0, \tag{2.3}$$

where u, w, p, R_h and R_v are the unknowns. Here the horizontal Reynolds stress tensor[3] $R_h : \mathbb{T}^2 \times [0,T] \to \mathcal{S}^{2\times 2}$ is a function of (x_1, x_2, t), while the vertical Reynolds stress tensor $R_v : \mathbb{T}^3 \times [0,T] \to \mathbb{R}^2$ is a function of (x_1, x_2, z, t), and which is mean-free with respect to z, i.e. $\int_0^1 R_v \, \mathrm{d}z = 0$. We will only work with smooth solutions to this system.

Remark 2.1. Notice that R_h is independent of z. Hence we have $\overline{R_h} = R_h$, see section 1.3.2. Moreover by definition R_v is mean-free with respect to z and thus $\widetilde{R_v} = R_v$.

The following definition is inspired by [27, Definition 2.1].

Definition 2.2. We say that a smooth solution (u, w, p, R_h, R_v) of the hydrostatic Euler-Reynolds system (2.1)-(2.3) is *well-prepared* if there exists a time interval $I \subseteq [0,T]$ and parameter $\tau > 0$ such that $R_h(x,t) = 0$, $R_v(x,t) = 0$ whenever $\operatorname{dist}(t, I^c) \leq \tau$.

Remark 2.3. In the definition of well-preparedness, the trivial case $I = [0,T]$ (i.e. without restrictions on the support of R_h and R_v) has not been excluded. In this case, the perturbations considered in the inductive proposition will be supported on the whole time interval, but the estimates stated in Proposition 2.4 below also hold when $I = [0,T]$. Including the trivial case in Definition 2.2 therefore allows us to phrase Proposition 2.4 in a more general way.

The core of the proof of Theorem 1.3 will revolve around proving the following inductive proposition.

Proposition 2.4. *Suppose* (u, w, p, R_h, R_v) *is a smooth solution of the hydrostatic Euler-Reynolds system* (2.1)-(2.3) *which is well-prepared with associated time interval* I *and parameter* $\tau > 0$. *Moreover consider parameters* $1 \leq q_1, q_2, q_3 \leq \infty$ *and* $0 < s_1, s_3$ *which satisfy the following constraints*[4]

$$q_2 > 2, \quad \frac{2}{q_1} > s_1 + 1, \quad \frac{2}{q_3} > s_3 + \frac{2}{q_2}. \tag{2.4}$$

Finally let $\delta, \epsilon > 0$ *be arbitrary. Then there exists another smooth solution* $(u + \overline{u}_p + \widetilde{u}_p, w + w_p, p + P, R_{h,1}, R_{v,1})$ *of the hydrostatic Euler-Reynolds system* (2.1)-(2.3) *which is well-prepared with respect to the same time interval* I *and parameter* $\tau/2$, *and has the following properties:*

1. $(\overline{u}_p, \widetilde{u}_p, w_p)(x,t) = (0,0,0)$ *whenever* $\operatorname{dist}(t, I^c) \leq \tau/2$.

[3] We denote the set of all symmetric 2×2 matrices by $\mathcal{S}^{2\times 2}$.

[4] Note that the constraints (2.4) are weaker than (1.15). Indeed from (1.15) we deduce

$$s_3 + \frac{2}{q_2} < s_1 + 1 < \frac{2}{q_1} \leq \frac{2}{q_3}.$$

2. *The following estimates are satisfied*[5][6]

$$\|R_{h,1}\|_{L^1(L^1)} \leq \epsilon, \tag{2.5}$$
$$\|R_{v,1}\|_{L^1(B^{-1}_{1,\infty})} \leq \epsilon, \tag{2.6}$$
$$\|\overline{u}_p\|_{L^{q_1}(H^{s_1})} \leq \epsilon, \tag{2.7}$$
$$\|\widetilde{u}_p\|_{L^{q_2-}(L^2)} \leq \epsilon, \tag{2.8}$$
$$\|\widetilde{u}_p\|_{L^{q_3-}(H^{s_3})} \leq \epsilon, \tag{2.9}$$
$$\|w_p\|_{L^{q_2'-}(L^2)} \leq \epsilon, \tag{2.10}$$
$$\|w_p\|_{L^{q_3'-}(H^{-s_3})} \leq \epsilon. \tag{2.11}$$
$$\|w_p\|_{L^{q_2'}(L^2)} \lesssim \|R_h\|_{L^1(L^1)}, \tag{2.12}$$
$$\|w_p\|_{L^{q_3'}(H^{-s_3})} \lesssim \|R_h\|_{L^1(L^1)}. \tag{2.13}$$

3. *Moreover, we have the following bounds*

$$\|\overline{u}_p\|_{L^2(L^2)} \lesssim \|R_h\|^{1/2}_{L^1(L^1)}, \tag{2.14}$$
$$\|w_p\widetilde{u}_p + w\widetilde{u}_p + w_p u\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{2.15}$$

Remark 2.5. Note that when writing $\overline{u}_p$, $\widetilde{u}_p$, we implicitly require $\overline{u}_p$ to be independent of z and $\widetilde{u}_p$ mean-free with respect to z, see section 1.3.1.

Remark 2.6. Another version of Proposition 2.4 where (2.12) and (2.13) are replaced by

$$\|\widetilde{u}_p\|_{L^{q_2}(L^2)} \lesssim \|R_h\|_{L^1(L^1)}, \tag{2.16}$$
$$\|\widetilde{u}_p\|_{L^{q_3}(H^{s_3})} \lesssim \|R_h\|_{L^1(L^1)}, \tag{2.17}$$

is true as well, see Remark 5.6. This way we obtain the endpoint time integrability for $\widetilde{u}$ rather than w in Theorem 1.3, see Remark 1.4.

Next we prove Theorem 1.3 using Proposition 2.4.

Proof of Theorem 1.3. We first take χ_1 and χ_2 to be a C^∞ partition of unity of $[0,T]$ such that $\chi_1 \equiv 1$ on $[0, 3T/8]$ and $\chi_2 \equiv 1$ on $[5T/8, T]$. Then we define (u_0, w_0, p_0) as follows

$$(u_0, w_0, p_0) := \chi_1(u_1, w_1, p_1) + \chi_2(u_2, w_2, p_2). \tag{2.18}$$

For a suitable choice of χ_1 and χ_2, (u_0, w_0, p_0) is no longer a solution of the hydrostatic Euler equations, but with a proper definition[7] of $R_{h,0}, R_{v,0}$ it solves the hydrostatic Euler-Reynolds system (2.1)-(2.3). Moreover $(u_0, w_0, p_0, R_{h,0}, R_{v,0})$ is well-prepared for the time interval $I := [T/4, 3T/4] \subset [0,T]$ and parameter $\tau_0 := T/16$.

[5] As usual we write $X \lesssim Y$ if $X \leq CY$ with a constant C. The implicit constant C in (2.12)-(2.15) does not depend on u, w, p, R_h, R_v or ϵ.

[6] As mentioned earlier, we quantify the '$-$' in $p-$ via $p- := \frac{1}{\frac{1}{p}+\delta}$, where $\delta > 0$ was fixed in the statement of the proposition.

[7] Using the inverse divergence operators from section 3.3, a precise definition of $R_{h,0}, R_{v,0}$ is straightforward.

Taking the sequence $\epsilon_n = 2^{-n}$, $\delta_n = \delta$ with a suitable choice of $\delta > 0$ (see below) and applying Proposition 2.4 inductively, we find a sequence of well-prepared solutions

$$\Big(u_0 + \sum_{k=1}^{n} \big(\overline{u}_k + \widetilde{u}_k\big), w_0 + \sum_{k=1}^{n} w_k, p_0 + \sum_{k=1}^{n} P_k, R_{h,n}, R_{v,n}\Big), \tag{2.19}$$

of the hydrostatic Euler-Reynolds system with a sequence of well-preparedness parameters $\{\tau_n\}$ (and the same time interval I). Note that $\tau_n \to 0$.

Estimates (2.5), (2.7)-(2.14) imply that the sequence $\Big\{\overline{u}_0 + \sum_{k=1}^{n} \overline{u}_k\Big\}$ is a Cauchy sequence in the space $L^2(\mathbb{T}^3 \times (0,T)) \cap L^{q_1}((0,T); H^{s_1}(\mathbb{T}^3))$, the sequence $\Big\{\widetilde{u}_0 + \sum_{k=1}^{n} \widetilde{u}_k\Big\}$ is Cauchy in $L^{q_2-}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3-}((0,T); H^{s_3}(\mathbb{T}^3))$ and the sequence $\Big\{w_0 + \sum_{k=1}^{n} w_k\Big\}$ is Cauchy in $L^{q_2'}((0,T); L^2(\mathbb{T}^3)) \cap L^{q_3'}((0,T); H^{-s_3}(\mathbb{T}^3))$. In particular, by choosing δ appropriately we can identify the limits $\overline{u}, \widetilde{u}$ and w, as $n \to \infty$, lying in the spaces stated in Theorem 1.3.

Now we define the pressure p by

$$p \coloneqq -(\Delta_h)^{-1}\big(\nabla_h \cdot (\nabla_h \cdot (\overline{u \otimes u}))\big), \tag{2.20}$$

where $u = \overline{u} + \widetilde{u}$.

Next, we check that the triple $(\overline{u} + \widetilde{u}, w, p)$ is a weak solution in the sense of Definition 1.1. We first show that $uw \in L^1((0,T); B^{-1}_{1,\infty}(\mathbb{T}^3))$. According to Lemma A.6, $\Big(\overline{u}_0 + \sum_{k=1}^{n} \overline{u}_k\Big)\Big(w_0 + \sum_{k=1}^{n} w_k\Big) \xrightarrow{n\to\infty} \overline{u}w$ in $L^1((0,T); B^{-1}_{1,\infty}(\mathbb{T}^3))$. Here we have also used that $1 \geq s_1 > s_3$ (which follows from (1.15)) as well as Lemma A.3, and the fact that $\frac{1}{q_1} + \frac{1}{q_3'} \leq 1$ (which follows from $q_3 \leq q_1$, see (1.15)) in order to obtain L^1 integrability in time. In addition, we have that $\Big(\widetilde{u}_0 + \sum_{k=1}^{n} \widetilde{u}_k\Big)\Big(w_0 + \sum_{k=1}^{n} w_k\Big) \xrightarrow{n\to\infty} \widetilde{u}w$ in $L^1((0,T); B^{-1}_{1,\infty}(\mathbb{T}^3))$ by estimates (2.6) and (2.15).

Furthermore, we observe that (1.13) immediately follows from the definition of p. Moreover since for any $n \in \mathbb{N}$ the quintuple (2.19) satisfies (2.3), we find that (u, w, p) complies with (1.14). In order to show (1.12) we define the abbreviations

$$u_n \coloneqq u_0 + \sum_{k=1}^{n} \big(\overline{u}_k + \widetilde{u}_k\big),$$
$$w_n \coloneqq w_0 + \sum_{k=1}^{n} w_k,$$
$$p_n \coloneqq p_0 + \sum_{k=1}^{n} P_k.$$

Since (2.19) satisfies (2.1), we observe that

$$\int_0^T \int_{\mathbb{T}^3} u_n \cdot \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \big(u_n \otimes u_n\big) : \nabla_h \varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$+\int_0^T \langle u_n w_n, \partial_z \varphi\rangle_{B^{-1}_{1,\infty}\times B^1_{\infty,1}} \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} p_n \nabla_h \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$= \int_0^T \int_{\mathbb{T}^3} R_{h,n} : \nabla_h \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle R_{v,n}, \partial_z \varphi\rangle_{B^{-1}_{1,\infty}\times B^1_{\infty,1}} \,\mathrm{d}t, \tag{2.21}$$

for any $n \in \mathbb{N}$ and any test function $\varphi \in \mathcal{D}(\mathbb{T}^3 \times (0,T))$. Note that (2.5) and (2.6) imply $R_{h,n} \xrightarrow{n\to\infty} 0$ in $L^1((0,T);L^1(\mathbb{T}^3))$ and $R_{v,n} \xrightarrow{n\to\infty} 0$ in $L^1((0,T);B^{-1}_{1,\infty}(\mathbb{T}^3))$, respectively. Hence by taking the limit we deduce from (2.21) that

$$\int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \big(u \otimes u\big) : \nabla_h \varphi \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \langle uw, \partial_z \varphi\rangle_{B^{-1}_{1,\infty}\times B^1_{\infty,1}} \,\mathrm{d}t = 0, \tag{2.22}$$

for any test function $\varphi \in \mathcal{D}(\mathbb{T}^3 \times (0,T))$ which is either mean-free with respect to z, or independent of z with $\nabla_h \cdot \varphi = 0$. Here we have used that for any $n \in \mathbb{N}$ and φ mean-free with respect to z,

$$\int_0^T \int_{\mathbb{T}^3} p_n \nabla_h \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{T}^2} \left[p_n \nabla_h \cdot \left(\int_{\mathbb{T}} \varphi \,\mathrm{d}z \right)\right] \mathrm{d}x_1\,\mathrm{d}x_2\,\mathrm{d}t = 0,$$

according to (2.2), and, furthermore, that for any $n \in \mathbb{N}$ and φ independent of z with $\nabla_h \cdot \varphi = 0$

$$\int_0^T \int_{\mathbb{T}^3} p_n \nabla_h \cdot \varphi \,\mathrm{d}x\,\mathrm{d}t = 0.$$

Now we are ready to prove (1.12). We may split the test function $\phi_1 = \overline{\phi}_1 + \widetilde{\phi}_1$ into the barotropic and baroclinic parts, and use the Helmholtz decomposition to find test functions φ, ψ, which are independent of z, and such that $\overline{\phi}_1 = \varphi + \nabla_h \psi$ and $\nabla_h \cdot \varphi = 0$. Then by (2.22) we have

$$\begin{aligned}
&\int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \phi_1 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} \big(u \otimes u\big) : \nabla_h \phi_1 \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \langle uw, \partial_z \phi_1\rangle_{B^{-s}_{1,\infty}\times B^s_{\infty,1}} \,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} p \nabla_h \cdot \phi_1 \,\mathrm{d}x\,\mathrm{d}t \\
&= \int_0^T \int_{\mathbb{T}^3} p \nabla_h \cdot \widetilde{\phi}_1 \,\mathrm{d}x\,\mathrm{d}t + \int_0^T \int_{\mathbb{T}^3} u \cdot \partial_t \nabla_h \psi \,\mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \int_{\mathbb{T}^3} \Big((u \otimes u) : \nabla_h (\nabla_h \psi) + p \Delta_h \psi \Big) \,\mathrm{d}x\,\mathrm{d}t \\
&= 0,
\end{aligned}$$

where the latter equality follows from the fact that p is independent of z, (1.14) and the definition of p in equation (2.20).

Finally, we observe that equation (1.16) follows from Proposition 2.4 because the time interval I of well-preparedness stays the same for the sequence (2.19). In particular, all the perturbations have support in the time interval I. Therefore since (u_0, w_0, p_0) agrees with (u_1, w_1, p_1) on $[0, T/4)$ and with (u_2, w_2, p_2) on $(3T/4, T]$, the constructed solution (u, w, p) will have the same properties, as no perturbations with support in $[0, T/4) \cup (3T/4, T]$ have been added. □

After recalling some preliminaries in section 3, we will prove Proposition 2.4 in sections 4-6.

3 Preliminaries

3.1 Outline

In this paper we are going to use Mikado flows as building blocks. These have been introduced in [31] and are built upon a geometric lemma which goes back to [71], see also [87, Lemma 3.3]. Later on *concentrated* Mikado flows have been introduced in [68] in order to construct solutions with Sobolev regularity. In this paper the term *Mikado flows* will always refer to such Mikado flows with concentration.

In the proof of Proposition 2.4 we will handle the two error terms R_h and R_v separately. To treat R_v we use two-dimensional Mikado flows and Mikado densities in two directions, whereas for R_h we use two-dimensional Mikado flows in several directions which are given by the above mentioned geometric lemma. We use the version of the Mikado flows which was introduced in [27]. We recall these flows in section 3.4. We call the perturbation which reduces the error R_v *vertical* and the one which reduces R_h *horizontal*.

The above mentioned concentration is represented by the spatial concentration parameters μ_h and μ_v which are used in the horizontal and vertical perturbation, respectively. Moreover, the perturbations will be highly oscillating flows, and this oscillation is represented by the spatial oscillation parameters σ_h and σ_v, which are again used in the horizontal and vertical perturbation, respectively.

Finally we will use *intermittent* flows. To this end we introduce temporal intermittency functions in section 3.5. These are time-dependent functions which contain the temporal concentration parameters κ_h, κ_v and temporal oscillation parameters ν_h, ν_v.

3.2 The parameters

As mentioned in section 3.1 we have two sets of four parameters, namely $\{\mu_h, \sigma_h, \kappa_h, \nu_h\}$ and $\{\mu_v, \sigma_v, \kappa_v, \nu_v\}$, so eight parameters in total. In addition to that, we work with two "master parameters" λ_h and λ_v, which the other parameters depend on via

$$\begin{aligned} \nu_i &= \lambda_i^{a_i}, \quad \sigma_i = \lambda_i^{b_i}, \\ \kappa_i &= \lambda_i^{c_i}, \quad \mu_i = \lambda_i \end{aligned} \tag{3.1}$$

for $i = h, v$ and fixed exponents $a_i, b_i, c_i > 0$. These exponents are determined in the following Lemma. We will later fix λ_h, λ_v. These parameters will be very large and such that $\sigma_h, \sigma_v \in \mathbb{N}$, as well as $\kappa_h, \kappa_v > 1$.

Lemma 3.1. *Let $1 \leq q_1, q_2, q_3 \leq \infty$ and $0 < s_1, s_3$ satisfy the following conditions*[8]

$$\frac{2}{q_1} > s_1 + 1, \quad \frac{2}{q_3} > s_3 + \frac{2}{q_2}. \tag{3.2}$$

[8] Obviously conditions (3.2) are weaker than constraints (2.4), which again are weaker than (1.15).

Then we can choose $a_i, b_i, c_i > 0$ for $i = h, v$ in (3.1) with the property that there exist $\gamma_h, \gamma_v > 0$ such that

$$\kappa_h^{1/2-1/q_1}(\sigma_h\mu_h)^{s_1} \leq \lambda_h^{-\gamma_h}, \tag{3.3}$$

$$\sigma_h^{-1}\nu_h\kappa_h^{1/2}\mu_h^{-1} \leq \lambda_h^{-\gamma_h}, \tag{3.4}$$

$$\kappa_v^{1/q_2-1/q_3}(\sigma_v\mu_v)^{s_3} = 1, \tag{3.5}$$

$$\sigma_v^{-1}\nu_v\kappa_v^{1/2}\mu_v^{-1} \leq \lambda_v^{-\gamma_v}, \tag{3.6}$$

$$\kappa_v^{-\delta} \leq \lambda_v^{-\gamma_v}, \tag{3.7}$$

and in addition $\mu_i, \sigma_i, \kappa_i, \nu_i \geq \lambda_i^{\gamma_i}$ for $i = h, v$.

Proof. We choose $0 < a_h, a_v < 1$ and set

$$\begin{aligned}
b_h &:= \frac{2s_1}{\frac{2}{q_1} - s_1 - 1},\\
b_v &:= \frac{s_3}{\frac{2}{q_3} - s_3 - \frac{2}{q_2}},\\
c_h &:= 2b_h,\\
c_v &:= 2b_v.
\end{aligned}$$

Notice that (3.2) ensures that $b_h, b_v > 0$. Consequently $c_h, c_v > 0$.

By taking logarithms, inequalities (3.3)-(3.7) are equivalent to

$$-\left(\frac{1}{2} - \frac{1}{q_1}\right)c_h - s_1(b_h + 1) \geq \gamma_h, \tag{3.8}$$

$$b_h - a_h - \frac{1}{2}c_h + 1 \geq \gamma_h, \tag{3.9}$$

$$-\left(\frac{1}{q_2} - \frac{1}{q_3}\right)c_v - s_3(b_v + 1) = 0, \tag{3.10}$$

$$b_v - a_v - \frac{1}{2}c_v + 1 \geq \gamma_v, \tag{3.11}$$

$$\delta c_v \geq \gamma_v. \tag{3.12}$$

Using the definition of b_v and c_v we immediately conclude that (3.10) is valid.

The required additional estimates $\mu_i, \sigma_i, \kappa_i, \nu_i \geq \lambda_i^{\gamma_i}$ for $i = h, v$ translate into the bounds $1, a_i, b_i, c_i \geq \gamma_i$. Since these upper bounds for γ_i are positive, it remains to show that the upper bounds given by the left-hand sides of (3.8), (3.9), (3.11) and (3.12) are positive as well.

It is obvious that $\delta c_v > 0$. Furthermore from our choice of a_i, b_i, c_i we obtain

$$b_i - a_i - \frac{1}{2}c_i + 1 = 1 - a_i > 0,$$

and

$$-\left(\frac{1}{2} - \frac{1}{q_1}\right)c_h - s_1(b_h + 1) = b_h\left(\frac{2}{q_1} - s_1 - 1\right) - s_1 = s_1 > 0.$$

□

Remark 3.2. When proving Proposition 2.4, inequality (3.3) ensures that $\|\overline{u}_p\|_{L^{q_1}(H^{s_1})}$ can be made small (see section 5.1.1), while inequality (3.5) guarantees that both $\|\widetilde{u}_p\|_{L^{q_3-}(H^{s_3})}$ and $\|w_p\|_{L^{q_3'-}(H^{-s_3})}$ can be made small (see section 5.1.2). Moreover (3.7) will be used at several points during the proof. Finally, inequalities (3.4) and (3.6) make sure that the temporal parts of the linear error are controlled, see section 6.3.1.

3.3 Inverse divergence operators

Like in most of the convex integration schemes in the context of fluid dynamics in the literature, we will need inverse divergence operators in order to define the new Reynolds stress tensors $R_{h,1}$ and $R_{v,1}$. In this context the first inverse divergence operator goes back to [33]. In this paper we will work with three inverse divergence operators. The horizontal inverse divergence $\mathcal{R}_h$ and its bilinear version $\mathcal{B}$ will be used to define the new horizontal Reynolds stress tensor $R_{h,1}$. Those operators are treated in sections 3.3.1 and 3.3.2, respectively. In order to determine the new vertical Reynolds stress tensor $R_{v,1}$ we need a "vertical inverse divergence" which is just an integral in z. It is introduced in section 3.3.3.

3.3.1 Horizontal inverse divergence

Our horizontal inverse divergence coincides with the two-dimensional inverse divergence from [27]. It is based upon the inverse divergence introduced in [33] and is defined as follows.

Definition 3.3. We define the map[9] $\mathcal{R}_h : C^\infty(\mathbb{T}^2;\mathbb{R}^2) \to C^\infty(\mathbb{T}^2;\mathcal{S}_0^{2\times 2})$ by[10]

$$(\mathcal{R}_h v)_{ij} \coloneqq \mathcal{R}_{ijk,h} v_k, \tag{3.13}$$

where

$$\mathcal{R}_{ijk,h} \coloneqq -\Delta_h^{-1}\partial_k\delta_{ij} + \Delta_h^{-1}\partial_i\delta_{jk} + \Delta_h^{-1}\partial_j\delta_{ik} \tag{3.14}$$

for $i,j,k \in \{1,2\}$.

The following Lemma, which can also be found in [27, Appendix B], summarizes some properties of the map $\mathcal{R}_h$.

Lemma 3.4. *1. The following identities hold*

$$\nabla_h \cdot (\mathcal{R}_h v) = v - \int_{\mathbb{T}^2} v \,\mathrm{d}x, \qquad \textit{for all } v \in C^\infty(\mathbb{T}^2;\mathbb{R}^2), \tag{3.15}$$

$$\mathcal{R}_h \Delta_h v = \nabla_h v + \nabla_h v^T, \qquad \textit{for all divergence-free } v \in C^\infty(\mathbb{T}^2;\mathbb{R}^2). \tag{3.16}$$

2. For $1 \le p \le \infty$, the operator $\mathcal{R}_h$ is bounded, i.e., for all $f \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ we have that

$$\|\mathcal{R}_h f\|_{L^p} \lesssim \|f\|_{L^p}. \tag{3.17}$$

[9] We denote the set of all symmetric 2×2 matrices with zero trace by $\mathcal{S}_0^{2\times 2}$.

[10] We are using the Einstein summation convention here, in particular we sum over $k = 1, 2$. Moreover, we recall that δ_{ij} is the Kronecker delta and in the definition of the inverse horizontal Laplacian Δ_h^{-1} we assume the spatial horizontal average to be zero (in order to ensure uniqueness).

If f is mean-free, i.e. $\fint_{\mathbb{T}^2} f \,\mathrm{d}x = 0$, *then*

$$\|\mathcal{R}_h f(\sigma\cdot)\|_{L^p} \lesssim \sigma^{-1}\|f\|_{L^p}, \qquad \text{for any } \sigma \in \mathbb{N}. \tag{3.18}$$

3. *The operator* $\mathcal{R}_h\nabla_h : C^\infty(\mathbb{T}^2;\mathbb{R}^{2\times 2}) \to C^\infty(\mathbb{T}^2;\mathcal{S}_0^{2\times 2})$ *is a Calderón-Zygmund operator, in particular for any* $1 < p < \infty$ *and all* $A \in C^\infty(\mathbb{T}^2;\mathbb{R}^{2\times 2})$ *we have*

$$\|\mathcal{R}_h\nabla_h \cdot A\|_{L^p} \lesssim \|A\|_{L^p}. \tag{3.19}$$

For the proof we refer to [27, Appendix B].

3.3.2 Horizontal bilinear inverse divergence

Next we recall the bilinear inverse divergence operator from [27]. For our purposes we call it *horizontal* bilinear inverse divergence.

Definition 3.5. We define $\mathcal{B} : C^\infty(\mathbb{T}^2;\mathbb{R}^2) \times C^\infty(\mathbb{T}^2;\mathbb{R}^{2\times 2}) \to C^\infty(\mathbb{T}^2;\mathcal{S}_0^{2\times 2})$ by

$$(\mathcal{B}(b,A))_{ij} = b_l\mathcal{R}_{ijk,h}A_{lk} - \mathcal{R}_h(\partial_i b_l \mathcal{R}_{ijk,h}A_{lk}), \tag{3.20}$$

or written without components (where we have abused notation)

$$\mathcal{B}(b,A) = b\mathcal{R}_h A - \mathcal{R}_h(\nabla_h b \mathcal{R}_h A). \tag{3.21}$$

We will also use the following Lemma from [27].

Lemma 3.6. *For* $1 \le p \le \infty$, $b \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ *and* $A \in C^\infty(\mathbb{T}^2;\mathbb{R}^{2\times 2})$ *with* $\fint_{\mathbb{T}^2} A \,\mathrm{d}x = 0$, *it holds that*

$$\nabla_h \cdot (\mathcal{B}(b,A)) = bA - \fint_{\mathbb{T}^2} bA \,\mathrm{d}x. \tag{3.22}$$

Moreover, we have the following estimate

$$\|\mathcal{B}(b,A)\|_{L^p} \lesssim \|b\|_{C^1}\|\mathcal{R}_h A\|_{L^p}. \tag{3.23}$$

The proof of Lemma 3.6 can be found in [27, Appendix B].

3.3.3 Vertical inverse divergence

Finally we introduce the vertical inverse divergence as follows.

Definition 3.7. We define the map[11] $\mathcal{R}_v : C^\infty_{0,z}(\mathbb{T}^3;\mathbb{R}^2) \to C^\infty(\mathbb{T}^3;\mathbb{R}^2)$ by

$$(\mathcal{R}_v v)(x_1,x_2,z) := \int_0^z v(x_1,x_2,z')\,\mathrm{d}z' - \int_0^1\int_0^{z'} v(x_1,x_2,z'')\,\mathrm{d}z''\,\mathrm{d}z'. \tag{3.24}$$

[11] We denote the space of all functions in $C^\infty(\mathbb{T}^3;\mathbb{R}^2)$ which have zero-mean with respect to z by $C^\infty_{0,z}(\mathbb{T}^3;\mathbb{R}^2)$.

The vertical inverse divergence operator has the following properties.

Lemma 3.8. *1. The following identities hold for any $v \in C^\infty_{0,z}(\mathbb{T}^3;\mathbb{R}^2)$*

$$\int_{\mathbb{T}} \mathcal{R}_v v \,\mathrm{d}z = 0, \tag{3.25}$$

$$\partial_z \mathcal{R}_v v = v, \tag{3.26}$$

$$\mathcal{R}_v(\partial_{zz} v) = \partial_z v. \tag{3.27}$$

2. For $1 \le p, q \le \infty$ and $s \in \mathbb{R}$, the operator $\mathcal{R}_v$ satisfies the following estimates

$$\|\mathcal{R}_v f\|_{L^p} \lesssim \|f\|_{L^p}, \tag{3.28}$$

$$\|\mathcal{R}_v f\|_{B^s_{p,q}} \lesssim \|f\|_{B^s_{p,q}}. \tag{3.29}$$

Moreover,

$$\|\mathcal{R}_v f(\sigma\cdot)\|_{L^p} \lesssim \sigma^{-1}\|f\|_{L^p} \qquad \text{for any } \sigma \in \mathbb{N}. \tag{3.30}$$

3. For any $1 \le p \le \infty$ and all $v \in C^\infty(\mathbb{T}^3;\mathbb{R}^2)$ we have

$$\|\mathcal{R}_v \partial_z v\|_{L^p} \lesssim \|v\|_{L^p}. \tag{3.31}$$

Proof. The identities (3.25), (3.26) are just a simple consequence of the definition of $\mathcal{R}_v$. We also observe that

$$\begin{aligned}\mathcal{R}_v(\partial_{zz} v)(x_1,x_2,z) &= \int_0^z \partial_{zz} v(x_1,x_2,z')\,\mathrm{d}z' - \int_0^1\int_0^{z'} \partial_{zz} v(x_1,x_2,z'')\,\mathrm{d}z''\,\mathrm{d}z' \\ &= \partial_z v(x_1,x_2,z) - \int_0^1 \partial_z v(x_1,x_2,z')\,\mathrm{d}z' = \partial_z v(x_1,x_2,z),\end{aligned}$$

i.e. (3.27).

Estimate (3.28) is established simply by moving the L^p norm inside the integral. In order to prove estimate (3.29), we first observe that

$$\mathcal{R}_v \Delta_j f = \Delta_j \mathcal{R}_v f, \tag{3.32}$$

which can be verified by a direct computation. Alternatively, thanks to equation (3.25) we have $(\widehat{\mathcal{R}_v f})_k = \frac{\widehat{f}_k}{2\pi i k_3}$ (where $k_3 \neq 0$) and then equation (3.32) follows by using the definition of the Littlewood-Paley blocks. From (3.32) and (3.28) we obtain

$$\|\Delta_j \mathcal{R}_v f\|_{L^p} = \|\mathcal{R}_v \Delta_j f\|_{L^p} \lesssim \|\Delta_j f\|_{L^p},$$

which implies (3.29).

To prove (3.30) we set $f_\sigma(x_1,x_2,z) := f(x_1,x_2,\sigma z)$ and compute

$$\mathcal{R}_v f_\sigma(x_1,x_2,z) = \int_0^z f_\sigma(x_1,x_2,z')\,\mathrm{d}z' - \int_0^1\int_0^{z'} f_\sigma(x_1,x_2,z'')\,\mathrm{d}z''\,\mathrm{d}z'$$

$$
\begin{aligned}
&= \sigma^{-1} \int_0^{\sigma z} f(x_1, x_2, z')\, \mathrm{d}z' - \sigma^{-1} \int_0^1 \int_0^{\sigma z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z' \\
&= \sigma^{-1} \int_0^{\sigma z} f(x_1, x_2, z')\, \mathrm{d}z' - \sigma^{-2} \int_0^{\sigma} \int_0^{z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z' \\
&= \sigma^{-1} \int_0^{\sigma z} f(x_1, x_2, z')\, \mathrm{d}z' - \sigma^{-1} \int_0^1 \int_0^{z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z'.
\end{aligned}
$$

Hence

$$
\begin{aligned}
&\|\mathcal{R}_v f_\sigma(x_1, x_2, \cdot)\|_{L^p(\mathbb{T})} \\
&\leq \sigma^{-1} \bigg(\int_{\mathbb{T}} \bigg| \int_0^{\sigma z} f(x_1, x_2, z')\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} + \sigma^{-1} \bigg(\int_{\mathbb{T}} \bigg| \int_0^1 \int_0^{z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} \\
&\leq \sigma^{-1} \bigg(\sigma^{-1} \int_{\sigma\mathbb{T}} \bigg| \int_0^{z} f(x_1, x_2, z')\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} + \sigma^{-1} \bigg(\int_{\mathbb{T}} \bigg| \int_0^1 \int_0^{z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} \\
&\leq \sigma^{-1} \bigg(\int_{\mathbb{T}} \bigg| \int_0^{z} f(x_1, x_2, z')\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} + \sigma^{-1} \bigg(\int_{\mathbb{T}} \bigg| \int_0^1 \int_0^{z'} f(x_1, x_2, z'')\, \mathrm{d}z''\, \mathrm{d}z' \bigg|^p \mathrm{d}z \bigg)^{1/p} \\
&\lesssim \sigma^{-1} \|f(x_1, x_2, \cdot)\|_{L^p(\mathbb{T})}.
\end{aligned}
$$

This implies

$$
\begin{aligned}
\|\mathcal{R}_v f(\sigma \cdot)\|_{L^p(\mathbb{T}^3)} &= \bigg(\int_{\mathbb{T}^2} \|\mathcal{R}_v f(\sigma x_1, \sigma x_2, \sigma \cdot)\|_{L^p(\mathbb{T})}^p \, \mathrm{d}x_1\, \mathrm{d}x_2 \bigg)^{1/p} \\
&\lesssim \sigma^{-1} \bigg(\int_{\mathbb{T}^2} \|f(\sigma x_1, \sigma x_2, \cdot)\|_{L^p(\mathbb{T})}^p \, \mathrm{d}x_1\, \mathrm{d}x_2 \bigg)^{1/p} \\
&= \sigma^{-1} \|f\|_{L^p(\mathbb{T}^3)},
\end{aligned}
$$

i.e. (3.30). The case $p = \infty$ follows in a similar fashion.

Finally observe that

$$
\begin{aligned}
\mathcal{R}_v \partial_z v &= \int_0^z \partial_z v\, \mathrm{d}z' - \int_0^1 \int_0^{z'} \partial_z v\, \mathrm{d}z''\, \mathrm{d}z' \\
&= v - \int_0^1 v\, \mathrm{d}z',
\end{aligned}
$$

which immediately yields (3.31). □

3.4 Building blocks for the perturbation

Next we recall the building blocks. We begin with the Mikado flows and Mikado densities which we use to handle R_v. We state their existence together with their most important properties in the following proposition. The construction of the Mikado flows and densities is nowadays standard and goes back to [31]. For the proof of the following proposition we refer to [27, Section 4.1].

Proposition 3.9. *For each $k \in \{1,2\}$ there exist functions $W_k \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ and $\phi_k \in C^\infty(\mathbb{T}^2;\mathbb{R})$ (referred to as the Mikado flows and Mikado densities respectively) depending on a parameter μ_v, with the following properties:*

1. *The functions W_k, ϕ_k have zero mean for all $k \in \{1,2\}$. Moreover*
$$\int_{\mathbb{T}^2} W_k \phi_k \,\mathrm{d}x = \mathbf{e}_k \quad \text{for all } k \in \{1,2\}, \tag{3.33}$$
where $\mathbf{e}_k$ denotes the k-th standard basis vector in $\mathbb{R}^2$, and by construction $W_k = \phi_k \mathbf{e}_k$.

2. *For any $k \in \{1,2\}$ there exists*[12] *$\Omega_k \in C^\infty(\mathbb{T}^2;\mathcal{A}^{2\times 2})$ with zero mean such that $W_k = \nabla_h \cdot \Omega_k$. In particular, $\nabla_h \cdot W_k = 0$. Moreover $\nabla_h \cdot (W_k \phi_k) = W_k \cdot \nabla_h \phi_k = 0$.*

3. *For all $s \geq 0$, $1 \leq p \leq \infty$ and $k, k' \in \{1,2\}$ with $k \neq k'$ the following estimates hold:*
$$\|\phi_k\|_{W^{s,p}(\mathbb{T}^2)} \lesssim \mu_v^{\frac{1}{2}-\frac{1}{p}+s}; \tag{3.34}$$
$$\|W_k\|_{W^{s,p}(\mathbb{T}^2)} \lesssim \mu_v^{\frac{1}{2}-\frac{1}{p}+s}; \tag{3.35}$$
$$\|\Omega_k\|_{W^{s,p}(\mathbb{T}^2)} \lesssim \mu_v^{-\frac{1}{2}-\frac{1}{p}+s}; \tag{3.36}$$
$$\|W_k \otimes W_{k'}\|_{L^p(\mathbb{T}^2)} \lesssim \mu_v^{1-\frac{2}{p}}. \tag{3.37}$$
Here the implicit constant may depend on s, p, but it does not depend on μ_v.

Let us now recall the Mikado flows which we will use to treat R_h. In the following proposition $B_{1/2}(\mathbb{I})$ denotes the closed ball in $\mathcal{S}^{2\times 2}$ around the identity matrix $\mathbb{I}$ with radius $1/2$. For the proof we refer to [27, Lemma 4.2, Theorem 4.3].

Proposition 3.10. *There exists $N \in \mathbb{N}$, $N \geq 3$ and for each $k \in \Lambda := \{3, ..., N\}$ there exists a flow $W_k \in C^\infty(\mathbb{T}^2;\mathbb{R}^2)$ (called Mikado flows) depending on a parameter μ_h and a function $\Gamma_k \in C^\infty(B_{1/2}(\mathbb{I});\mathbb{R})$, with the following properties:*

1. *The flows W_k have zero mean, i.e. $\int_{\mathbb{T}^2} W_k \,\mathrm{d}x = 0$, for all $k \in \Lambda$. Moreover*
$$\sum_{k\in\Lambda} \Gamma_k^2(R) \int_{\mathbb{T}^2} W_k \otimes W_k \,\mathrm{d}x = R \quad \text{for all } R \in B_{1/2}(\mathbb{I}). \tag{3.38}$$

2. *For any $k \in \Lambda$ there exists $\Omega_k \in C^\infty(\mathbb{T}^2;\mathcal{A}^{2\times 2})$ with zero mean such that $W_k = \nabla_h \cdot \Omega_k$. In particular, $\nabla_h \cdot W_k = 0$. Moreover, $\nabla_h \cdot (W_k \otimes W_k) = W_k \cdot \nabla_h W_k = 0$.*

3. *For all $s \geq 0$, $1 \leq p \leq \infty$ and $k, k' \in \Lambda$ with $k \neq k'$ the following estimates hold:*
$$\|W_k\|_{W^{s,p}(\mathbb{T}^2)} \lesssim \mu_h^{\frac{1}{2}-\frac{1}{p}+s}; \tag{3.39}$$
$$\|\Omega_k\|_{W^{s,p}(\mathbb{T}^2)} \lesssim \mu_h^{-\frac{1}{2}-\frac{1}{p}+s}; \tag{3.40}$$
$$\|W_k \otimes W_{k'}\|_{L^p(\mathbb{T}^2)} \lesssim \mu_h^{1-\frac{2}{p}}. \tag{3.41}$$
Here the implicit constant may depend on s, p, but it does not depend on μ_h.

[12] We denote the set of all skew-symmetric 2×2 matrices by $\mathcal{A}^{2\times 2}$.

The following Lemma is a simple corollary of (3.34)-(3.36), (3.39) and (3.40).

Lemma 3.11. *Let $\sigma \in \mathbb{N}$. Then we have the following bounds for all $s \geq 0$, $1 \leq p \leq \infty$ and $k \in \{1, 2\}$, $k' \in \Lambda$:*

$$\|\phi_k(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_v)^s \mu_v^{\frac{1}{2}-\frac{1}{p}}, \tag{3.42}$$

$$\|W_k(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_v)^s \mu_v^{\frac{1}{2}-\frac{1}{p}}, \tag{3.43}$$

$$\|W_{k'}(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_h)^s \mu_h^{\frac{1}{2}-\frac{1}{p}}, \tag{3.44}$$

$$\|\Omega_k(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_v)^s \mu_v^{-\frac{1}{2}-\frac{1}{p}}, \tag{3.45}$$

$$\|\Omega_{k'}(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_h)^s \mu_h^{-\frac{1}{2}-\frac{1}{p}}. \tag{3.46}$$

Proof. For any $s \in \mathbb{N}_0$, the estimates simply follow from taking derivatives and (3.34)-(3.36), (3.39) and (3.40). Then by interpolation we obtain the desired estimates for any $s \geq 0$. □

3.5 Intermittency

As was done in [26–28] we now introduce the temporal intermittency functions, which differ for the horizontal and vertical perturbations. We first fix a non-negative function $G \in C_c^\infty((0, 1/2))$ with

$$\int_{[0,1]} G^2(t)\,\mathrm{d}t = 1. \tag{3.47}$$

3.5.1 Horizontal temporal intermittency functions

We set

$$g_h(t) \coloneqq \kappa_h^{1/2} G(\kappa_h t),$$

more precisely, g_h is the 1-periodic extension of the right-hand side (where we require that $\kappa_h > 1$). Note that from (3.47) we obtain the normalisation identity

$$\int_{[0,1]} g_h^2\,\mathrm{d}t = 1, \tag{3.48}$$

and furthermore it is straightforward to verify that

$$\|g_h\|_{L^p([0,1])} \lesssim \kappa_h^{1/2-1/p}, \tag{3.49}$$

for any $p \in [1, \infty]$. Subsequently, we introduce the temporal correction function

$$h_h(t) \coloneqq \int_0^t \left(g_h^2(\tau) - 1\right) \mathrm{d}\tau.$$

Due to (3.48), h_h is 1-periodic and we have

$$\|h_h\|_{L^\infty([0,1])} \leq 1. \tag{3.50}$$

3.5.2 Vertical temporal intermittency functions

The vertical temporal oscillation functions are given by the 1-periodic extension (assuming that $\kappa_v > 1$) of

$$g_{v,1}^{-}(t) \coloneqq \kappa_v^{1/q_2} G(\kappa_v t), \quad g_{v,1}^{+}(t) \coloneqq \kappa_v^{1-1/q_2} G(\kappa_v t).$$

The corresponding temporal correction function is defined by

$$h_{v,1}(t) \coloneqq \int_0^t \left(g_{v,1}^{-}(\tau) g_{v,1}^{+}(\tau) - 1 \right) \mathrm{d}\tau.$$

In addition to that we need temporal oscillation functions where the argument of G is shifted. Those are defined as the 1-periodic extension of

$$g_{v,2}^{-}(t) \coloneqq \kappa_v^{1/q_2} G(\kappa_v (t - 1/2)), \quad g_{v,2}^{+}(t) \coloneqq \kappa_v^{1-1/q_2} G(\kappa_v (t - 1/2))$$

with corresponding correction function

$$h_{v,2}(t) \coloneqq \int_0^t \left(g_{v,2}^{-}(\tau) g_{v,2}^{+}(\tau) - 1 \right) \mathrm{d}\tau.$$

Since G has compact support in $(0, 1/2)$ and $\kappa_v > 1$, the functions $g_{v,1}^{\pm}$ and $g_{v,2}^{\pm}$ have disjoint supports.

Note that due to the fact that $q_2 > 2$, we have $1/q_2 < 1 - 1/q_2$ which justifies the notation $g_{v,k}^{-}$, $g_{v,k}^{+}$ for $k = 1, 2$.

Similarly to the horizontal temporal functions which we introduced in section 3.5.1, we have the following estimates for any $p \in [1, \infty]$ and $k \in \{1, 2\}$

$$\|g_{v,k}^{-}\|_{L^p([0,1])} \lesssim \kappa_v^{1/q_2 - 1/p}, \tag{3.51}$$

$$\|g_{v,k}^{+}\|_{L^p([0,1])} \lesssim \kappa_v^{1-1/q_2-1/p}, \tag{3.52}$$

$$\|h_{v,k}\|_{L^\infty([0,1])} \leq 1. \tag{3.53}$$

Finally in a similar manner one can show that

$$\|g_{v,k}^{-}\|_{W^{n,p}([0,1])} \lesssim \kappa_v^{1/q_2 + n - 1/p} \tag{3.54}$$

for any $n \in \mathbb{N}_0$ and $p \in [1, \infty]$.

4 Velocity perturbation and new Reynolds stress tensor

In sections 4, 5 and 6 we prove Proposition 2.4 hence we suppose that the assumptions of Proposition 2.4 hold.

The perturbation will be written as

$$\overline{u}_p = u_{p,h} + u_{c,h} + u_{t,h}, \tag{4.1}$$

$$\widetilde{u}_p = u_{p,v} + u_{c,v} + u_{t,v}, \tag{4.2}$$

$$w_p = w_{p,v} + w_{t,v}, \tag{4.3}$$

where $u_{p,h}$ and $u_{p,v}$ are referred to as the horizontal and vertical principal parts of the perturbation, while $u_{c,h}$, $u_{c,v}$, $u_{t,h}$ and $u_{t,v}$ are referred to as the horizontal and vertical spatial and temporal correctors.

Remark 4.1. We would like to remark that $u_{p,h}, u_{c,h}$ and $u_{t,h}$ do no depend on z, while $u_{p,v}, u_{c,v}$ and $u_{t,v}$ are mean-free with respect to z. Therefore, the first three are indeed a barotropic perturbation, while the latter three are a baroclinic perturbation. This is already hidden in (4.1) and (4.2).

In sections 4.1 and 4.3 we define the horizontal perturbation $\overline{u}_p$ and the vertical perturbation $\widetilde{u}_p$, respectively. The pressure perturbation P is determined in section 4.2. Finally we define the new Reynolds stress tensors $R_{h,1}$ and $R_{v,1}$ in section 4.4.

4.1 The horizontal perturbation

We begin by constructing the horizontal perturbation which consists (see above) of a principal part $u_{p,h}$, a spatial corrector $u_{c,h}$ and a temporal corrector $u_{t,h}$.

In order to construct $u_{p,h}$, we introduce a cutoff function χ. First we choose $\widetilde{\chi} \in C^\infty([0,\infty))$ to be increasing and satisfying

$$\widetilde{\chi}(\sigma) = \begin{cases} 4\|R_h\|_{L^1(L^1)} & \text{if } 0 \leq \sigma \leq \|R_h\|_{L^1(L^1)}, \\ 4\sigma & \text{if } \sigma \geq 2\|R_h\|_{L^1(L^1)}. \end{cases}$$

Next we define the function

$$\chi(x,t) := \widetilde{\chi}\big(|R_h(x,t)|\big).$$

It is straightforward to check that $\mathbb{I} - \frac{R_h}{\chi} \in B_{1/2}(\mathbb{I})$ for all $(x,t) \in \mathbb{T}^2 \times [0,T]$. This means in particular that we can evaluate the functions Γ_k (see Proposition 3.10) at $\mathbb{I} - \frac{R_h}{\chi}$.

We now introduce a temporal smooth cutoff function $\theta \in C^\infty([0,T];[0,1])$ which satisfies

$$\theta(t) = \begin{cases} 1 & \text{if } \operatorname{dist}(t, I^c) \geq \tau, \\ 0 & \text{if } \operatorname{dist}(t, I^c) \leq \frac{1}{2}\tau, \end{cases} \tag{4.4}$$

in order to achieve the desired property of the supports of the perturbations $\overline{u}_p, \widetilde{u}_p, w_p$. The horizontal principal perturbation is then defined by

$$u_{p,h}(x,t) := \sum_{k\in\Lambda} a_k(x,t) W_k(\sigma_h x), \tag{4.5}$$

where the W_k are given by Proposition 3.10 and the amplitude functions are

$$a_k(x,t) := \theta(t) g_h(\nu_h t) \chi^{1/2}(x,t) \Gamma_k\left(\mathbb{I} - \frac{R_h(x,t)}{\chi(x,t)}\right). \tag{4.6}$$

Notice that $u_{p,h}$ does not need to be divergence free. To overcome this we define the corrector $u_{c,h}$ as

$$u_{c,h} \coloneqq \sigma_h^{-1} \sum_{k\in\Lambda} \nabla_h a_k \cdot \Omega_k(\sigma_h x). \tag{4.7}$$

Hence

$$u_{p,h} + u_{c,h} = \sigma_h^{-1} \sum_{k\in\Lambda} \nabla_h \cdot (a_k(x,t)\Omega_k(\sigma_h x)), \tag{4.8}$$

which implies

$$\nabla_h \cdot (u_{p,h} + u_{c,h}) = \sigma_h^{-1} \sum_{k\in\Lambda} (\nabla_h \otimes \nabla_h) : (a_k(x,t)\Omega_k(\sigma_h x)) = 0, \tag{4.9}$$

as Ω_k is skew-symmetric. Moreover, using the definition of θ in (4.4), we have $u_{p,h} = u_{c,h} = 0$ whenever $\operatorname{dist}(t, I^c) \le \tau/2$.

Next, we define the horizontal temporal corrector to be

$$u_{t,h} \coloneqq \nu_h^{-1} h_h(\nu_h t)(\nabla_h \cdot R_h - \nabla_h \Delta_h^{-1}[(\nabla_h \otimes \nabla_h) : R_h]). \tag{4.10}$$

It is straightforward to check that $(\nabla_h \otimes \nabla_h) : R_h$ is mean-free (so that the inverse Laplacian Δ_h^{-1} can be applied to this expression), $\nabla_h \cdot u_{t,h} = 0$, and that $u_{t,h} = 0$ whenever $\operatorname{dist}(t, I^c) \le \tau/2$.

Finally notice, that $u_{p,h}$, $u_{c,h}$ and $u_{t,h}$ are indeed independent of z.

4.2 The pressure perturbation

The pressure perturbation is defined as follows

$$P \coloneqq -\theta^2 g_h^2(\nu_h t)\chi + \nu_h^{-1}\Delta_h^{-1}(\nabla_h \otimes \nabla_h) : \partial_t(h_h(\nu_h t)R_h). \tag{4.11}$$

Note that $\partial_z P = 0$, since R_h, and hence also χ, are independent of z.

4.3 The vertical perturbation

We denote the components of the vertical Reynolds stress tensor as $R_{v,k}$, $k = 1, 2$, i.e.,

$$R_v = \begin{pmatrix} R_{v,1} \\ R_{v,2} \end{pmatrix}.$$

This allows us to define the vertical principal perturbation as

$$u_{p,v}(x,t) \coloneqq -\sum_{k=1}^{2} \frac{g_{v,k}^-(\nu_v t)\theta(t)R_{v,k}(x,t)W_k(\sigma_v x)}{\|R_h\|_{L^1(L^1)}}, \tag{4.12}$$

$$w_{p,v}(x,t) \coloneqq \sum_{k=1}^{2} g_{v,k}^+(\nu_v t)\theta(t)\phi_k(\sigma_v x)\|R_h\|_{L^1(L^1)}, \tag{4.13}$$

where W_k and ϕ_k are given by Proposition 3.9.

We now introduce the vertical spatial corrector in order to make the perturbation divergence free. We set

$$u_{c,v} := -\frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \nabla_h \big(g_{v,k}^-(\nu_v t)\theta R_{v,k}\big)\Omega_k(\sigma_v x). \tag{4.14}$$

Observe that

$$u_{p,v} + u_{c,v} = -\frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \nabla_h \cdot \big(g_{v,k}^-(\nu_v t)\theta R_{v,k}\Omega_k(\sigma_v x)\big), \tag{4.15}$$

and hence

$$\nabla_h \cdot (u_{p,v} + u_{c,v}) = -\frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} (\nabla_h \otimes \nabla_h) : \big(g_{v,k}^-(\nu_v t)\theta R_{v,k}\Omega_k(\sigma_v x)\big) = 0,$$

since Ω_k is skew-symmetric. Notice that $w_{p,v}$ is independent of z. Again according to the definition of θ in (4.4), $u_{p,v} = u_{c,v} = w_{p,v} = 0$ whenever $\mathrm{dist}(t, I^c) \le \tau/2$.

Moreover, we introduce the vertical temporal corrector to be

$$u_{t,v} := \nu_v^{-1} \sum_{k=1}^{2} h_{v,k}(\nu_v t)\partial_z R_{v,k}\mathbf{e}_k. \tag{4.16}$$

Since $u_{t,v}$ does not need to be divergence free, we introduce the corrector

$$w_{t,v} := -\nu_v^{-1} \sum_{k=1}^{2} h_{v,k}(\nu_v t)\partial_k R_{v,k}. \tag{4.17}$$

It is then simple to check that $\nabla_h \cdot u_{t,v} + \partial_z w_{t,v} = 0$. Similar to $u_{t,h}$, see above, we have $u_{t,v} = 0$ and $w_{t,v} = 0$ whenever $\mathrm{dist}(t, I^c) \le \tau/2$.

Finally notice, that $u_{p,v}$, $u_{c,v}$ and $u_{t,v}$ are mean-free with respect to z because R_v is mean-free with respect to z.

4.4 New Reynolds stress tensors

The goal of this section is to define the new Reynolds stress tensors R_h and R_v. These will consist of several pieces.

4.4.1 Horizontal oscillation error

Let us first define

$$R_{\text{far}} := \sum_{k,k' \in \Lambda, k \neq k'} a_k a_{k'} W_k(\sigma_h x) \otimes W_{k'}(\sigma_h x), \tag{4.18}$$

$$R_{\text{osc},x,h} := \sum_{k \in \Lambda} \mathcal{B}\bigg(\nabla_h(a_k^2), W_k(\sigma_h x) \otimes W_k(\sigma_h x) - \int_{\mathbb{T}^2} W_k \otimes W_k \,\mathrm{d}x\bigg), \tag{4.19}$$

$$R_{\text{osc},t,h} := \nu_h^{-1} h_h(\nu_h t)\partial_t R_h. \tag{4.20}$$

where $\mathcal{B}$ is the bilinear inverse divergence operator from section 3.3.2. Moreover, we set

$$R_{\text{osc},h} = R_{\text{osc},x,h} + R_{\text{osc},t,h} + R_{\text{far}}. \tag{4.21}$$

Lemma 4.2. *We have*

$$\partial_t u_{t,h} + \nabla_h \cdot (u_{p,h} \otimes u_{p,h} + R_h) + \nabla_h P = \nabla_h \cdot R_{\text{osc},h}. \tag{4.22}$$

Proof. Let us first look at the term $\nabla_h \cdot (u_{p,h} \otimes u_{p,h} + R_h)$. We may write

$$\nabla_h \cdot (u_{p,h} \otimes u_{p,h} + R_h) = \nabla_h \cdot \left(\sum_{k\in\Lambda} a_k^2 W_k(\sigma_h x) \otimes W_k(\sigma_h x) + R_h \right) + \nabla_h \cdot R_{\text{far}}.$$

Using the definition of the a_k, items 1 and 2 of Proposition 3.10 and Lemma 3.6 we find

$$\begin{aligned}
&\nabla_h \cdot \left(\sum_{k\in\Lambda} a_k^2 W_k(\sigma_h x) \otimes W_k(\sigma_h x) + R_h \right) \\
&= \nabla_h \cdot \left[\sum_{k\in\Lambda} a_k^2 \left(W_k(\sigma_h x) \otimes W_k(\sigma_h x) - \int_{\mathbb{T}^2} W_k \otimes W_k \,\mathrm{d}x + \int_{\mathbb{T}^2} W_k \otimes W_k \,\mathrm{d}x \right) + R_h \right] \\
&= \sum_{k\in\Lambda} \nabla_h (a_k^2) \cdot \left(W_k(\sigma_h x) \otimes W_k(\sigma_h x) - \int_{\mathbb{T}^2} W_k \otimes W_k \,\mathrm{d}x \right) \\
&\qquad + \theta^2 g_h^2(\nu_h t) \nabla_h \chi + (1 - \theta^2 g_h^2(\nu_h t)) \nabla_h \cdot R_h \\
&= \nabla_h \cdot R_{\text{osc},x,h} + \theta^2 g_h^2(\nu_h t) \nabla_h \chi + (1 - \theta^2 g_h^2(\nu_h t)) \nabla_h \cdot R_h.
\end{aligned}$$

Next we compute

$$\begin{aligned}
\partial_t u_{t,h} &= \nu_h^{-1} \partial_t (h_h(\nu_h t)) \nabla_h \cdot R_h + \nu_h^{-1} h_h(\nu_h t) \nabla_h \cdot \partial_t R_h \\
&\qquad - \nu_h^{-1} \nabla_h \Delta_h^{-1} (\nabla_h \otimes \nabla_h) : \partial_t (h_h(\nu_h t) R_h)) \\
&= (g_h^2(\nu_h t) - 1) \nabla_h \cdot R_h + \nabla_h \cdot R_{\text{osc},t,h} - \nu_h^{-1} \nabla_h \Delta_h^{-1} (\nabla_h \otimes \nabla_h) : \partial_t (h_h(\nu_h t) R_h).
\end{aligned}$$

Hence we have shown

$$\partial_t u_{t,h} + \nabla_h \cdot (u_{p,h} \otimes u_{p,h} + R_h) + \nabla_h P = \nabla_h \cdot (R_{\text{osc},x,h} + R_{\text{osc},t,h} + R_{\text{far}}) + g_h^2(\nu_h t)(1 - \theta^2) \nabla_h \cdot R_h.$$

If $\operatorname{dist}(t, I^c) \leq \tau$, then $R_h = 0$ by well-preparedness. If $\operatorname{dist}(t, I^c) \geq \tau$ then $\theta(t) = 1$ and hence $1 - \theta^2 = 0$. This completes the proof of (4.22).

□

4.4.2 Vertical oscillation error

We define

$$R_{\text{osc},x,v} := - \sum_{k=1}^{2} g_{v,k}^-(\nu_v t) g_{v,k}^+(\nu_v t) \theta^2 R_{v,k} \left(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k W_k \,\mathrm{d}x \right), \tag{4.23}$$

$$R_{\text{osc},t,v} := \nu_v^{-1} \sum_{k=1}^{2} h_{v,k}(\nu_v t) \partial_t R_{v,k} \mathbf{e}_k. \tag{4.24}$$

and set

$$R_{\mathrm{osc},v} = R_{\mathrm{osc},x,v} + R_{\mathrm{osc},t,v}. \tag{4.25}$$

Lemma 4.3. *We have*

$$\partial_t u_{t,v} + \partial_z (w_{p,v} u_{p,v} + R_v) = \partial_z R_{\mathrm{osc},v}. \tag{4.26}$$

Proof. First we observe that

$$w_{p,v} u_{p,v} = -\sum_{k=1}^{2} g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) \theta^2 R_{v,k} \phi_k(\sigma_v x) W_k(\sigma_v x),$$

since $g_{v,1}^{\pm} g_{v,2}^{\pm} = 0$, see section 3.5.2. Hence we obtain using Proposition 3.9

$$\begin{aligned}
&\partial_z (w_{p,v} u_{p,v} + R_v) \\
&= -\sum_{k=1}^{2} g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) \theta^2 \partial_z R_{v,k} \Big(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k W_k \,\mathrm{d}x \Big) \\
&\qquad - \sum_{k=1}^{2} \big(g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) - 1 \big) \partial_z R_{v,k} \mathbf{e}_k + \sum_{k=1}^{2} g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) (1-\theta^2) \partial_z R_{v,k} \mathbf{e}_k \\
&= \partial_z R_{\mathrm{osc},x,v} - \sum_{k=1}^{2} \big(g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) - 1 \big) \partial_z R_{v,k} \mathbf{e}_k.
\end{aligned}$$

Here we used that $(1-\theta^2)\partial_z R_{v,k} = 0$, see the proof of Lemma 4.2. Moreover a straightforward computation shows

$$\begin{aligned}
\partial_t u_{t,v} - \sum_{k=1}^{2} \big(g_{v,k}^{-}(\nu_v t) g_{v,k}^{+}(\nu_v t) - 1 \big) \partial_z R_{v,k} \mathbf{e}_k &= \nu_v^{-1} \sum_{k=1}^{2} h_{v,k}(\nu_v t) \partial_t \partial_z R_{v,k} \mathbf{e}_k \\
&= \partial_z R_{\mathrm{osc},t,v},
\end{aligned}$$

which finishes the proof of Lemma 4.3. □

4.4.3 Linear errors

Next we define the horizontal and vertical linear errors by

$$\begin{aligned}
R_{\mathrm{lin},h} := \mathcal{R}_h \Big[\partial_t \big(u_{p,h} + u_{c,h} \big) + \nabla_h \cdot \Big(&\overline{u} \otimes \big(u_{p,h} + u_{c,h} + u_{t,h} \big) + \overline{u \otimes \big(u_{p,v} + u_{c,v} + u_{t,v} \big)} \\
&+ \big(u_{p,h} + u_{c,h} + u_{t,h} \big) \otimes \overline{u} + \overline{\big(u_{p,v} + u_{c,v} + u_{t,v} \big) \otimes u} \Big) \Big],
\end{aligned}$$

and

$$\begin{aligned}
R_{\mathrm{lin},v} := \mathcal{R}_v \Big[\partial_t \big(u_{p,v} + u_{c,v} \big) + \nabla_h \cdot \Big(&\widetilde{u} \otimes \big(u_{p,h} + u_{c,h} + u_{t,h} \big) + \widetilde{u \otimes \big(u_{p,v} + u_{c,v} + u_{t,v} \big)} \\
&+ \big(u_{p,h} + u_{c,h} + u_{t,h} \big) \otimes \widetilde{u} + \widetilde{\big(u_{p,v} + u_{c,v} + u_{t,v} \big) \otimes u} \Big) \\
&+ \partial_z \Big(w \big(u_{p,h} + u_{c,h} + u_{t,h} + u_{p,v} + u_{c,v} + u_{t,v} \big) + \big(w_{p,v} + w_{t,v} \big) u \Big) \Big].
\end{aligned}$$

Note that the arguments of the operators $\mathcal{R}_h$ and $\mathcal{R}_v$ satisfy the required properties, i.e. they are independent of z and mean-free with respect to z, respectively.

For convenience let us write

$$\begin{aligned} R_{\text{lin},t,h} &:= \mathcal{R}_h \partial_t \big(u_{p,h} + u_{c,h}\big), \\ R_{\text{lin},t,v} &:= \mathcal{R}_v \partial_t \big(u_{p,v} + u_{c,v}\big), \end{aligned}$$

and

$$\begin{aligned} R_{\text{lin},x,h} &:= R_{\text{lin},h} - R_{\text{lin},t,h}, \\ R_{\text{lin},x,v} &:= R_{\text{lin},v} - R_{\text{lin},t,v}. \end{aligned}$$

4.4.4 Corrector errors

Finally, we define the horizontal and vertical corrector errors by

$$\begin{aligned} R_{\text{cor},h} := \mathcal{R}_h \Bigg[\nabla_h \cdot \bigg(&\big(u_{c,h} + u_{t,h}\big) \otimes \big(u_{c,h} + u_{t,h}\big) + u_{p,h} \otimes \big(u_{c,h} + u_{t,h}\big) + \big(u_{c,h} + u_{t,h}\big) \otimes u_{p,h} \\ &+ \overline{\big(u_{p,v} + u_{c,v} + u_{t,v}\big) \otimes \big(u_{p,v} + u_{c,v} + u_{t,v}\big)} \bigg) \Bigg] \end{aligned}$$

and

$$\begin{aligned} R_{\text{cor},v} := \mathcal{R}_v \Bigg[\nabla_h \cdot \bigg(&\widetilde{\big(u_{p,v} + u_{c,v} + u_{t,v}\big) \otimes \big(u_{p,v} + u_{c,v} + u_{t,v}\big)} \\ &+ \big(u_{p,h} + u_{c,h} + u_{t,h}\big) \otimes \big(u_{p,v} + u_{c,v} + u_{t,v}\big) \\ &+ \big(u_{p,v} + u_{c,v} + u_{t,v}\big) \otimes \big(u_{p,h} + u_{c,h} + u_{t,h}\big) \bigg) \\ + \partial_z \bigg(&w_{t,v} \big(u_{p,h} + u_{c,h} + u_{t,h} + u_{p,v} + u_{c,v} + u_{t,v}\big) \\ &+ w_{p,v} \big(u_{c,v} + u_{t,v}\big) \bigg) \Bigg]. \end{aligned}$$

As in section 4.4.3 we remark that the arguments of the operators $\mathcal{R}_h$ and $\mathcal{R}_v$ satisfy the required properties, i.e. they are independent of z and mean-free with respect to z, respectively.

4.4.5 Conclusion

The new Reynolds stress tensors $R_{h,1}, R_{v,1}$ are then given by

$$R_{h,1} := R_{\text{osc},h} + R_{\text{lin},h} + R_{\text{cor},h}, \qquad R_{v,1} := R_{\text{osc},v} + R_{\text{lin},v} + R_{\text{cor},v}.$$

First, we note that by definition $R_{\text{osc},h}$, $R_{\text{lin},h}$ and $R_{\text{cor},h}$ (and consequently also $R_{h,1}$) are independent of z. Moreover, by definition $R_{\text{osc},v}$ is mean-free with respect to z, and so are $R_{\text{lin},v}$ and $R_{\text{cor},v}$ according to Lemma 3.8 (consequently $R_{v,1}$ has the same property).

Next we remark that $R_{h,1}(x,t) = R_{v,1}(x,t) = 0$ whenever $\operatorname{dist}(t, I^c) \le \frac{\tau}{2}$. Indeed, for the oscillation errors $R_{\text{osc},h}$, $R_{\text{osc},v}$ this follows from the fact that $R_h = R_v = 0$ whenever $\operatorname{dist}(t, I^c) \le \tau$, and the definition of θ, see (4.4). The fact that $R_{\text{lin},h} = 0$, $R_{\text{lin},v} = 0$, $R_{\text{cor},h} = 0$, and $R_{\text{cor},v} = 0$ whenever $\operatorname{dist}(t, I^c) \le \frac{\tau}{2}$ immediately follows from item 1 of Proposition 2.4, which we already proved in section 4.1 and 4.3

Finally, with the help of

- the fact that $\nabla_h \cdot (u + \overline{u}_p + \widetilde{u}_p) + \partial_z(w + w_p) = 0$,
- Lemmas 4.2 and 4.3,
- Lemmas 3.4 and 3.8,
- the fact that $\int_{\mathbb{T}^2} \big(u_{p,h} + u_{c,h}\big) \, \mathrm{d}x = 0$, see (4.8),
- and $\partial_z u_{p,h} = \partial_z u_{c,h} = \partial_z u_{t,h} = \partial_z w_{p,v} = 0$, see section 4.1 and section 4.3,

a long but straightforward computation shows

$$
\begin{aligned}
&\partial_t(u + \overline{u}_p + \widetilde{u}_p) + (u + \overline{u}_p + \widetilde{u}_p) \cdot \nabla_h(u + \overline{u}_p + \widetilde{u}_p) + (w + w_p)\partial_z(u + \overline{u}_p + \widetilde{u}_p) + \nabla_h(p + P) \\
&= \partial_t\big(\overline{u}_p + \widetilde{u}_p\big) + \nabla_h \cdot \Big(u \otimes \big(\overline{u}_p + \widetilde{u}_p\big) + \big(\overline{u}_p + \widetilde{u}_p\big) \otimes u + \big(\overline{u}_p + \widetilde{u}_p\big) \otimes \big(\overline{u}_p + \widetilde{u}_p\big)\Big) \\
&\qquad + \partial_z\Big(w\big(\overline{u}_p + \widetilde{u}_p\big) + w_p u + w_p\big(\overline{u}_p + \widetilde{u}_p\big)\Big) + \nabla_h P + \nabla_h \cdot R_h + \partial_z R_v \\
&= \nabla_h \cdot R_{h,1} + \partial_z R_{v,1}.
\end{aligned}
$$

Hence $(u + \overline{u}_p + \widetilde{u}_p, w + w_p, p + P, R_{h,1}, R_{v,1})$ solves (2.1).

5 Estimates on the perturbation

In the remaining sections we will use the following convention: for quantities Q_1 and Q_2 we write $Q_1 \lesssim Q_2$ if there exists a constant C such that $Q_1 \le C Q_2$. In general we require that C does not depend on (u, w, p, R_h, R_v). However if the right-hand side Q_2 only contains powers of the parameters $\mu_i, \sigma_i, \kappa_i, \nu_i, \lambda_i$ $(i = h, v)$, then the implicit constant C may depend on (u, w, p, R_h, R_v).

5.1 Principal perturbation

5.1.1 Horizontal principal perturbation

First, we estimate the horizontal part of the principal perturbation. We recall the following Lemma from [27].

Lemma 5.1. *We have the following estimates for all $n, m \in \mathbb{N}_0$, $p \in [1, \infty]$*

$$
\|\partial_t^n \nabla^m a_k\|_{L^p(L^\infty)} \lesssim (\nu_h \kappa_h)^n \kappa_h^{1/2 - 1/p}, \tag{5.1}
$$

$$\|a_k(t)\|_{L^2} \lesssim \theta(t) g_h(\nu_h t) \bigg(\int_{\mathbb{T}^2} \chi(x,t)\,\mathrm{d}x \bigg)^{1/2}, \qquad \textit{for any } t \in [0,T]. \tag{5.2}$$

The implicit constant in (5.1) *might depend on* u *or* R_h*, whereas the implicit constant in* (5.2) *neither depends on* t *nor on* u *or* R_h*.*

For the proof we refer to [27, Lemma 5.2].

With Lemma 5.1 at hand we are ready to prove the estimates on the horizontal principal perturbation.

Lemma 5.2. *If* λ_h *is chosen sufficiently large (depending on* R_h*), then the horizontal principal perturbation satisfies the following estimates*

$$\|u_{p,h}\|_{L^2(L^2)} \lesssim \|R_h\|_{L^1(L^1)}^{1/2}, \tag{5.3}$$

$$\|u_{p,h}\|_{L^{q_1}(H^{s_1})} \lesssim \lambda_h^{-\gamma_h}. \tag{5.4}$$

Proof. By applying the improved Hölder inequality in Lemma B.1 and Lemmas 3.11 and 5.1, we find that

$$\begin{aligned}
\|u_{p,h}\|_{L^2(L^2)} &\le \sum_{k\in\Lambda} \|a_k W_k(\sigma_h \cdot)\|_{L^2(L^2)} \lesssim \sum_{k\in\Lambda} \Big(\|a_k\|_{L^2(L^2)} \|W_k\|_{L^2} + \sigma_h^{-1/2} \|a_k\|_{L^2(C^1)} \|W_k\|_{L^2} \Big) \\
&\lesssim \bigg\| g_h(\nu_h \cdot) \bigg(\int_{\mathbb{T}^2} \chi(\cdot,x)\,\mathrm{d}x \bigg)^{1/2} \bigg\|_{L^2} + C_{u,R_h} \sigma_h^{-1/2}
\end{aligned}$$

with a constant C_{u,R_h} depending on u, R_h. Since $t \mapsto \big(\int_{\mathbb{T}^2} \chi(\cdot,x)\,\mathrm{d}x \big)^{1/2}$ is smooth, we can apply Lemma B.1 once again to obtain

$$\begin{aligned}
&\bigg\| g_h(\nu_h \cdot) \bigg(\int_{\mathbb{T}^2} \chi(\cdot,x)\,\mathrm{d}x \bigg)^{1/2} \bigg\|_{L^2} \\
&\lesssim \bigg\| \bigg(\int_{\mathbb{T}^2} \chi(\cdot,x)\,\mathrm{d}x \bigg)^{1/2} \bigg\|_{L^2} \|g_h\|_{L^2} + \nu_h^{-1/2} \bigg\| \bigg(\int_{\mathbb{T}^2} \chi(\cdot,x)\,\mathrm{d}x \bigg)^{1/2} \bigg\|_{C^1} \|g_h\|_{L^2} \\
&\lesssim \|\chi\|_{L^1(L^1)}^{1/2} + C_{u,R_h} \nu_h^{-1/2},
\end{aligned}$$

where we made use of (3.48). Since $\chi(x,t) \le 4|R_h(x,t)| + 4\|R_h\|_{L^1(L^1)}$, we have

$$\|\chi\|_{L^1(L^1)} \lesssim \|R_h\|_{L^1(L^1)} + \|R_h\|_{L^1(L^1)} \lesssim \|R_h\|_{L^1(L^1)}.$$

So have shown that

$$\|u_{p,h}\|_{L^2(L^2)} \lesssim \|R_h\|_{L^1(L^1)}^{1/2} + C_{u,R_h} \nu_h^{-1/2} + C_{u,R_h} \sigma_h^{-1/2},$$

which implies (5.3) by taking λ_h sufficiently large (depending on R_h).

Furthermore by applying Lemmas 3.1, 3.11 and 5.1 we find that

$$\|u_{p,h}\|_{L^{q_1}(H^{s_1})} \le \sum_{k\in\Lambda} \|a_k\|_{L^{q_1}(W^{1,\infty})} \|W_k(\sigma_h \cdot)\|_{H^{s_1}} \lesssim \kappa_h^{1/2 - 1/q_1} (\sigma_h \mu_h)^{s_1} \le \lambda_h^{-\gamma_h}$$

which proves (5.4). □

5.1.2 Vertical principal perturbation

Let us first show the following Lemma.

Lemma 5.3. *For all* $1 \leq p \leq \infty$ *and* $k \in \{1,2\}$ *we have*

$$\left\| \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \, \mathrm{d}x \right) R_{v,k} \right\|_{L^p(B^{-1}_{1,\infty})} \lesssim \sigma_v^{-1}. \tag{5.5}$$

Proof. Using equation (3.15) we find

$$\begin{aligned}
&\left(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \, \mathrm{d}x \right) R_{v,k} \\
&= R_{v,k} \nabla_h \cdot \left[\mathcal{R}_h \left(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right] \\
&= \nabla_h \cdot \left[R_{v,k} \mathcal{R}_h \left(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right] \\
&\quad - \nabla_h R_{v,k} \cdot \mathcal{R}_h \left(\phi_k(\sigma_v x) W_k(\sigma_v x) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right).
\end{aligned}$$

Next, we observe that according to Lemma 3.11

$$\left\| \phi_k W_k - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \, \mathrm{d}x \right\|_{L^1} \lesssim \|\phi_k W_k\|_{L^1} + 1 \leq \|\phi_k\|_{L^2} \|W_k\|_{L^2} + 1 \lesssim 1. \tag{5.6}$$

Then by Lemmas 3.4 and A.3, and inequality (5.6) we obtain

$$\begin{aligned}
&\left\| \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \, \mathrm{d}x \right) R_{v,k} \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\lesssim \left\| \nabla_h \cdot \left[R_{v,k} \mathcal{R}_h \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right] \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\qquad + \left\| \nabla_h R_{v,k} \cdot \mathcal{R}_h \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\lesssim \left\| R_{v,k} \mathcal{R}_h \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right\|_{L^p(L^1)} \\
&\qquad + \left\| \nabla_h R_{v,k} \cdot \mathcal{R}_h \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right\|_{L^p(L^1)} \\
&\lesssim \left(\|R_{v,k}\|_{L^p(L^\infty)} + \|\nabla_h R_{v,k}\|_{L^p(L^\infty)} \right) \left\| \mathcal{R}_h \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right) \right\|_{L^1} \\
&\lesssim \sigma_v^{-1} C_{R_v} \left\| \phi_k W_k - \int_{\mathbb{T}^2} \phi_k W_k \, \mathrm{d}x \right\|_{L^1} \lesssim \sigma_v^{-1}.
\end{aligned}$$

□

Remark 5.4. We present an alternative proof of Lemma 5.3 in the appendix, see section A.3.

Next we estimate the vertical principal perturbation.

Lemma 5.5. *If λ_v is chosen sufficiently large (depending on R_v), then the vertical principal perturbation satisfies the following inequalities*

$$\|u_{p,v}\|_{L^{q_3-}(H^{s_3})} \lesssim \lambda_v^{-\gamma_v}, \tag{5.7}$$

$$\|u_{p,v}\|_{L^{q_2-}(L^2)} \lesssim \lambda_v^{-\gamma_v}, \tag{5.8}$$

$$\|w_{p,v}\|_{L^{q_3'-}(H^{-s_3})} \lesssim \lambda_v^{-\gamma_v}, \tag{5.9}$$

$$\|w_{p,v}\|_{L^{q_2'-}(L^2)} \lesssim \lambda_v^{-\gamma_v}, \tag{5.10}$$

$$\|w_{p,v}\|_{L^{q_3'}(H^{-s_3})} \lesssim \|R_h\|_{L^1(L^1)}, \tag{5.11}$$

$$\|w_{p,v}\|_{L^{q_2'}(L^2)} \lesssim \|R_h\|_{L^1(L^1)}, \tag{5.12}$$

$$\|w_{p,v}u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{5.13}$$

Proof. According to (3.51) and Lemmas 3.1 and 3.11, we obtain

$$\begin{aligned}\|u_{p,v}\|_{L^{q_3-}(H^{s_3})} &\le \frac{1}{\|R_h\|_{L^1(L^1)}}\sum_{k=1}^{2}\|g^-_{v,k}(\nu_v\cdot)\|_{L^{q_3-}}\|R_v\|_{L^\infty(W^{1,\infty})}\|W_k(\sigma_v\cdot)\|_{H^{s_3}}\\ &\lesssim \kappa_v^{1/q_2-1/q_3-\delta}(\sigma_v\mu_v)^{s_3} = \kappa_v^{-\delta} \le \lambda_v^{-\gamma_v}.\end{aligned}$$

Similarly,

$$\begin{aligned}\|u_{p,v}\|_{L^{q_2-}(L^2)} &\le \frac{1}{\|R_h\|_{L^1(L^1)}}\sum_{k=1}^{2}\|g^-_{v,k}(\nu_v\cdot)\|_{L^{q_2-}}\|R_v\|_{L^\infty(L^\infty)}\|W_k(\sigma_v\cdot)\|_{L^2}\\ &\lesssim \kappa_v^{1/q_2-1/q_2-\delta} = \kappa_v^{-\delta} \le \lambda_v^{-\gamma_v}.\end{aligned} \tag{5.14}$$

So we have shown (5.7), (5.8).

Next, notice that in accordance with Proposition 3.9 and Lemma 3.11 (keeping in mind that $s_3 \le 1$ and hence $-s_3+1 \ge 0$)

$$\begin{aligned}\|\phi_k(\sigma_v\cdot)\|_{H^{-s_3}} &= \|W_k(\sigma_v\cdot)\|_{H^{-s_3}} = \sigma_v^{-1}\big\|\nabla_h\cdot\big[\Omega_k(\sigma_v\cdot)\big]\big\|_{H^{-s_3}}\\ &\lesssim \sigma_v^{-1}\|\Omega_k(\sigma_v\cdot)\|_{H^{-s_3+1}} \lesssim \sigma_v^{-1}(\sigma_v\mu_v)^{-s_3+1}\mu_v^{-1} = (\sigma_v\mu_v)^{-s_3}.\end{aligned}$$

Together with (3.52) and Lemma 3.1 this yields

$$\begin{aligned}\|w_{p,v}\|_{L^{q_3'-}(H^{-s_3})} &\le \sum_{k=1}^{2}\|\phi_k(\sigma_v\cdot)\|_{H^{-s_3}}\|R_h\|_{L^1(L^1)}\|g^+_{v,k}(\nu_v\cdot)\|_{L^{q_3'-}}\\ &\lesssim \kappa_v^{1-1/q_2-1/q_3'-\delta}(\sigma_v\mu_v)^{-s_3} = \kappa_v^{-\delta} \le \lambda_v^{-\gamma_v}.\end{aligned} \tag{5.15}$$

Similarly, using Lemma 3.11, we obtain

$$\begin{aligned}\|w_{p,v}\|_{L^{q_2'-}(L^2)} &\le \sum_{k=1}^{2}\|\phi_k(\sigma_v\cdot)\|_{L^2}\|R_h\|_{L^1(L^1)}\|g^+_{v,k}(\nu_v\cdot)\|_{L^{q_2'-}}\\ &\lesssim \kappa_v^{1-1/q_2-1/q_2'-\delta} = \kappa_v^{-\delta} \le \lambda_v^{-\gamma_v}.\end{aligned} \tag{5.16}$$

Hence we have proven (5.9), (5.10). Additionally, from (5.15) and (5.16) we see that (5.11) and (5.12) hold.

Finally, we derive estimate (5.13) for the product $u_{p,v}w_{p,v}$. Because $g^{\pm}_{v,1}g^{\pm}_{v,2} = 0$, see section 3.5.2, and the improved Hölder inequality (Lemma B.1) we have

$$\begin{aligned}
\|w_{p,v}u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} &\lesssim \sum_{k=1}^{2} \left\| g^-_{v,k}(\nu_v\cdot)g^+_{v,k}(\nu_v\cdot)\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \sum_{k=1}^{2} \|g^-_{v,k}g^+_{v,k}\|_{L^1}\left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{L^1(B^{-1}_{1,\infty})} \\
&\quad + \sum_{k=1}^{2} \nu_v^{-1}\|g^-_{v,k}g^+_{v,k}\|_{L^1}\left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{C^1(B^{-1}_{1,\infty})}.
\end{aligned}$$

First, observe that

$$\|g^-_{v,k}g^+_{v,k}\|_{L^1} \leq \|g^-_{v,k}\|_{L^2}\|g^+_{v,k}\|_{L^2} \lesssim \kappa_v^{1/q_2-1/2+1-1/q_2-1/2} = 1, \tag{5.17}$$

according to (3.51), (3.52). Next, we estimate

$$\begin{aligned}
\left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{C^1(B^{-1}_{1,\infty})} &\lesssim \left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{C^1(L^1)} \\
&\lesssim \|R_{v,k}\|_{C^1(L^\infty)}\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)\|_{L^1} \\
&\lesssim C_{R_v}\|\phi_k(\sigma_v\cdot)\|_{L^2}\|W_k(\sigma_v\cdot)\|_{L^2} \\
&\lesssim C_{R_v},
\end{aligned}$$

where C_{R_v} is a constant depending on R_v, and where we used Lemmas 3.11 and A.3. Moreover, we obtain by Lemma 5.3

$$\begin{aligned}
&\left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot)R_{v,k}\right\|_{L^1(B^{-1}_{1,\infty})} \\
&\leq \left\|\left(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right)R_{v,k}\right\|_{L^1(B^{-1}_{1,\infty})} \\
&\quad + \left|\int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right|\|R_v\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \sigma_v^{-1} + \|R_v\|_{L^1(B^{-1}_{1,\infty})}.
\end{aligned}$$

Hence we have shown

$$\|w_{p,v}u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})} + C_{R_v}\left(\sigma_v^{-1} + \nu_v^{-1}\right) \tag{5.18}$$

which implies (5.13) by choosing λ_v sufficiently large (depending on R_v). □

Remark 5.6. As already mentioned in Remark 2.6 we can establish (2.16), (2.17) instead of (2.12), (2.13). To this end we have to replace (4.12) and (4.13) by

$$u_{p,v}(x,t) := -\sum_{k=1}^{2} g^-_{v,k}(\nu_v t)\theta(t)R_{v,k}(x,t)W_k(\sigma_v x)\frac{\|R_h\|_{L^1(L^1)}}{\|R_v\|_{L^\infty(W^{1,\infty})}},$$

$$w_{p,v}(x,t) := \sum_{k=1}^{2} g^{+}_{v,k}(\nu_v t)\theta(t)\phi_k(\sigma_v x)\frac{\|R_v\|_{L^\infty(W^{1,\infty})}}{\|R_h\|_{L^1(L^1)}}.$$

Then (5.11), (5.12) are no longer true. Instead we find

$$\begin{aligned}
\|u_{p,v}\|_{L^{q_3}(H^{s_3})} &\le \frac{\|R_h\|_{L^1(L^1)}}{\|R_v\|_{L^\infty(W^{1,\infty})}} \sum_{k=1}^{2} \|g^{-}_{v,k}(\nu_v\cdot)\|_{L^{q_3}} \|R_v\|_{L^\infty(W^{1,\infty})} \|W_k(\sigma_v\cdot)\|_{H^{s_3}} \\
&\lesssim \|R_h\|_{L^1(L^1)} \kappa_v^{1/q_2-1/q_3} (\sigma_v\mu_v)^{s_3} = \|R_h\|_{L^1(L^1)}, \\
\|u_{p,v}\|_{L^{q_2}(L^2)} &\le \frac{\|R_h\|_{L^1(L^1)}}{\|R_v\|_{L^\infty(W^{1,\infty})}} \sum_{k=1}^{2} \|g^{-}_{v,k}(\nu_v\cdot)\|_{L^{q_2}} \|R_v\|_{L^\infty(L^\infty)} \|W_k(\sigma_v\cdot)\|_{L^2} \\
&\lesssim \|R_h\|_{L^1(L^1)} \kappa_v^{1/q_2-1/q_2} = \|R_h\|_{L^1(L^1)}.
\end{aligned}$$

Further modifications are straightforward.

5.2 Spatial correctors

Lemma 5.7. *The spatial correctors satisfy the following estimates*

$$\begin{aligned}
\|u_{c,h}\|_{L^{q_1}(H^{s_1})} + \|u_{c,h}\|_{L^2(L^\infty)} &\lesssim \lambda_h^{-\gamma_h}, \\
\|u_{c,v}\|_{L^{q_2}(L^\infty)} + \|u_{c,v}\|_{L^{q_3}(H^{s_3})} &\lesssim \lambda_v^{-\gamma_v}.
\end{aligned}$$

Proof. By using Lemmas 3.1, 3.11 and 5.1 as well as estimate (3.51), one gets that

$$\begin{aligned}
\|u_{c,h}\|_{L^{q_1}(H^{s_1})} &\le \sigma_h^{-1} \sum_{k\in\Lambda} \|\nabla a_k\|_{L^{q_1}(W^{1,\infty})} \|\Omega_k(\sigma_h\cdot)\|_{H^{s_1}} \lesssim \sigma_h^{-1} \kappa_h^{1/2-1/q_1} (\sigma_h\mu_h)^{s_1} \mu_h^{-1} \lesssim \lambda_h^{-\gamma_h}, \\
\|u_{c,h}\|_{L^2(L^\infty)} &\le \sigma_h^{-1} \sum_{k\in\Lambda} \|\nabla a_k\|_{L^2(L^\infty)} \|\Omega_k(\sigma_h\cdot)\|_{L^\infty} \lesssim \sigma_h^{-1} \mu_h^{-1/2} \lesssim \lambda_h^{-\gamma_h}, \\
\|u_{c,v}\|_{L^{q_2}(L^\infty)} &\le \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \|g^{-}_{v,k}(\nu_v\cdot)\|_{L^{q_2}} \|\nabla_h R_{v,k}\|_{L^\infty(L^\infty)} \|\Omega_k(\sigma_v\cdot)\|_{L^\infty} \\
&\lesssim \sigma_v^{-1} \mu_v^{-1/2} \lesssim \lambda_v^{-\gamma_v}, \\
\|u_{c,v}\|_{L^{q_3}(H^{s_3})} &\le \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \|g^{-}_{v,k}(\nu_v\cdot)\|_{L^{q_3}} \|\nabla_h R_{v,k}\|_{L^\infty(W^{1,\infty})} \|\Omega_k(\sigma_v\cdot)\|_{H^{s_3}} \\
&\lesssim \sigma_v^{-1} \kappa_v^{1/q_2-1/q_3} (\sigma_v\mu_v)^{s_3} \mu_v^{-1} \lesssim \lambda_v^{-\gamma_v}.
\end{aligned}$$

□

5.3 Temporal correctors

Lemma 5.8. *The temporal correctors satisfy the estimates*

$$\|u_{t,h}\|_{L^\infty(W^{n,\infty})} \lesssim \lambda_h^{-\gamma_h},$$

$$\|u_{t,v}\|_{L^\infty(W^{n,\infty})} \lesssim \lambda_v^{-\gamma_v},$$
$$\|w_{t,v}\|_{L^\infty(W^{n,\infty})} \lesssim \lambda_v^{-\gamma_v},$$

where $n \in \mathbb{N}$ is arbitrary, and the implicit constant may depend on n.

Proof. Using (3.50) and (3.53) we obtain

$$\begin{aligned}
\|u_{t,h}\|_{L^\infty(W^{n,\infty})} &\leq \nu_h^{-1}\|h_h(\nu_h\cdot)\|_{L^\infty} C_{R_h} \lesssim \nu_h^{-1} \leq \lambda_h^{-\gamma_h},\\
\|u_{t,v}\|_{L^\infty(W^{n,\infty})} &\leq \nu_v^{-1}\|h_{v,k}(\nu_v\cdot)\|_{L^\infty} C_{R_v} \lesssim \nu_v^{-1} \lesssim \lambda_v^{-\gamma_v},\\
\|w_{t,v}\|_{L^\infty(W^{n,\infty})} &\leq \nu_v^{-1}\|h_{v,k}(\nu_v\cdot)\|_{L^\infty} C_{R_v} \lesssim \nu_v^{-1} \lesssim \lambda_v^{-\gamma_v}.
\end{aligned}$$

□

5.4 Conclusion

We have already shown in section 4 that $\overline{u}_p$, $\widetilde{u}_p$ and w_p satisfy

$$\nabla_h \cdot (\overline{u}_p + \widetilde{u}_p) + \partial_z w_p = 0,$$

as well as item 1 of Proposition 2.4. Hence $(u+\overline{u}_p+\widetilde{u}_p, w+w_p)$ fulfills (2.3). Additionally, we have shown in section 4 that $\partial_z P = 0$ and hence $p + P$ satisfies (2.2). Moreover, we proved that (2.1) holds. Consequently $(u + \overline{u}_p + \widetilde{u}_p, w + w_p, p + P, R_{h,1}, R_{v,1})$ is indeed a solution of the Euler-Reynolds system (2.1)-(2.3). We also showed in section 4 that $(u + \overline{u}_p + \widetilde{u}_p, w + w_p, p + P, R_{h,1}, R_{v,1})$ is well-prepared for the time interval I and parameter $\tau/2$.

Furthermore, estimates (2.7)-(2.14) of Proposition 2.4 are a simple consequence of Lemmas 5.2, 5.5, 5.7 and 5.8, where one has to choose λ_h, λ_v sufficiently large, depending on R_h and R_v, respectively.

In addition, estimate (2.15) can be derived from Lemmas 5.5, 5.7, 5.8 as well. Indeed, Lemma 5.5 already proves $\|w_{p,v}u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}$. Moreover, from Lemmas A.3, 5.5, 5.7 and 5.8 we obtain

$$\begin{aligned}
\|w_{p,v}(u_{c,v} + u_{t,v})\|_{L^1(B^{-1}_{1,\infty})} &\lesssim \|w_{p,v}(u_{c,v} + u_{t,v})\|_{L^1(L^1)}\\
&\lesssim \|w_{p,v}\|_{L^{q_2'}(L^2)}(\|u_{c,v}\|_{L^{q_2}(L^2)} + \|u_{t,v}\|_{L^{q_2}(L^2)}) \lesssim \lambda_v^{-\gamma_v}.
\end{aligned}$$

Similarly (from the proof of Lemma 5.5 we obtain $\|u_{p,v}\|_{L^{q_2}(L^2)} \lesssim C_{R_h,R_v}$)

$$\|w_{t,v}\widetilde{u}_p\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|w_{t,v}\widetilde{u}_p\|_{L^1(L^1)} \lesssim \|w_{t,v}\|_{L^{q_2'}(L^2)}\|\widetilde{u}_p\|_{L^{q_2}(L^2)} \lesssim \lambda_v^{-\gamma_v}.$$

Finally, Lemmas 5.5, 5.7 and 5.8 yield

$$\begin{aligned}
\|w\widetilde{u}_p + w_p u\|_{L^1(B^{-1}_{1,\infty})} &\lesssim \|w\widetilde{u}_p + w_p u\|_{L^1(L^1)}\\
&\lesssim \|w\|_{L^\infty(L^\infty)}\|\widetilde{u}_p\|_{L^1(L^1)} + \|w_p\|_{L^1(L^1)}\|u\|_{L^\infty(L^\infty)} \lesssim \lambda_v^{-\gamma_v}.
\end{aligned}$$

Hence, if λ_v is chosen sufficiently large, depending on R_v, we obtain (2.15).

6 Estimates on the stress tensor

In order to finish the proof of Proposition 2.4 it remains to show estimates (2.5), (2.6). These two estimates simply follow from Lemmas 6.1, 6.2, 6.3 and 6.4, which we prove in this section, below.

6.1 Oscillation error

6.1.1 Horizontal part

Lemma 6.1. *If λ_h is chosen sufficiently large (depending on R_h), then the horizontal oscillation error satisfies*

$$\|R_{\mathrm{osc},h}\|_{L^1(L^1)} \le \frac{\epsilon}{3}. \tag{6.1}$$

Proof. Using Lemmas 3.1, 3.4, 3.6 and 5.1 we estimate $R_{\mathrm{osc},x,h}$ as follows

$$\begin{aligned}
\|R_{\mathrm{osc},x,h}\|_{L^1(L^1)} &= \left\| \sum_{k\in\Lambda} \mathcal{B}\Big(\nabla_h(a_k^2), W_k(\sigma_h\cdot)\otimes W_k(\sigma_h\cdot) - \int_{\mathbb{T}^2} W_k\otimes W_k \,\mathrm{d}x\Big)\right\|_{L^1(L^1)} \\
&\le \sum_{k\in\Lambda} \|\nabla_h(a_k^2)\|_{L^1(C^1)} \left\| \mathcal{R}_h\Big(W_k(\sigma_h\cdot)\otimes W_k(\sigma_h\cdot) - \int_{\mathbb{T}^2} W_k\otimes W_k\,\mathrm{d}x\Big)\right\|_{L^1} \\
&\le \sigma_h^{-1}\sum_{k\in\Lambda} \|\nabla_h(a_k^2)\|_{L^1(C^1)} \left\| W_k\otimes W_k - \int_{\mathbb{T}^2} W_k\otimes W_k\,\mathrm{d}x\right\|_{L^1} \\
&\lesssim \sigma_h^{-1}\kappa_h^{-1/2}.
\end{aligned}$$

Here we have used that (similar to (5.6))

$$\left\| W_k\otimes W_k - \int_{\mathbb{T}^2} W_k\otimes W_k\,\mathrm{d}x\right\|_{L^1} \lesssim \|W_k\otimes W_k\|_{L^1} + 1 \le \|W_k\|_{L^2}^2 + 1 \lesssim 1, \tag{6.2}$$

according to Lemma 3.11.

Next, we obtain from (3.50)

$$\|R_{\mathrm{osc},t,h}\|_{L^1(L^1)} \le \nu_h^{-1}\|h_h(\nu_h\cdot)\|_{L^\infty}\|\partial_t R_h\|_{L^1(L^1)} \lesssim \nu_h^{-1}.$$

Finally, using Lemma 5.1 and Proposition 3.10 we get

$$\begin{aligned}
\|R_{\mathrm{far}}\|_{L^1(L^1)} &\le \left\| \sum_{k,k'\in\Lambda, k\ne k'} a_k a_{k'} W_k(\sigma_h\cdot)\otimes W_{k'}(\sigma_h\cdot)\right\|_{L^1(L^1)} \\
&\lesssim \sum_{k,k'\in\Lambda, k\ne k'} \|a_k\|_{L^2(L^\infty)}\|a_{k'}\|_{L^2(L^\infty)} \left\| W_k(\sigma_h\cdot)\otimes W_{k'}(\sigma_h\cdot)\right\|_{L^1} \\
&\lesssim \sum_{k,k'\in\Lambda, k\ne k'} \|W_k\otimes W_{k'}\|_{L^1} \lesssim \mu_h^{-1}.
\end{aligned}$$

By choosing λ_h large enough (depending on R_h), we conclude with (6.1). □

6.1.2 Vertical part

Lemma 6.2. *If λ_v is chosen sufficiently large (depending on R_v), then the vertical oscillation error satisfies*

$$\|R_{\text{osc},v}\|_{L^1(B^{-1}_{1,\infty})} \le \frac{\epsilon}{3}. \tag{6.3}$$

Proof. Using (5.17) and Lemma 5.3 we find

$$\begin{aligned}
&\|R_{\text{osc},x,v}\|_{L^1(B^{-1}_{1,\infty})}\\
&\lesssim \sum_{k=1}^{2} \left\| g^-_{v,k}(\nu_v \cdot) g^+_{v,k}(\nu_v \cdot)\right\|_{L^1} \|\theta^2\|_{L^\infty} \left\| R_{v,k} \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k W_k \,\mathrm{d}x \right)\right\|_{L^\infty(B^{-1}_{1,\infty})}\\
&\lesssim \sigma_v^{-1}.
\end{aligned}$$

For the temporal part of the error we obtain by (3.53)

$$\|R_{\text{osc},t,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_{\text{osc},t,v}\|_{L^\infty(L^\infty)} = \nu_v^{-1} \sum_{k=1}^{2} \|h_{v,k}(\nu_v \cdot)\|_{L^\infty} \|\partial_t R_{v,k}\|_{L^\infty(L^\infty)} \lesssim \nu_v^{-1}.$$

Consequently (6.3) follows by choosing λ_v sufficiently large, depending on R_v. □

6.2 Corrector error

Lemma 6.3. *If λ_h and λ_v are sufficiently large (depending on R_h and R_v, respectively), then the corrector errors satisfy the estimates*

$$\|R_{\text{cor},h}\|_{L^1(L^1)} \le \frac{\epsilon}{3}, \tag{6.4}$$

$$\|R_{\text{cor},v}\|_{L^1(B^{-1}_{1,\infty})} \le \frac{\epsilon}{3}. \tag{6.5}$$

Proof. First, we estimate $R_{\text{cor},h}$. Since estimate (3.19) does not hold for $p = 1$, we have to introduce a suitable $r > 1$. Let us fix $1 < r \le 2$ such that $1 - \frac{1}{2}\delta c_v \le \frac{1}{r}$, where $c_v > 0$ is given by Lemma 3.1. More precisely, if $1 - \frac{1}{2}\delta c_v > 0$, we choose $1 < r \le \min\left\{2, \frac{1}{1-\frac{1}{2}\delta c_v}\right\}$ which is possible due to $1 - \frac{1}{2}\delta c_v < 1$. On the other hand, if $1 - \frac{1}{2}\delta c_v \le 0$, we simply take $1 < r \le 2$. Moreover, we set $\frac{1}{\tilde{r}} = \frac{1}{r} - \frac{1}{2}$. Then (similar to (5.14)), we obtain by (3.1), (3.51) and Lemmas 3.1 and 3.11

$$\begin{aligned}
\|u_{p,v}\|_{L^{q_2-}(L^{\tilde{r}})} &\le \frac{1}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \|g^-_{v,k}(\nu_v \cdot)\|_{L^{q_2-}} \|R_v\|_{L^\infty(L^\infty)} \|W_k(\sigma_v \cdot)\|_{L^{\tilde{r}}}\\
&\lesssim \kappa_v^{1/q_2 - 1/q_2 - \delta} \mu_v^{1/2 - 1/\tilde{r}} = \kappa_v^{-\delta} \mu_v^{1-1/r} \lesssim \kappa_v^{-\delta} \mu_v^{\frac{1}{2}\delta c_v} = \kappa_v^{-\delta + \frac{1}{2}\delta} \lesssim \lambda_v^{-\frac{1}{2}\gamma_v}.
\end{aligned} \tag{6.6}$$

Now we are ready to estimate $R_{\mathrm{cor},h}$. Using Lemmas 3.4, 5.2, 5.5, 5.7 and 5.8, and bound (6.6) we get[13]

$$
\begin{aligned}
\|R_{\mathrm{cor},h}\|_{L^1(L^1)} &\lesssim \|R_{\mathrm{cor},h}\|_{L^1(L^r)} \\
&\lesssim \Big\| \big(u_{c,h}+u_{t,h}\big)\otimes\big(u_{c,h}+u_{t,h}\big) + u_{p,h}\otimes\big(u_{c,h}+u_{t,h}\big) + \big(u_{c,h}+u_{t,h}\big)\otimes u_{p,h} \\
&\qquad + \overline{\big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big)}\Big\|_{L^1(L^r)} \\
&\lesssim \|u_{p,h}\|_{L^2(L^2)}\Big(\|u_{c,h}\|_{L^2(L^\infty)} + \|u_{t,h}\|_{L^2(L^\infty)}\Big) + \|u_{c,h}\|^2_{L^2(L^\infty)} + \|u_{t,h}\|^2_{L^2(L^\infty)} \\
&\qquad + \|u_{p,v}\|_{L^2(L^2)}\|u_{p,v}\|_{L^2(L^{\tilde r})} + \|u_{c,v}\|^2_{L^2(L^\infty)} + \|u_{t,v}\|^2_{L^2(L^\infty)} \\
&\lesssim \|R_h\|^{1/2}_{L^1(L^1)}\lambda_h^{-\gamma_h} + \lambda_h^{-2\gamma_h} + \lambda_v^{-\frac{3}{2}\gamma_v} + \lambda_v^{-2\gamma_v},
\end{aligned}
$$

which implies (6.4) as soon as λ_h and λ_v are suitably large (depending on R_h and R_v, respectively).

Finally, according to Lemmas 3.8, 5.2, 5.5, 5.7, 5.8 and A.3

$$
\begin{aligned}
&\|R_{\mathrm{cor},v}\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \Big\| \nabla_h \cdot \Big(\overline{\big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big)} \\
&\qquad\qquad + \big(u_{p,h}+u_{c,h}+u_{t,h}\big)\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big) \\
&\qquad\qquad + \big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes\big(u_{p,h}+u_{c,h}+u_{t,h}\big)\Big)\Big\|_{L^1(B^{-1}_{1,\infty})} \\
&\qquad + \Big\| w_{t,v}\big(u_{p,h}+u_{c,h}+u_{t,h}+u_{p,v}+u_{c,v}+u_{t,v}\big) + w_{p,v}\big(u_{c,v}+u_{t,v}\big)\Big\|_{L^1(L^1)} \\
&\lesssim \Big\| \overline{\big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big)} \\
&\qquad\qquad + \big(u_{p,h}+u_{c,h}+u_{t,h}\big)\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big) \\
&\qquad\qquad + \big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes\big(u_{p,h}+u_{c,h}+u_{t,h}\big)\Big\|_{L^1(L^1)} \\
&\qquad + \Big\| w_{t,v}\big(u_{p,h}+u_{c,h}+u_{t,h}+u_{p,v}+u_{c,v}+u_{t,v}\big) + w_{p,v}\big(u_{c,v}+u_{t,v}\big)\Big\|_{L^1(L^1)} \\
&\lesssim \|u_{p,v}\|^2_{L^2(L^2)} + \|u_{c,v}\|^2_{L^2(L^2)} + \|u_{t,v}\|^2_{L^2(L^2)} \\
&\qquad + \Big(\|u_{p,h}\|_{L^2(L^2)} + \|u_{c,h}\|_{L^2(L^2)} + \|u_{t,h}\|_{L^2(L^2)}\Big)\Big(\|u_{p,v}\|_{L^2(L^2)} + \|u_{c,v}\|_{L^2(L^2)} + \|u_{t,v}\|_{L^2(L^2)}\Big) \\
&\qquad + \|w_{t,v}\|_{L^2(L^2)}\Big(\|u_{p,h}\|_{L^2(L^2)} + \|u_{c,h}\|_{L^2(L^2)} + \|u_{t,h}\|_{L^2(L^2)}\Big) \\
&\qquad + \|w_{t,v}\|_{L^2(L^2)}\Big(\|u_{p,v}\|_{L^2(L^2)} + \|u_{c,v}\|_{L^2(L^2)} + \|u_{t,v}\|_{L^2(L^2)}\Big)
\end{aligned}
$$

[13]To be precise in the following we will use $q_2- > 2$. Note that $q_2 > 2$ by assumption (2.4) and we may assume without loss of generality that δ is small enough such that $q_2- > 2$. Indeed shrinking δ makes the result in Proposition 2.4 stronger.

$$+ \|w_{p,v}\|_{L^{q_2'}(L^2)}\Big(\|u_{c,v}\|_{L^{q_2}(L^2)} + \|u_{t,v}\|_{L^{q_2}(L^2)}\Big)$$
$$\lesssim \lambda_v^{-2\gamma_v} + \big(\|R_h\|_{L^1(L^1)}^{1/2} + \lambda_h^{-\gamma_h}\big)\lambda_v^{-\gamma_v} + \|R_h\|_{L^1(L^1)}\lambda_v^{-\gamma_v}.$$

In these estimates we have used the fact that the time interval $[0,T]$ is finite. Then (6.5) follows by choosing λ_h and λ_v large enough (again depending on R_h and R_v, respectively). □

6.3 Linear error

Lemma 6.4. *If λ_h and λ_v are chosen sufficiently large (depending on R_h and R_v, respectively), then the linear errors satisfy the estimates*

$$\|R_{\text{lin},h}\|_{L^1(L^1)} \leq \frac{\epsilon}{3}, \tag{6.7}$$
$$\|R_{\text{lin},v}\|_{L^1(B^{-1}_{1,\infty})} \leq \frac{\epsilon}{3}. \tag{6.8}$$

In order to prove this Lemma, we consider the time derivative (see section 6.3.1) and advective terms (see section 6.3.2) separately.

Proof of Lemma 6.4. We simply conclude using Lemmas 6.5 and 6.6 below by choosing λ_h and λ_v large enough. □

6.3.1 Time derivative

Lemma 6.5. *For the time derivative part of the linear error, the following bounds hold*

$$\|R_{\text{lin},t,h}\|_{L^1(L^1)} \lesssim \lambda_h^{-\gamma_h}, \tag{6.9}$$
$$\|R_{\text{lin},t,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \lambda_v^{-\gamma_v}. \tag{6.10}$$

Proof. According to (4.8) we have

$$\partial_t(u_{p,h} + u_{c,h}) = \sigma_h^{-1}\sum_{k\in\Lambda}\nabla_h\cdot(\partial_t a_k(x,t)\Omega_k(\sigma_h x)).$$

Using Lemmas 3.1, 3.4, 3.11 and 5.1 we thus find

$$\|R_{\text{lin},t,h}\|_{L^1(L^1)} \lesssim \|\mathcal{R}_h\partial_t(u_{p,h}+u_{c,h})\|_{L^1(L^2)}$$
$$\lesssim \sigma_h^{-1}\sum_{k\in\Lambda}\|\partial_t a_k\|_{L^1(L^\infty)}\|\Omega_k(\sigma_h\cdot)\|_{L^2} \leq \sigma_h^{-1}\nu_h\kappa_h^{1/2}\mu_h^{-1} \lesssim \lambda_h^{-\gamma_h}.$$

Similarly (4.15) implies

$$\partial_t(u_{p,v}+u_{c,v}) = -\frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}}\sum_{k=1}^{2}\nabla_h\cdot\partial_t\big(g_{v,k}^-(\nu_v t)\theta R_{v,k}\Omega_k(\sigma_v x)\big).$$

Hence from Lemmas 3.1, 3.8, 3.11 and A.3, the assumption $q_2 > 2$, and estimate (3.54) we obtain

$$
\begin{aligned}
&\|R_{\text{lin},t,v}\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \Big\| \mathcal{R}_v \nabla_h \cdot \partial_t \big(g^-_{v,k}(\nu_v \cdot) \theta R_{v,k} \Omega_k(\sigma_v \cdot) \big) \Big\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \Big\| \nabla_h \cdot \partial_t \big(g^-_{v,k}(\nu_v \cdot) \theta R_{v,k} \Omega_k(\sigma_v \cdot) \big) \Big\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \Big\| \partial_t \big(g^-_{v,k}(\nu_v \cdot) \theta R_{v,k} \Omega_k(\sigma_v \cdot) \big) \Big\|_{L^1(L^1)} \\
&\lesssim \frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}} \sum_{k=1}^{2} \|g^-_{v,k}(\nu_v \cdot)\|_{W^{1,1}} \|\theta\|_{W^{1,\infty}} \|R_{v,k}\|_{W^{1,\infty}(L^\infty)} \|\Omega_k(\sigma_v \cdot)\|_{L^1} \\
&\lesssim \sigma_v^{-1} \nu_v \kappa_v^{1/q_2} \mu_v^{-3/2} \lesssim \lambda_v^{-\gamma_v}.
\end{aligned}
$$

□

6.3.2 Advective terms

Lemma 6.6. *For the advective part of the linear error, the following bounds hold*

$$
\begin{aligned}
\|R_{\text{lin},x,h}\|_{L^1(L^1)} &\lesssim \lambda_h^{-\gamma_h} + \lambda_v^{-\gamma_v}, \\
\|R_{\text{lin},x,v}\|_{L^1(B^{-1}_{1,\infty})} &\lesssim \lambda_h^{-\gamma_h} + \lambda_v^{-\gamma_v}.
\end{aligned}
$$

Proof. Lemmas 3.4, 5.2, 5.5, 5.7 and 5.8 yield

$$
\begin{aligned}
&\|R_{\text{lin},x,h}\|_{L^1(L^1)} \\
&= \bigg\| \mathcal{R}_h \Big[\nabla_h \cdot \Big(\overline{u} \otimes \big(u_{p,h} + u_{c,h} + u_{t,h}\big) + \overline{u \otimes \big(u_{p,v} + u_{c,v} + u_{t,v}\big)} \\
&\qquad + \big(u_{p,h} + u_{c,h} + u_{t,h}\big) \otimes \overline{u} + \overline{\big(u_{p,v} + u_{c,v} + u_{t,v}\big) \otimes u} \Big) \Big] \bigg\|_{L^1(L^1)} \\
&\lesssim \bigg\| \overline{u} \otimes \big(u_{p,h} + u_{c,h} + u_{t,h}\big) + \overline{u \otimes \big(u_{p,v} + u_{c,v} + u_{t,v}\big)} \\
&\qquad + \big(u_{p,h} + u_{c,h} + u_{t,h}\big) \otimes \overline{u} + \overline{\big(u_{p,v} + u_{c,v} + u_{t,v}\big) \otimes u} \bigg\|_{L^1(L^2)} \\
&\lesssim \|u\|_{L^\infty(L^\infty)} \Big(\|u_{p,h}\|_{L^1(L^2)} + \|u_{c,h}\|_{L^1(L^2)} + \|u_{t,h}\|_{L^1(L^2)} \Big) \\
&\qquad + \|u\|_{L^\infty(L^\infty)} \Big(\|u_{p,v}\|_{L^1(L^2)} + \|u_{c,v}\|_{L^1(L^2)} + \|u_{t,v}\|_{L^1(L^2)} \Big) \\
&\lesssim \lambda_h^{-\gamma_h} + \lambda_v^{-\gamma_v}.
\end{aligned}
$$

For the vertical advective terms we have according to Lemmas 3.8, 5.2, 5.5, 5.7, 5.8 and A.3

$$\begin{aligned}
&\|R_{\mathrm{lin},x,v}\|_{L^1(B^{-1}_{1,\infty})}\\
&= \Bigg\|\mathcal{R}_v\Bigg[\nabla_h\cdot\Bigg(\widetilde{u}\otimes\big(u_{p,h}+u_{c,h}+u_{t,h}\big)+\widetilde{u\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big)}\\
&\qquad\qquad+\big(u_{p,h}+u_{c,h}+u_{t,h}\big)\otimes\widetilde{u}+\widetilde{\big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes u}\Bigg)\\
&\qquad+\partial_z\Bigg(w\big(u_{p,h}+u_{c,h}+u_{t,h}+u_{p,v}+u_{c,v}+u_{t,v}\big)+\big(w_{p,v}+w_{t,v}\big)u\Bigg)\Bigg]\Bigg\|_{L^1(B^{-1}_{1,\infty})}\\
&\lesssim \Bigg\|\widetilde{u}\otimes\big(u_{p,h}+u_{c,h}+u_{t,h}\big)+\widetilde{u\otimes\big(u_{p,v}+u_{c,v}+u_{t,v}\big)}\\
&\qquad\qquad+\big(u_{p,h}+u_{c,h}+u_{t,h}\big)\otimes\widetilde{u}+\widetilde{\big(u_{p,v}+u_{c,v}+u_{t,v}\big)\otimes u}\Bigg\|_{L^1(L^1)}\\
&\qquad+\Bigg\|w\big(u_{p,h}+u_{c,h}+u_{t,h}+u_{p,v}+u_{c,v}+u_{t,v}\big)+\big(w_{p,v}+w_{t,v}\big)u\Bigg\|_{L^1(L^1)}\\
&\lesssim \Big(\|u\|_{L^\infty(L^\infty)}+\|w\|_{L^\infty(L^\infty)}\Big)\Big(\|u_{p,h}\|_{L^1(L^2)}+\|u_{c,h}\|_{L^1(L^2)}+\|u_{t,h}\|_{L^1(L^2)}\Big)\\
&\qquad+\Big(\|u\|_{L^\infty(L^\infty)}+\|w\|_{L^\infty(L^\infty)}\Big)\Big(\|u_{p,v}\|_{L^1(L^2)}+\|u_{c,v}\|_{L^1(L^2)}+\|u_{t,v}\|_{L^1(L^2)}\Big)\\
&\qquad+\|u\|_{L^\infty(L^\infty)}\Big(\|w_{p,v}\|_{L^1(L^1)}+\|w_{t,v}\|_{L^1(L^1)}\Big)\\
&\lesssim \lambda_h^{-\gamma_h}+\lambda_v^{-\gamma_v}.
\end{aligned}$$

□

7 The viscous primitive equations

In this section we consider the viscous primitive equations (1.17)-(1.19). We begin by stating the viscous primitive-Reynolds system

$$\partial_t u-\nu_h^*\Delta_h u-\nu_v^*\partial_{zz}u+u\cdot\nabla_h u+w\partial_z u+\nabla_h p=\nabla_h\cdot R_h+\partial_z R_v, \tag{7.1}$$

$$\partial_z p=0, \tag{7.2}$$

$$\nabla_h\cdot u+\partial_z w=0. \tag{7.3}$$

We prove the following version of Proposition 2.4. Theorem 1.10 can be proven in exactly the same fashion as Theorem 1.3.

Proposition 7.1. *Suppose (u,w,p,R_h,R_v) is a smooth solution of the viscous primitive-Reynolds system (7.1)-(7.3), which is well-prepared with associated time interval I and parameter $\tau>0$. Moreover consider parameters $1\le q_1,q_2,q_3\le\infty$ and $0<s_1,s_3$ which satisfy*

the following constraints[14]

$$q_2 > 2, \quad \frac{2}{q_1} > s_1 + 1, \quad \frac{2}{q_3} > s_3 + \frac{2}{q_2}, \quad s_3 > \frac{1}{2\left(1 - \frac{1}{q_2}\right)}\left(\frac{1}{q_3} - \frac{1}{q_2}\right). \tag{7.4}$$

Finally let $\delta, \epsilon > 0$ be arbitrary. Then there exists another smooth solution $(u + \overline{u}_p + \widetilde{u}_p, w + w_p, p + P, R_{h,1}, R_{v,1})$ of the viscous primitive-Reynolds system (7.1)-(7.3) which is well-prepared with respect to the same time interval I and parameter $\tau/2$, and has the following properties:

1. $(\overline{u}_p, \widetilde{u}_p, w_p)(x,t) = (0,0,0)$ *whenever* $\operatorname{dist}(t, I^c) \le \tau/2$.
2. *The perturbation and Reynolds stress tensors satisfy the following estimates*

$$\|R_{h,1}\|_{L^1(L^1)} \le \epsilon, \tag{7.5}$$
$$\|R_{v,1}\|_{L^1(B^{-1}_{1,\infty})} \le \epsilon, \tag{7.6}$$
$$\|\overline{u}_p\|_{L^1(W^{1,1})} \le \epsilon, \tag{7.7}$$
$$\|\overline{u}_p\|_{L^{q_1}(H^{s_1})} \le \epsilon, \tag{7.8}$$
$$\|\widetilde{u}_p\|_{L^1(W^{1,1})} \le \epsilon, \tag{7.9}$$
$$\|\widetilde{u}_p\|_{L^{q_2-}(L^2)} \le \epsilon, \tag{7.10}$$
$$\|\widetilde{u}_p\|_{L^{q_3-}(H^{s_3})} \le \epsilon, \tag{7.11}$$
$$\|w_p\|_{L^{q_2'-}(L^2)} \le \epsilon, \tag{7.12}$$
$$\|w_p\|_{L^{q_3'-}(H^{-s_3})} \le \epsilon. \tag{7.13}$$

3. *Moreover, we have the following bounds*

$$\|\overline{u}_p\|_{L^2(L^2)} \lesssim \|R_h\|^{1/2}_{L^1(L^1)}, \tag{7.14}$$
$$\|w_p\widetilde{u}_p + w\widetilde{u}_p + w_p u\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{7.15}$$

In order to prove Proposition 7.1, we need the following version of Lemma 3.1.

Lemma 7.2. *Let $1 \le q_1, q_2, q_3 \le \infty$ and $0 < s_1, s_3$ satisfy the conditions (7.4). Then we can choose $a_i, b_i, c_i > 0$ for $i = h, v$ in (3.1) with the property that there exist $\gamma_h, \gamma_v > 0$ such that*

$$\kappa_h^{-1/2}\sigma_h\mu_h^{1/2} \le \lambda_h^{-\gamma_h}, \tag{7.16}$$
$$\kappa_v^{1/q_2-1}\sigma_v\mu_v^{1/2} \le \lambda_v^{-\gamma_v}, \tag{7.17}$$

in addition to (3.3)-(3.7) and $\mu_i, \sigma_i, \kappa_i, \nu_i \ge \lambda_i^{\gamma_i}$ for $i = h, v$.

Proof. Similar to the proof of Lemma 3.1, it suffices to show that there is a choice of $a_i, b_i, c_i > 0$ for $i = h, v$ such that

$$-\left(\frac{1}{2} - \frac{1}{q_1}\right)c_h - s_1(b_h + 1) > 0, \tag{7.18}$$

[14] Again the constraints (7.4) are weaker than (1.20), cf. the footnote in Proposition 2.4.

$$b_h - a_h - \frac{1}{2}c_h + 1 > 0, \tag{7.19}$$

$$-\left(\frac{1}{q_2} - \frac{1}{q_3}\right)c_v - s_3(b_v + 1) = 0, \tag{7.20}$$

$$b_v - a_v - \frac{1}{2}c_v + 1 > 0, \tag{7.21}$$

$$\frac{1}{2}c_h - b_h - \frac{1}{2} > 0, \tag{7.22}$$

$$-\left(\frac{1}{q_2} - 1\right)c_v - b_v - \frac{1}{2} > 0. \tag{7.23}$$

Let us first fix $0 < a_h < 1/2$, and then $b_h > 0$ such that

$$b_h\left(s_1 + 1 - \frac{2}{q_1}\right) < -s_1 + (2 - 2a_h)\left(\frac{1}{q_1} - \frac{1}{2}\right). \tag{7.24}$$

Note that such a choice is possible since $s_1 + 1 - \frac{2}{q_1} < 0$ according to (7.4). Because (7.4) implies $q_1 < 2$, (7.24) is equivalent to

$$\frac{s_1(b_h + 1)}{\frac{1}{q_1} - \frac{1}{2}} < 2b_h + 2 - 2a_h. \tag{7.25}$$

Moreover as $a_h < 1/2$, we have

$$2b_h + 1 < 2b_h + 2 - 2a_h. \tag{7.26}$$

From (7.25) and (7.26) we deduce that there exists $c_h > 0$ with

$$\begin{aligned} \frac{s_1(b_h + 1)}{\frac{1}{q_1} - \frac{1}{2}} &< c_h, \\ 2b_h + 2 - 2a_h &> c_h, \\ 2b_h + 1 &< c_h, \end{aligned}$$

which are equivalent to (7.18), (7.19) and (7.22) respectively.

Next we choose $a_v, b_v, c_v > 0$. We simply deduce from (7.4) that

$$\frac{s_3\left(1 - \frac{1}{q_2}\right)}{\frac{1}{q_3} - \frac{1}{q_2}} > \frac{1}{2}.$$

Thus we can choose $0 < b_v \ll 1$ such that

$$b_v\left(-1 + \frac{s_3\left(1 - \frac{1}{q_2}\right)}{\frac{1}{q_3} - \frac{1}{q_2}}\right) - \frac{1}{2} + \frac{s_3\left(1 - \frac{1}{q_2}\right)}{\frac{1}{q_3} - \frac{1}{q_2}} > 0. \tag{7.27}$$

Then we fix

$$c_v := \frac{s_3\,(b_v + 1)}{\frac{1}{q_3} - \frac{1}{q_2}},$$

which is positive as $\frac{1}{q_3}-\frac{1}{q_2}>0$ which in turn follows from (7.4). The choice of c_v immediately implies (7.20), while (7.27) is equivalent to (7.23). Finally (7.4) ensures

$$(b_v+1)\left(1-\frac{s_3}{2\left(\frac{1}{q_3}-\frac{1}{q_2}\right)}\right)>0.$$

This is equivalent to

$$b_v-\frac{1}{2}c_v+1>0,$$

which in turn allows for the choice of a small $a_v>0$ such that (7.21) holds. □

Now we can prove Proposition 7.1.

Proof of Proposition 7.1. We make the same choice of perturbations $\overline{u}_p$, $\widetilde{u}_p$, w_p, P as we did in the inviscid case, see section 4. The only errors that change compared to the inviscid case are the linear errors, which now contain the additional terms

$$\mathcal{R}_h(\nu_h^*\Delta_h\overline{u}_p),\qquad \mathcal{R}_v\big(\nu_h^*\Delta_h\widetilde{u}_p+\nu_v^*\partial_{zz}(\overline{u}_p+\widetilde{u}_p)\big),\tag{7.28}$$

respectively. Thus the validity of (7.8), (7.10)-(7.15) follows immediately from sections 4-6.

As in the proof of (5.4) one can deduce from (7.16) that

$$\|u_{p,h}\|_{L^1(W^{1,1})}\le\sum_{k\in\Lambda}\|a_k\|_{L^1(W^{1,\infty})}\|W_k(\sigma_h\cdot)\|_{W^{1,1}}\lesssim\kappa_h^{-1/2}\sigma_h\mu_h^{1/2}\le\lambda_h^{-\gamma_h}.$$

Analogously we find

$$\|u_{c,h}\|_{L^1(W^{1,1})}\le\sigma_h^{-1}\sum_{k\in\Lambda}\|\nabla a_k\|_{L^1(W^{1,\infty})}\|\Omega_k(\sigma_h\cdot)\|_{W^{1,1}}\lesssim\sigma_h^{-1}\kappa_h^{-1/2}\sigma_h\mu_h^{-1/2}\lesssim\lambda_h^{-\gamma_h}.$$

Similarly we obtain from (7.17)

$$\begin{aligned}\|u_{p,v}\|_{L^1(W^{1,1})}&\le\frac{1}{\|R_h\|_{L^1(L^1)}}\sum_{k=1}^{2}\|g_{v,k}^-(\nu_v\cdot)\|_{L^1}\|R_v\|_{L^\infty(W^{1,\infty})}\|W_k(\sigma_v\cdot)\|_{W^{1,1}}\\&\lesssim\kappa_v^{1/q_2-1}\sigma_v\mu_v^{1/2}\le\lambda_v^{-\gamma_v}\end{aligned}$$

and

$$\begin{aligned}\|u_{c,v}\|_{L^1(W^{1,1})}&\le\frac{\sigma_v^{-1}}{\|R_h\|_{L^1(L^1)}}\sum_{k=1}^{2}\|g_{v,k}^-(\nu_v\cdot)\|_{L^1}\|\nabla_hR_{v,k}\|_{L^\infty(W^{1,\infty})}\|\Omega_k(\sigma_v\cdot)\|_{W^{1,1}}\\&\lesssim\sigma_v^{-1}\kappa_v^{1/q_2-1}\sigma_v\mu_v^{-1/2}\le\lambda_v^{-\gamma_v}.\end{aligned}$$

Since $\|u_{t,h}\|_{L^1(W^{1,1})}\lesssim\lambda_h^{-\gamma_h}$ and $\|u_{t,v}\|_{L^1(W^{1,1})}\lesssim\lambda_v^{-\gamma_v}$, which follow immediately from Lemma 5.8, we deduce (7.7) and (7.9) by choosing λ_h,λ_v sufficiently large (depending on R_h and R_v, respectively).

In order to show (7.5) and (7.6), it remains to estimate the terms in (7.28). Using Lemmas 3.4, 3.8 and A.3, as well as $\|\overline{u}_p\|_{L^1(W^{1,1})} \lesssim \lambda_h^{-\gamma_h}$ and $\|\widetilde{u}_p\|_{L^1(W^{1,1})} \lesssim \lambda_v^{-\gamma_v}$ which we established above, we find

$$\begin{aligned}\|\mathcal{R}_h(\nu_h^*\Delta_h\overline{u}_p)\|_{L^1(L^1)} &= \nu_h^*\|\nabla_h\overline{u}_p + \nabla_h\overline{u}_p^T\|_{L^1(L^1)} \lesssim \|\overline{u}_p\|_{L^1(W^{1,1})} \lesssim \lambda_h^{-\gamma_h},\\ \|\mathcal{R}_v(\nu_h^*\Delta_h\widetilde{u}_p + \nu_v^*\partial_{zz}(\overline{u}_p + \widetilde{u}_p))\|_{L^1(B^{-1}_{1,\infty})} &\lesssim \|\Delta_h\widetilde{u}_p\|_{L^1(B^{-1}_{1,\infty})} + \|\partial_z(\overline{u}_p + \widetilde{u}_p)\|_{L^1(B^{-1}_{1,\infty})}\\ &\lesssim \|\nabla_h\widetilde{u}_p\|_{L^1(L^1)} + \|\overline{u}_p + \widetilde{u}_p\|_{L^1(L^1)}\\ &\lesssim \|\overline{u}_p\|_{L^1(W^{1,1})} + \|\widetilde{u}_p\|_{L^1(W^{1,1})} \lesssim \lambda_h^{-\gamma_h} + \lambda_v^{-\gamma_v}.\end{aligned}$$

□

8 Two-dimensional hydrostatic Euler equations

In this section we will develop a convex integration scheme for the two-dimensional hydrostatic Euler equations (1.21)-(1.23), which is somewhat different in nature to the scheme in the three-dimensional case. A similar scheme will be established in section 9 for the (two-dimensional) Prandtl equations (1.25)-(1.27).

In two dimensions the hydrostatic Euler-Reynolds system (2.1)-(2.3) reduces to

$$\begin{aligned}\partial_t u + u\partial_{x_1}u + w\partial_z u + \partial_{x_1}p &= \partial_{x_1}R_h + \partial_z R_v,\\ \partial_z p &= 0,\\ \partial_{x_1}u + \partial_z w &= 0.\end{aligned}$$

We observe that in contrast to the three-dimensional case u, R_h and R_v are now just scalar quantities, where R_h does not depend on z and R_v is mean-free with respect to z. The former allows to include R_h as part of the pressure. In other words by setting

$$p' = p - R_h$$

we may assume without loss of generality that $R_h = 0$ (up to a redefinition of the pressure). So all in all the two-dimensional hydrostatic Euler-Reynolds system we will work with, reads

$$\partial_t u + u\partial_{x_1}u + w\partial_z u + \partial_{x_1}p = \partial_z R_v, \tag{8.1}$$

$$\partial_z p = 0, \tag{8.2}$$

$$\partial_{x_1}u + \partial_z w = 0. \tag{8.3}$$

with unknowns u, w, p and R_v. As R_h is no longer there, the only task is to minimize R_v. For this reason we will only have a baroclinic perturbation $\widetilde{u}_p$ in Proposition 8.1 below.

In this section, we will prove the following version of the inductive proposition (cf. Proposition 2.4). Theorem 1.18 then follows exactly as Theorem 1.3.

Proposition 8.1. *Suppose (u, w, p, R_v) is a smooth solution of the two-dimensional hydrostatic Euler-Reynolds system (8.1)-(8.3), which is well-prepared with associated time interval I and parameter $\tau > 0$. Moreover, consider parameters $1 \leq q_2, q_3 \leq \infty$ and $0 < s_3$ which* *satisfy the constraints in* (1.24)*. Finally, let* $\delta, \epsilon > 0$ *be arbitrary. Then there exists another smooth solution* $(u+\widetilde{u}_p, w+w_p, p+P, R_{v,1})$ *of the two-dimensional hydrostatic Euler-Reynolds system* (8.1)-(8.3) *which is well-prepared with respect to the same time interval* I *and parameter* $\tau/2$*, and has the following properties:*

1. $(\widetilde{u}_p, w_p)(x,t) = (0,0)$ *whenever* $\text{dist}(t, I^c) \le \tau/2$.
2. *It satisfies the following estimates*

$$\|R_{v,1}\|_{L^1(B^{-1}_{1,\infty})} \le \epsilon, \tag{8.4}$$

$$\|\widetilde{u}_p\|_{L^{q_2-}(L^2)} \le \epsilon, \tag{8.5}$$

$$\|\widetilde{u}_p\|_{L^{q_3-}(H^{s_3})} \le \epsilon, \tag{8.6}$$

$$\|w_p\|_{L^{q_2'-}(L^2)} \le \epsilon, \tag{8.7}$$

$$\|w_p\|_{L^{q_3'-}(H^{-s_3})} \le \epsilon. \tag{8.8}$$

3. *Moreover, we have that*

$$\|w_p\widetilde{u}_p + w\widetilde{u}_p + w_p u\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{8.9}$$

Remark 8.2. The smooth solution $(u + \widetilde{u}_p, w + w_p, p + P, R_{v,1})$ constructed in the proof of Proposition 8.1 is even well-prepared for the parameter τ (rather than $\tau/2$). Moreover item 1 of Proposition 8.1 is even satisfied whenever $\text{dist}(t, I^c) \le \tau$.

8.1 Preliminaries

Due to the fact that there is no longer an error R_h in the two-dimensional hydrostatic Euler-Reynolds system (8.1)-(8.3), there is no need for a barotropic perturbation $\overline{u}_p$. Thus there is only a baroclinic perturbation $\widetilde{u}_p$ in Proposition 8.1. For this reason only the 'vertical' parameters $\mu_v, \sigma_v, \kappa_v, \nu_v$ are used in sections 8 and 9. Regarding the two-dimensional hydrostatic Euler equations (1.21)-(1.23), we need the following version of Lemma 3.1.

Lemma 8.3. *Let* $1 \le q_2, q_3 \le \infty$ *and* $0 < s_3$ *satisfy the constraints* (1.24)*. Then we can choose* $a_v, b_v, c_v > 0$ *in* (3.1) *with the property that there exists* $\gamma_v > 0$ *such that*

$$\kappa_v^{2/q_2-1}\sigma_v\mu_v \le \lambda_v^{-\gamma_v}, \tag{8.10}$$

$$\sigma_v^{-1}\nu_v\kappa_v^{1/q_2}\mu_v^{-3/2} \le \lambda_v^{-\gamma_v}, \tag{8.11}$$

in addition to (3.5)*,* (3.7) *and* $\mu_v, \sigma_v, \kappa_v, \nu_v \ge \lambda_v^{\gamma_v}$*.*

Proof. Similar to the proof of Lemma 3.1, it suffices to show that there is a choice of $a_v, b_v, c_v > 0$ such that

$$-\left(\frac{1}{q_2} - \frac{1}{q_3}\right)c_v - s_3(b_v + 1) = 0, \tag{8.12}$$

$$b_v - a_v - \frac{1}{q_2} c_v + \frac{3}{2} > 0, \tag{8.13}$$

$$-\left(\frac{2}{q_2} - 1\right) c_v - b_v - 1 > 0. \tag{8.14}$$

We know from (1.24) that

$$\frac{3}{2} - \frac{s_3}{q_2 \left(\frac{1}{q_3} - \frac{1}{q_2}\right)} > 0.$$

Hence we can choose $0 < b_v \ll 1$ such that

$$b_v \left(1 - \frac{s_3}{q_2 \left(\frac{1}{q_3} - \frac{1}{q_2}\right)}\right) + \frac{3}{2} - \frac{s_3}{q_2 \left(\frac{1}{q_3} - \frac{1}{q_2}\right)} > 0.$$

By setting

$$c_v := \frac{s_3 \left(b_v + 1\right)}{\frac{1}{q_3} - \frac{1}{q_2}},$$

the latter implies that

$$b_v - \frac{1}{q_2} c_v + \frac{3}{2} > 0,$$

which in turn allows for the choice of a small $a_v > 0$ such that (8.13) holds. The definition of c_v immediately implies (8.12). Finally (1.24) guarantees that

$$\left(1 - \frac{2}{q_2}\right) \frac{s_3}{\left(\frac{1}{q_3} - \frac{1}{q_2}\right)} - 1 > 0,$$

hence

$$(b_v + 1) \left(\left(1 - \frac{2}{q_2}\right) \frac{s_3}{\left(\frac{1}{q_3} - \frac{1}{q_2}\right)} - 1\right) > 0,$$

which is equivalent to (8.14). □

Remark 8.4. Notice that compared to Lemma 3.1 we have replaced (3.6) by (8.11), where the latter is a weaker restriction than the former. Indeed it is simple to see that (3.6) implies (8.11) provided $q_2 > 2$. Note furthermore that (8.11) suffices to prove Lemma 6.5. Moreover, the additional inequality (8.10) is needed to deal with an additional spatial corrector for the vertical velocity.

Moreover, we need a two-dimensional version of the vertical inverse divergence operator $\mathcal{R}_v$, cf. Definition 3.7.

Definition 8.5. We define the map[15] $\mathcal{R}_v : C^\infty_{0,z}(\mathbb{T}^2;\mathbb{R}) \to C^\infty(\mathbb{T}^2;\mathbb{R})$ by

$$(\mathcal{R}_v v)(x_1, z) := \int_0^z v(x_1, z')\, \mathrm{d}z' - \int_0^1 \int_0^{z'} v(x_1, z'')\, \mathrm{d}z''\, \mathrm{d}z'. \tag{8.15}$$

[15]Again we denote the space of all functions in $C^\infty(\mathbb{T}^2;\mathbb{R})$ which have zero-mean with respect to z by $C^\infty_{0,z}(\mathbb{T}^2;\mathbb{R})$.

Note that the vertical inverse divergence defined in Definition 8.5 has the same properties as stated in Lemma 3.8.

Regarding the building blocks we will use the following version of Proposition 3.9.

Proposition 8.6. *There exists a function $\phi \in C^\infty(\mathbb{T};\mathbb{R})$ (referred to as the Mikado density) depending on a parameter μ_v, with the following properties.*

1. *The function ϕ has zero mean. Moreover $\fint_{\mathbb{T}} \phi^2 \,\mathrm{d}x = 1$.*
2. *There exists $\Omega \in C^\infty(\mathbb{T};\mathbb{R})$ with zero mean such that $\phi = \partial_{x_1}\Omega$.*
3. *For all $s \geq 0$ and $1 \leq p \leq \infty$ the following estimates hold:*

$$\|\phi\|_{W^{s,p}(\mathbb{T})} \lesssim \mu_v^{\frac{1}{2}-\frac{1}{p}+s},$$
$$\|\Omega\|_{W^{s,p}(\mathbb{T})} \lesssim \mu_v^{-\frac{1}{2}-\frac{1}{p}+s}.$$

Here the implicit constant may depend on s, p but it does not depend on μ_v.

Similar to Proposition 3.9, Proposition 8.6 can be proven as in [27, Section 4.1]. In fact, the function ϕ in Proposition 8.6 coincides with the function ϕ_2 from Proposition 3.9.

Analogously to Lemma 3.11 the estimates

$$\|\phi(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_v)^s \mu_v^{\frac{1}{2}-\frac{1}{p}}, \tag{8.16}$$
$$\|\Omega(\sigma\cdot)\|_{W^{s,p}} \lesssim (\sigma\mu_v)^s \mu_v^{-\frac{1}{2}-\frac{1}{p}}, \tag{8.17}$$

hold for all $\sigma \in \mathbb{N}$, $s \geq 0$ and $1 \leq p \leq \infty$.

In what follows we will always write $\phi(x)$, but let us clarify that actually ϕ only depends on x_1.

Finally, we remark that for the two-dimensional convex integration scheme developed in this section, we will work with the same temporal intermittency functions as defined in section 3.5.2.

8.2 Definition of the perturbation

The velocity perturbation will be written as

$$\widetilde{u}_p = u_{p,v} + u_{t,v},$$
$$w_p = w_{p,v} + w_{c,v} + w_{t,v},$$

so in contrast to the three-dimensional case, a spatial corrector $u_{c,v}$ is not needed, however we will have spatial corrector $w_{c,v}$ for the vertical velocity.

We make the following choice for the principal part of the perturbation

$$u_{p,v}(x,t) := -g^{-}_{v,2}(\nu_v t)\theta(t) R_v(x,t)\phi(\sigma_v x), \tag{8.18}$$
$$w_{p,v}(x,t) := g^{+}_{v,2}(\nu_v t)\theta(t)\phi(\sigma_v x). \tag{8.19}$$

Remark 8.7. In order to achieve endpoint time integrability for w (cf. Remark 1.20) one has to multiply the right-hand side of (8.19) by $\|R_v\|_{L^1(B^{-1}_{1,\infty})}$ and divide the right-hand side of (8.18) by the same factor. Further modifications are straightforward. In order to get endpoint time integrability for u one proceeds as described in Remarks 1.4, 2.6 and 5.6.

Then we introduce a corrector for the vertical velocity in order to get a divergence-free perturbation:

$$w_{c,v} := \mathcal{R}_v\Big(g^-_{v,2}(\nu_v t)\theta\partial_{x_1}(R_v\phi(\sigma_v x))\Big), \tag{8.20}$$

where $\mathcal{R}_v$ is now given by Definition 8.5. Note that R_v is mean-free with respect to z and hence the operator $\mathcal{R}_v$ can be applied to the expression in (8.20). With Lemma 3.8 it is simple to see that $\partial_{x_1}u_{p,v} + \partial_z w_{p,v} = 0$.

Finally, we introduce a temporal corrector of the form

$$u_{t,v} := \nu_v^{-1}h_{v,2}(\nu_v t)\partial_z R_v. \tag{8.21}$$

To keep the whole velocity field divergence-free, we now must introduce a temporal corrector for the vertical velocity

$$w_{t,v} := -\nu_v^{-1}h_{v,2}(\nu_v t)\partial_{x_1} R_v. \tag{8.22}$$

We observe that $\partial_{x_1}\widetilde{u}_p + \partial_z w_p = 0$, item 1 of Proposition 8.1 holds and $u_{p,v}$ and $u_{t,v}$ are indeed mean-free with respect to z.

8.3 The new Reynolds stress tensor

As in the three-dimensional scheme, the new Reynolds stress tensor $R_{v,1}$ will be written as

$$R_{v,1} = R_{\mathrm{osc},v} + R_{\mathrm{lin},v} + R_{\mathrm{cor},v}.$$

First, we set $R_{\mathrm{osc},v} = R_{\mathrm{osc},x,v} + R_{\mathrm{osc},t,v}$, where

$$\begin{aligned} R_{\mathrm{osc},x,v} &:= -g^-_{v,2}(\nu_v t)g^+_{v,2}(\nu_v t)\theta^2 R_v\left(\phi^2(\sigma_v x) - \int_{\mathbb{T}}\phi^2(x)\,\mathrm{d}x\right), \\ R_{\mathrm{osc},t,v} &:= \nu_v^{-1}h_{v,2}(\nu_v t)\partial_t R_v. \end{aligned}$$

Exactly as in Lemma 4.3 one can show that

$$\partial_t u_{t,v} + \partial_z(w_{p,v}u_{p,v} + R_v) = \partial_z R_{\mathrm{osc},v}. \tag{8.23}$$

Next, we define the linear and corrector errors by

$$R_{\mathrm{lin},v} := \mathcal{R}_v\left[\partial_t u_{p,v} + 2\partial_{x_1}\left(\widetilde{u\big(u_{p,v} + u_{t,v}\big)}\right) + \partial_z\left(w\big(u_{p,v} + u_{t,v}\big) + \big(w_{p,v} + w_{c,v} + w_{t,v}\big)u\right)\right]$$

and

$$R_{\mathrm{cor},v} := \mathcal{R}_v\left[\partial_{x_1}\left(\widetilde{\big(u_{p,v} + u_{t,v}\big)^2}\right) + \partial_z\left(\big(w_{c,v} + w_{t,v}\big)\big(u_{p,v} + u_{t,v}\big) + w_{p,v}u_{t,v}\right)\right].$$

Note that the arguments of the operator $\mathcal{R}_v$ are indeed mean-free with respect to z.

In the three-dimensional convex integration scheme we introduced a pressure perturbation, see section 4.2. An analogue of this perturbation is not needed in two dimensions as there is no barotropic perturbation. However, the pressure has to absorb some terms that were covered by the horizontal Reynolds stress tensor in the three-dimensional scheme. To this end we define

$$P := -2\overline{u\big(u_{p,v} + u_{t,v}\big)} - \overline{\big(u_{p,v} + u_{t,v}\big)^2}. \tag{8.24}$$

We observe that $\partial_z P = 0$.

Let us finally remark that it is straightforward to see that $R_{v,1}$ is mean-free with respect to z, that $R_{v,1}(x,t) = 0$ whenever $\operatorname{dist}(t, I^c) \leq \frac{\tau}{2}$, and that $(u + \widetilde{u}_p, w + w_p, p + P, R_{v,1})$ solves (8.1).

8.4 Estimates on the perturbation

Now we claim the following estimates on the perturbation.

Lemma 8.8. *If λ_v is chosen sufficiently large (depending on R_v), then we have that*

$$\|u_{p,v}\|_{L^{q_2-}(L^2)} \lesssim \lambda_v^{-\gamma_v}, \tag{8.25}$$

$$\|u_{p,v}\|_{L^{q_3-}(H^{s_3})} \lesssim \lambda_v^{-\gamma_v}, \tag{8.26}$$

$$\|w_{p,v}\|_{L^{q_2'-}(L^2)} \lesssim \lambda_v^{-\gamma_v}, \tag{8.27}$$

$$\|w_{p,v}\|_{L^{q_3'-}(H^{-s_3})} \lesssim \lambda_v^{-\gamma_v}, \tag{8.28}$$

$$\|w_{p,v} u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{8.29}$$

Proof. In fact the proof of (8.25)-(8.28) can be taken verbatim from Lemma 5.5. Estimate (8.29) can also be proven in a similar way as in Lemma 5.5. To this end we need a version of Lemma 5.3, namely the estimate

$$\left\| \left(\phi^2(\sigma_v \cdot) - \int_{\mathbb{T}} \phi^2 \,\mathrm{d}x \right) R_v \right\|_{L^p(B^{-1}_{1,\infty})} \lesssim \sigma_v^{-1}, \tag{8.30}$$

which holds for any $1 \leq p \leq \infty$. To show (8.30), we mimic the proof of Lemma 5.3. This requires to introduce a one-dimensional horizontal inverse divergence operator $\mathcal{R}_h$, which is defined analogously to the vertical inverse divergence, cf. Definition 8.5, specifically

$$\mathcal{R}_h v(x) := \int_0^x v(x') \,\mathrm{d}x' - \int_0^1 \int_0^{x'} v(x'') \,\mathrm{d}x'' \,\mathrm{d}x'.$$

Consequently $\mathcal{R}_h$ has the properties stated in Lemma 3.8. Using property (3.30) we are able to prove (8.30) and thus (8.29) follows. □

The spatial and temporal correctors can be estimated as follows.

Lemma 8.9. *The spatial and temporal correctors satisfy the following estimates*

$$\|w_{c,v}\|_{L^{q_2'}(L^2)} + \|w_{c,v}\|_{L^{q_3'}(H^{-s_3})} \lesssim \lambda_v^{-\gamma_v}, \tag{8.31}$$

$$\|u_{t,v}\|_{L^\infty(W^{n,\infty})} \lesssim \lambda_v^{-\gamma_v}, \tag{8.32}$$

$$\|w_{t,v}\|_{L^\infty(W^{n,\infty})} \lesssim \lambda_v^{-\gamma_v}, \tag{8.33}$$

where $n \in \mathbb{N}$ is arbitrary, and the implicit constant may depend on n.

Proof. We only prove (8.31), as the proof of the estimates for the temporal correctors (8.32), (8.33) is similar to the proof of Lemma 5.8. We obtain using Lemmas 3.8 and 8.3, and equations (3.51), (8.16)

$$\begin{aligned}\|w_{c,v}\|_{L^{q_2'}(L^2)} &\lesssim \|g_{v,2}^-(\nu_v\cdot)\|_{L^{q_2'}} \|\theta\|_{L^\infty} \|R_v\|_{L^\infty(W^{1,\infty})} \|\phi(\sigma_v\cdot)\|_{H^1} \\ &\lesssim \kappa_v^{1/q_2 - 1/q_2'} \sigma_v\mu_v = \kappa_v^{2/q_2-1}\sigma_v\mu_v \le \lambda_v^{-\gamma_v}.\end{aligned}$$

Analogously

$$\begin{aligned}\|w_{c,v}\|_{L^{q_3'}(H^{-s_3})} &\lesssim \|g_{v,2}^-(\nu_v\cdot)\|_{L^{q_3'}} \|\theta\|_{L^\infty} \|R_v\|_{L^\infty(W^{1,\infty})} \|\phi(\sigma_v\cdot)\|_{H^{1-s_3}} \\ &\lesssim \kappa_v^{1/q_2-1/q_3'}(\sigma_v\mu_v)^{1-s_3} = \big(\kappa_v^{2/q_2-1}\sigma_v\mu_v\big)\big(\kappa_v^{1/q_3-1/q_2}(\sigma_v\mu_v)^{-s_3}\big) \le \lambda_v^{-\gamma_v}.\end{aligned}$$

Here we have used that $\mathcal{R}_v$ is bounded in H^{-s_3}, see (3.29) and keeping in mind that $H^{-s_3} = B^{-s_3}_{2,2}$, and $s_3 \le 1$. □

Lemmas 8.8 and 8.9 show that estimates (8.5)-(8.9) hold.

8.5 Estimates on the Reynolds stress tensor

In order to finish the proof of Proposition 8.1, it remains to show (8.4).

Lemma 8.10. *If λ_v is chosen sufficiently large (depending on R_v), then the errors satisfy the following estimates*

$$\|R_{\mathrm{osc},v}\|_{L^1(B^{-1}_{1,\infty})} \le \frac{\epsilon}{3},$$

$$\|R_{\mathrm{cor},v}\|_{L^1(B^{-1}_{1,\infty})} \le \frac{\epsilon}{3},$$

$$\|R_{\mathrm{lin},v}\|_{L^1(B^{-1}_{1,\infty})} \le \frac{\epsilon}{3}.$$

Proof. Lemma 8.10 can be proven similarly to Lemmas 6.2-6.6, with only small modifications: To obtain the required estimate for $R_{\mathrm{osc},x,v}$ one has to use (8.30) rather than Lemma 5.3. Moreover, since in two dimensions there is no spatial corrector $u_{c,v}$, the time derivative part of the linear error $R_{\mathrm{lin},t,v}$ must be estimated slightly differently compared to Lemma 6.5. To this end we write

$$\begin{aligned}u_{p,v} &= -g_{v,2}^-(\nu_v t)\theta R_v \phi(\sigma_v x) \\ &= -\sigma_v^{-1} g_{v,2}^-(\nu_v t)\theta R_v \partial_{x_1}\big[\Omega(\sigma_v x)\big] \\ &= \partial_{x_1}\big[-\sigma_v^{-1} g_{v,2}^-(\nu_v t)\theta R_v \Omega(\sigma_v x)\big] + \sigma_v^{-1} g_{v,2}^-(\nu_v t)\theta(\partial_{x_1} R_v)\Omega(\sigma_v x).\end{aligned}$$

Hence we find (by using inequality (8.11))

$$
\begin{aligned}
&\left\|\mathcal{R}_v\left[\partial_t u_{p,v}\right]\right\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \|\partial_t u_{p,v}\|_{L^1(B^{-1}_{1,\infty})} \\
&\lesssim \left\|\partial_t\left[\sigma_v^{-1} g_{v,2}^-(\nu_v\cdot)\theta R_v \Omega(\sigma_v\cdot)\right]\right\|_{L^1(L^1)} + \left\|\partial_t\left[\sigma_v^{-1} g_{v,2}^-(\nu_v\cdot)\theta (\partial_{x_1} R_v)\Omega(\sigma_v\cdot)\right]\right\|_{L^1(L^1)} \\
&\lesssim \sigma_v^{-1}\|g_{v,2}^-(\nu_v\cdot)\|_{W^{1,1}}\|\theta\|_{W^{1,\infty}}\|R_v\|_{W^{1,\infty}(W^{1,\infty})}\|\Omega(\sigma_v\cdot)\|_{L^1} \\
&\lesssim \sigma_v^{-1}\nu_v\kappa_v^{1/q_2}\mu_v^{-3/2} \leq \lambda_v^{\gamma_v}.
\end{aligned}
$$

□

This finishes the proof of Proposition 8.1.

9 The two-dimensional Prandtl equations

In this section we will study the following Prandtl-Reynolds system

$$\partial_t u - \nu_v^* \partial_{zz} u + u\partial_{x_1} u + w\partial_z u + \partial_{x_1} p = \partial_z R_v, \tag{9.1}$$

$$\partial_z p = 0, \tag{9.2}$$

$$\partial_{x_1} u + \partial_z w = 0, \tag{9.3}$$

with unknowns (u, w, p, R_v). As in section 8 there is no horizontal Reynolds stress tensor R_h. In this setting we have the following version of the inductive proposition (cf. Proposition 2.4). As before, the proof of Theorem 1.23 then works in the same way as the proof of Theorem 1.3.

Proposition 9.1. *Suppose (u, w, p, R_v) is a smooth solution of the Prandtl-Reynolds system (9.1)-(9.3), which is well-prepared with associated time interval I and parameter $\tau > 0$. Moreover, consider parameters $1 \leq q_2, q_3 \leq \infty$ and $0 < s_3$ which satisfy the constraints in (1.28). Finally, let $\delta, \epsilon > 0$ be arbitrary. Then there exists another smooth solution $(u+\widetilde{u}_p, w+w_p, p+P, R_{v,1})$ of the Prandtl-Reynolds system (9.1)-(9.3), which is well-prepared with respect to the same time interval I and parameter $\tau/2$, and has the following properties:*

1. *$(\widetilde{u}_p, w_p)(x,t) = (0,0)$ whenever $\operatorname{dist}(t, I^c) \leq \tau/2$.*
2. *The perturbation and the new Reynolds stress tensor satisfy the following estimates*

$$\|R_{v,1}\|_{L^1(B^{-1}_{1,\infty})} \leq \epsilon, \tag{9.4}$$

$$\|\widetilde{u}_p\|_{L^1(W^{1,1})} \leq \epsilon, \tag{9.5}$$

$$\|\widetilde{u}_p\|_{L^{q_2-}(L^2)} \leq \epsilon, \tag{9.6}$$

$$\|\widetilde{u}_p\|_{L^{q_3-}(H^{s_3})} \leq \epsilon, \tag{9.7}$$

$$\|w_p\|_{L^{q_2'-}(L^2)} \leq \epsilon, \tag{9.8}$$

$$\|w_p\|_{L^{q_3'-}(H^{-s_3})} \leq \epsilon, \tag{9.9}$$

3. Finally, the products of the vertical and horizontal perturbations satisfy that

$$\|w_p\widetilde{u}_p + w\widetilde{u}_p + w_p u\|_{L^1(B^{-1}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}. \tag{9.10}$$

Remark 9.2. Remark 8.2 also holds in the context of the Prandtl equations (1.25)-(1.27).

Proof of Proposition 9.1. In order to prove Proposition 9.1 we modify the proof of Proposition 8.1 in the same way as we did in the three-dimensional case, cf. proof of Proposition 7.1. Specifically we choose $\widetilde{u}_p$, w_p and P as in the proof of Proposition 8.1, while the linear error now contains the additional term

$$\mathcal{R}_v(\nu_v^*\partial_{zz}\widetilde{u}_p). \tag{9.11}$$

Then it follows from section 8 that estimates (9.6)-(9.10) hold. In order to show (9.5) we proceed as in the proof of Proposition 7.1. To this end we need the additional parameter estimate

$$\kappa_v^{1/q_2-1}\sigma_v\mu_v^{1/2} \leq \lambda_v^{-\gamma_v}.$$

We can achieve this as soon as

$$\frac{s_3\left(1-\frac{1}{q_2}\right)}{\frac{1}{q_3}-\frac{1}{q_2}} > \frac{1}{2}, \tag{9.12}$$

see the proof of Lemma 7.2. Using

$$\frac{1}{1-\frac{2}{q_2}} > \frac{1}{2\left(1-\frac{1}{q_2}\right)}$$

we see that (9.12) holds according to (1.28).

It remains to estimate the additional term in (9.11). This works exactly as in the proof of Proposition 7.1. □

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A Littlewood-Paley theory, Besov spaces and paradifferential calculus

In this appendix, we state some basic definitions from Littlewood-Paley theory and paradifferential calculus, which will be used throughout the paper. More details can be found in [3, 55, 70, 83].

A.1 Littlewood-Paley theory and Besov spaces

We first introduce a dyadic partition of unity $\{\rho_j\}_{j=-1}^{\infty}$ as follows

$$\rho_0(\xi) = \rho(\xi), \quad \rho_j(\xi) = \rho(2^{-j}\xi) \text{ for } j = 1, 2, \ldots, \quad \rho_{-1}(\xi) = 1 - \sum_{j=0}^{\infty} \rho_j(\xi).$$

Then for $f \in \mathcal{S}'(\mathbb{T}^3)$ the Littlewood-Paley blocks are given by

$$\widehat{\Delta_j f}(\xi) = \rho_j(\xi)\widehat{f}(\xi), \quad j = -1, 0, \ldots.$$

The Besov space $B^s_{p,q}(\mathbb{T}^3)$ is then defined in terms of the Littlewood-Paley based norm, which reads for $q < \infty$

$$\|f\|_{B^s_{p,q}} := \|\Delta_{-1} f\|_{L^p} + \left(\sum_{j=0}^{\infty} 2^{sjq} \|\Delta_j f\|^q_{L^p} \right)^{1/q}.$$

If $q = \infty$, the norm is defined as follows

$$\|f\|_{B^s_{p,\infty}} := \|\Delta_{-1} f\|_{L^p} + \sup_{j \geq 0} \left(2^{sj} \|\Delta_j f\|_{L^p} \right).$$

It is also possible to define Besov spaces using difference quotients. We first define the forward finite difference operator

$$\Delta^1_h f(x) = f(x + h) - f(x).$$

The higher order finite differences are defined inductively. For $m \geq 2$, we define that

$$\Delta^m_h f(x) = \Delta^1_h (\Delta^{m-1}_h f(x)).$$

Let $s > 0$ and $1 \leq p, q \leq \infty$ and let $\lfloor s \rfloor$ denote the integer part of s, then Besov norm may be defined as follows (if $q < \infty$)

$$\|f\|_{B^s_{p,q}} = \|f\|_{L^p} + \left(\int_{\mathbb{T}^3} \|\Delta^{\lfloor s \rfloor + 1}_h f\|^q_{L^p} \frac{\mathrm{d}h}{|h|^{3+sq}} \right)^{1/q}.$$

If $q = \infty$, the Besov norm is given by

$$\|f\|_{B^s_{p,\infty}} := \|f\|_{L^p} + \sup_{h \in \mathbb{R}^3 \setminus \{0\}} \frac{\|\Delta^{\lfloor s \rfloor + 1}_h f\|_{L^p}}{|h|^s}.$$

These two different definitions of the Besov norm are equivalent.

Remark A.1. Note that the index h in the notation for the finite differences Δ_h^m represents the shift. This is in contrast to the main body of this paper where the index h always means "horizontal".

Remark A.2. The reader should note that $B^s_{p,p}(\mathbb{T}^3) = W^{s,p}(\mathbb{T}^3)$ when $s \in \mathbb{R}\backslash\mathbb{Z}$ and $1 \le p \le \infty$. This is stated in [1, Equation 3.5] for example. The equivalence of function spaces in the case $0 < s < 1$ can also be found in [88, Definition 32.2] (when the interpolation space definition of Besov spaces is used) and in [86, Proposition 2] and [8, Page 1686] (where the Sobolev spaces are defined using the Sobolev-Slobodeckij seminorm). We note that the case $0 < s < 1$ can easily be extended to all $s > 0$ with $s \notin \mathbb{N}$ by using Theorem 2.3 in [83]. Finally, we recall that $B^s_{2,2}(\mathbb{T}^3) = H^s(\mathbb{T}^3)$ for all $s \in \mathbb{R}$ (even when s is an integer), see [3, Page 99]. The latter is used several times in this paper.

Next we recall some essential estimates regarding the Besov norm.

Lemma A.3. *For any* $1 \le p, q, q_1, q_2 \le \infty$, $\alpha \in \mathbb{R}$, $\delta > 0$ *the following estimates hold*

$$\|f\|_{B^0_{p,\infty}} \lesssim \|f\|_{L^p} \lesssim \|f\|_{B^0_{p,1}}, \tag{A.1}$$

$$\|f\|_{B^\alpha_{p,q_1}} \lesssim \|f\|_{B^{\alpha+\delta}_{p,q_2}}, \tag{A.2}$$

$$\|f\|_{B^\alpha_{p,q_1}} \lesssim \|f\|_{B^\alpha_{p,q_2}}, \qquad \text{if } q_1 \ge q_2 \tag{A.3}$$

$$\|\partial_i f\|_{B^{\alpha-1}_{p,q}} \lesssim \|f\|_{B^\alpha_{p,q}}. \tag{A.4}$$

Proof. Estimates (A.1), (A.2), (A.3) and (A.4) can be found in [83] Propositions 2.1, 2.2, 2.3 and Theorem 2.2, respectively. □

Note that (A.2) implies that

$$\|f\|_{B^\alpha_{p,q}} \lesssim \|f\|_{B^\beta_{p,q}}$$

for any $\alpha \le \beta$.

A.2 Paradifferential calculus

We recall Bony's product decomposition

$$fg = T_f g + T_g f + R(f,g). \tag{A.5}$$

The terms $T_f g$ and $T_g f$ are called paraproducts and are given by

$$T_f g = \sum_{j=-1}^{\infty} \sum_{i=-1}^{j-2} \Delta_i f \Delta_j g, \quad T_g f = \sum_{j=-1}^{\infty} \sum_{i=-1}^{j-2} \Delta_i g \Delta_j f. \tag{A.6}$$

The term $R(f,g)$ is referred to as the resonance term and is defined as

$$R(f,g) = \sum_{|k-j|\le 1} \Delta_k f \Delta_j g. \tag{A.7}$$

We will also use the notation $T(f,g) \coloneqq T_f(g)$ and $T(g,f) \coloneqq T_g(f)$.

One can estimate the three terms of the product decomposition separately. For the paraproducts we have the following estimates.

Lemma A.4 (Lemma 2.1 in [79]). *Let $\alpha, \beta \in \mathbb{R}$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

- *For any $f \in L^{p_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q}(\mathbb{T}^3)$ we have that*

$$\|T_f(g)\|_{B^{\beta}_{p,q}} \lesssim \|f\|_{L^{p_1}} \|g\|_{B^{\beta}_{p_2,q}}.$$

- *If $\alpha < 0$ then for any $f \in B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$ we have*

$$\|T_f(g)\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}.$$

We recall the following estimate on the resonance term.

Lemma A.5 (Theorem 2.85 in [3]). *Let $\alpha, \beta \in \mathbb{R}$ and $1 \le p, p_1, p_2, q, q_1, q_2 \le \infty$ with*

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}, \qquad \frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}.$$

- *If $\alpha + \beta > 0$ then we have for any $f \in B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$*

$$\|R(f,g)\|_{B^{\alpha+\beta}_{p,q}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}.$$

- *If $\alpha + \beta = 0$ and $q = 1$ then we have for any $f \in B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$*

$$\|R(f,g)\|_{B^{0}_{p,\infty}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}.$$

Combining these estimates leads to the following result.

Lemma A.6. *Let $\alpha < 0 < \beta$, $\beta + \alpha > 0$, and $1 \le p_1, p_2, p, q_1, q_2 \le \infty$ with*

$$\frac{1}{p_1} + \frac{1}{p_2} = \frac{1}{p}.$$

- *We have for any $f \in B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g \in B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$*

$$\|fg\|_{B^{\alpha}_{p,q_1}} \lesssim \|f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}}. \tag{A.8}$$

- *Let $\{f_n\}, \{g_n\}$ be sequences such that $f_n \to f$ in $B^{\alpha}_{p_1,q_1}(\mathbb{T}^3)$ and $g_n \to g$ in $B^{\beta}_{p_2,q_2}(\mathbb{T}^3)$. Then $f_n g_n \to fg$ in B^{α}_{p,q_1}.*

Proof. Estimate (A.8) is a simple consequence of Lemmas A.4 and A.5, see also [70, Prop. A.7] for a similar proof.

The convergence claimed in the second bullet point can be easily deduced from (A.8). Indeed we have

$$\begin{aligned}\|f_n g_n - fg\|_{B^{\alpha}_{p,q_1}} &\lesssim \|f_n(g_n - g)\|_{B^{\alpha}_{p,q_1}} + \|(f_n - f)g\|_{B^{\alpha}_{p,q_1}} \\ &\lesssim \|f_n\|_{B^{\alpha}_{p_1,q_1}} \|g_n - g\|_{B^{\beta}_{p_2,q_2}} + \|f_n - f\|_{B^{\alpha}_{p_1,q_1}} \|g\|_{B^{\beta}_{p_2,q_2}} \to 0.\end{aligned}$$

□

A.3 Alternative proof of Lemma 5.3

In this section we present an alternative proof of Lemma 5.3 using a paraproduct decomposition and Lemmas A.4 and A.5.

Alternative Proof of Lemma 5.3. We start by decomposing the term under consideration in terms of Bony's decomposition, i.e.

$$\begin{aligned}
&\left\| \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right) R_{v,k} \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\leq \left\| T\left(\left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right), R_{v,k} \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\quad + \left\| T\left(R_{v,k}, \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right) \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\quad + \left\| R\left(\left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right), R_{v,k} \right) \right\|_{L^p(B^{-1}_{1,\infty})} .
\end{aligned}$$

Using Lemma A.4 we find

$$\begin{aligned}
&\left\| T\left(\left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right), R_{v,k} \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\lesssim \left\| \phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right\|_{B^{-1}_{1,\infty}} \| R_{v,k} \|_{L^p(B^0_{\infty,\infty})} .
\end{aligned} \tag{A.9}$$

Another application of Lemma A.4 yields

$$\begin{aligned}
&\left\| T\left(R_{v,k}, \left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right) \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\lesssim \left\| \phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right\|_{B^{-1}_{1,\infty}} \| R_{v,k} \|_{L^p(L^\infty)} .
\end{aligned}$$

Finally Lemmas A.3 and A.5 lead to

$$\begin{aligned}
&\left\| R\left(\left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right), R_{v,k} \right) \right\|_{L^p(B^{-1}_{1,\infty})} \\
&\lesssim \left\| R\left(\left(\phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right), R_{v,k} \right) \right\|_{L^p(B^{s}_{1,\infty})} \\
&\lesssim \left\| \phi_k(\sigma_v \cdot) W_k(\sigma_v \cdot) - \int_{\mathbb{T}^2} \phi_k(x) W_k(x) \,\mathrm{d}x \right\|_{B^{-1}_{1,\infty}} \| R_{v,k} \|_{L^p(B^{1+s}_{\infty,\infty})}
\end{aligned}$$

where $0 < s \ll 1$ can be chosen arbitrary.

To conclude we proceed similar to the original proof. Due Lemmas 3.4 and A.3, and estimate (5.6)

$$\begin{aligned}
&\left\|\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right\|_{B^{-1}_{1,\infty}} \\
&= \left\|\nabla_h\cdot\left[\mathcal{R}_h\left(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right)\right]\right\|_{B^{-1}_{1,\infty}} \\
&\lesssim \left\|\mathcal{R}_h\left(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right)\right\|_{B^{0}_{1,\infty}} \\
&\lesssim \left\|\mathcal{R}_h\left(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right)\right\|_{L^1} \\
&\lesssim \sigma_v^{-1}\left\|\phi_kW_k - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\right\|_{L^1} \lesssim \sigma_v^{-1}.
\end{aligned}$$

This finishes the proof of Lemma 5.3. □

B Other estimates

B.1 Improved Hölder inequality

Let us recall the following estimate from [68].

Lemma B.1 (Improved Hölder inequality)**.** *For any $\sigma\in\mathbb{N}$, $1\le p\le\infty$ and all functions $f\in C^1(\mathbb{T}^d)$, $g\in L^p(\mathbb{T}^d)$ it holds that*

$$\Big|\|f(\cdot)g(\sigma\cdot)\|_{L^p} - \|f\|_{L^p}\|g\|_{L^p}\Big| \lesssim \sigma^{-1/p}\|f\|_{C^1}\|g\|_{L^p}. \tag{B.1}$$

The proof can be found in [68, Lemma 2.1].

B.2 Oscillatory paraproduct estimate

We are going to prove a version of Lemma B.1 in the case of Besov spaces.

Lemma B.2. *For any $\sigma\in\mathbb{N}$, $1\le p,q\le\infty$, $s\in\mathbb{R}$, $0<\epsilon\le 1$ and all functions $f\in L^p(\mathbb{T}^3)$, $g\in B^{s+\epsilon}_{p,q}(\mathbb{T}^3)\cap B^{s+1+\epsilon}_{\infty,q}(\mathbb{T}^3)$ it holds that*

$$\|T(f_\sigma,g)\|_{B^s_{p,q}} \lesssim \|f\|_{L^p}\|g\|_{B^{s+\epsilon}_{p,q}} + \sigma^{-1/p}\|f\|_{L^p}\|g\|_{B^{s+1+\epsilon}_{\infty,q}}, \tag{B.2}$$

where we define $f_\sigma(x) \coloneqq f(\sigma x)$.

Proof. We first recall the low-frequency cut-off operator S_j from [3]:

$$S_jf \coloneqq \sum_{i=-1}^{j-1}\Delta_i f.$$

Hence we may write $T(f_\sigma, g) = \sum_{j=-1}^{\infty} S_{j-1}f_\sigma \Delta_j g = \sum_{j=1}^{\infty} S_{j-1}f_\sigma \Delta_j g$, where the latter equation follows from the fact that $S_{-2}f_\sigma = S_{-1}f_\sigma = 0$. In order to estimate the Besov norm of $T(f_\sigma, g)$, we will use [3, Lemma 2.69]. To be able to use this lemma we need to show that $\operatorname{supp}\big(\mathcal{F}(S_{j-1}f_\sigma \Delta_j g)\big)$ lies in $2^j\mathcal{C}$ for any $j \in \mathbb{N}_0$, where $\mathcal{C}$ is a fixed annulus.

In order to show this, let $j \in \mathbb{N}$ and $i \in \{-1, ..., j-2\}$. By construction of the dyadic partition of unity, there exist radii $0 < r < r_0 < R$ such that the support of ρ_{-1} is contained in the ball with radius r_0, the support of ρ_0 is contained in the annulus with inner radius r and outer radius R, and $r_0 < 2r < R < 4r$. Next we observe

$$\operatorname{supp}\big(\mathcal{F}(\Delta_i f_\sigma \Delta_j g)\big) = \operatorname{supp}\big((\rho_i \widehat{f_\sigma}) * (\rho_j \widehat{g})\big).$$

For $i = -1$ this yields that $\operatorname{supp}\big(\mathcal{F}(\Delta_i f_\sigma \Delta_j g)\big)$ is contained in the annulus with inner radius $2^j(r - 2^{-j}r_0)$ and outer radius $2^j(R + 2^{-j}r_0)$, which is in turn a subset of the annulus with inner and outer radii $2^j(r - \frac{1}{2}r_0)$ and $2^j(R + \frac{1}{2}r_0)$. Note that the inner radius is positive due to $2r > r_0$. Similarly for $i \geq 0$ we obtain that $\operatorname{supp}\big(\mathcal{F}(\Delta_i f_\sigma \Delta_j g)\big)$ is contained in the annulus with inner radius $2^j(r - 2^{i-j}R)$ and outer radius $2^j(R + 2^{i-j}R)$, which is in turn subset of the annulus with inner and outer radii $2^j(r - \frac{1}{4}R)$ and $2^j(R + \frac{1}{4}R)$. Again note that $4r > R$ implies that the inner radius is positive. Hence there exists an annulus $\mathcal{C}$ such that

$$\operatorname{supp}\big(\mathcal{F}(S_{j-1}f_\sigma \Delta_j g)\big) \subset 2^j\mathcal{C}.$$

Hence we may apply Lemma 2.69 from [3] to conclude that

$$\|T(f_\sigma, g)\|_{B^s_{p,q}} \lesssim \Big\| \big(2^{js}\|S_{j-1}f_\sigma \Delta_j g\|_{L^p}\big)_{j\in\mathbb{N}} \Big\|_{l^q(\mathbb{N})}. \tag{B.3}$$

In order to estimate $\|S_{j-1}f_\sigma \Delta_j g\|_{L^p}$ we define $\phi_i := \mathcal{F}^{-1}\rho_i$. Hence we have $\Delta_i f = \phi_i * f$. A direct computation shows $\|\phi_i\|_{L^1} = \|\phi_0\|_{L^1}$ for any $i \in \mathbb{N}$. Hence we obtain for $1 \leq p < \infty$ and for any i, j by Minkowski's integral inequality and Lemma B.1

$$\begin{aligned}
\|\Delta_i f_\sigma \Delta_j g\|_{L^p} &= \bigg(\int_{\mathbb{R}^n} \bigg| \Delta_j g(x) \bigg(\int_{\mathbb{R}^n} \phi_i(y) f_\sigma(x-y)\,\mathrm{d}y \bigg) \bigg|^p \mathrm{d}x \bigg)^{1/p} \\
&\leq \int_{\mathbb{R}^n} \bigg(\int_{\mathbb{R}^n} |f_\sigma(x-y)\phi_i(y)\Delta_j g(x)|^p \,\mathrm{d}x \bigg)^{1/p} \mathrm{d}y \\
&= \int_{\mathbb{R}^n} |\phi_i(y)| \|f(\cdot\sigma - \sigma y)\Delta_j g(\cdot)\|_{L^p} \,\mathrm{d}y \\
&\lesssim \int_{\mathbb{R}^n} |\phi_i(y)| \Big(\|f(\cdot - \sigma y)\|_{L^p} \|\Delta_j g\|_{L^p} + \sigma^{-1/p} \|\Delta_j g\|_{C^1} \|f(\cdot - \sigma y)\|_{L^p} \Big) \mathrm{d}y \\
&\leq \|\phi_0\|_{L^1} \Big(\|f\|_{L^p} \|\Delta_j g\|_{L^p} + \sigma^{-1/p} \|\Delta_j g\|_{C^1} \|f\|_{L^p} \Big) \\
&\lesssim \|f\|_{L^p} \|\Delta_j g\|_{L^p} + \sigma^{-1/p} \|\Delta_j g\|_{C^1} \|f\|_{L^p}.
\end{aligned}$$

For the case $p = \infty$ one obtains the same result, the details are left to the reader. Hence

$$\|S_{j-1}f_\sigma \Delta_j g\|_{L^p} \leq \sum_{i=-1}^{j-2} \|\Delta_i f_\sigma \Delta_j g\|_{L^p}$$

$$\lesssim j\Big(\|f\|_{L^p}\|\Delta_j g\|_{L^p} + \sigma^{-1/p}\|\Delta_j g\|_{C^1}\|f\|_{L^p}\Big) \tag{B.4}$$

for any $j \in \mathbb{N}$. Combining (B.3) and (B.4) we get for any $1 \le q < \infty$

$$\begin{aligned}
&\|T(f_\sigma, g)\|_{B^s_{p,q}} \\
&\lesssim \Big(\sum_{j=1}^{\infty} 2^{jsq}\|S_{j-1}f_\sigma \Delta_j g\|^q_{L^p}\Big)^{1/q} \\
&\lesssim \Big(\sum_{j=1}^{\infty} 2^{jsq}j^q\Big(\|f\|_{L^p}\|\Delta_j g\|_{L^p} + \sigma^{-1/p}\|\Delta_j g\|_{C^1}\|f\|_{L^p}\Big)^q\Big)^{1/q} \\
&\lesssim \|f\|_{L^p}\Big(\sum_{j=1}^{\infty} 2^{j(s+\epsilon)q}j^q 2^{-j\epsilon q}\|\Delta_j g\|^q_{L^p}\Big)^{1/q} + \sigma^{-1/p}\|f\|_{L^p}\Big(\sum_{j=1}^{\infty} 2^{j(s+\epsilon)q}j^q 2^{-j\epsilon q}\|\Delta_j g\|^q_{C^1}\Big)^{1/q} \\
&\lesssim \|f\|_{L^p}\|g\|_{B^{s+\epsilon}_{p,q}} + \sigma^{-1/p}\|f\|_{L^p}\|g\|_{B^{s+1+\epsilon}_{\infty,q}},
\end{aligned}$$

where we have used that $\|\Delta_j g\|_{C^1} \le \|\nabla\Delta_j g\|_{L^\infty} + \|\Delta_j g\|_{L^\infty}$, $\nabla\Delta_j g = \Delta_j \nabla g$ and Lemma A.3. For the case $q = \infty$ we proceed analogously. □

Lemma B.2 can be used to prove the following slightly weaker version of Lemma 5.3.

Lemma B.3. *For all $1 \le p \le \infty$, $\delta > 0$ and $k \in \{1, 2\}$ we have*

$$\Big\|\Big(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\Big)R_{v,k}\Big\|_{L^p(B^{-1-\delta}_{1,\infty})} \lesssim \sigma_v^{-1} + \|R_v\|_{L^p(B^{-1}_{1,\infty})}. \tag{B.5}$$

Proof. Compared to the proof presented in section A.3 the only difference is how we handle the paraproduct in (A.9). We use Lemma B.2 and estimate (5.6) to find

$$\begin{aligned}
&\Big\|T\Big(\Big(\phi_k(\sigma_v\cdot)W_k(\sigma_v\cdot) - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\Big), R_{v,k}\Big)\Big\|_{L^p(B^{-1-\delta}_{1,\infty})} \\
&\lesssim \Big\|\phi_k W_k - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\Big\|_{L^1}\|R_v\|_{L^p(B^{-1}_{1,\infty})} \\
&\qquad + \sigma_v^{-1}\Big\|\phi_k W_k - \int_{\mathbb{T}^2}\phi_k(x)W_k(x)\,\mathrm{d}x\Big\|_{L^1}\|R_v\|_{L^p(B^0_{\infty,\infty})} \\
&\lesssim \|R_v\|_{L^p(B^{-1}_{1,\infty})} + C_{R_v}\sigma_v^{-1}.
\end{aligned}$$

□

With Lemma B.3 one can show

$$\|w_{p,v}u_{p,v}\|_{L^1(B^{-1-\delta}_{1,\infty})} \lesssim \|R_v\|_{L^1(B^{-1}_{1,\infty})}$$

which is slightly weaker than (5.13). This finally allows to prove a version of Theorem 1.3 where the regularity parameter is $s = 1+$.

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