Hölder regularity of the pressure for weak solutions of the 3D Euler equations in bounded domains

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Abstract

We consider the three-dimensional incompressible Euler equations on a bounded domain Ω with C^3 boundary. We prove that if the velocity field $u \in C^{0,\alpha}(\Omega)$ with $\alpha > 0$ (where we are omitting the time dependence), it follows that the pressure $p \in C^{0,\alpha}(\Omega)$. In order to prove this result we use a local parametrisation of the boundary and a very weak formulation of the boundary condition for the pressure, as was introduced in [C. Bardos and E.S. Titi, Philos. Trans. Royal Soc. A, 380 (2022), 20210073]. Moreover, we provide an example illustrating the necessity of this new very weak formulation of the boundary condition for the pressure. This result is of importance for the proof of the first half of the Onsager Conjecture, the sufficient conditions for energy conservation of weak solutions to the three-dimensional incompressible Euler equations in bounded domains.

Keywords: Incompressible Euler equations, Onsager's conjecture, pressure regularity, boundary regularity, weak formulation

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1 Introduction

Since the introduction of techniques from convex integration to fluid mechanics, especially the incompressible Euler equations, it is known that there exist weak solutions of the Euler equations [15, 17]. The existence of weak solutions of the general Cauchy problem was first proven in [38]. Since then, the techniques of convex integration have been steadily improved

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to construct Hölder continuous solutions of the Euler equations in a sequence of papers [4, 5, 13, 16, 27] (and see references therein). Reviews of these techniques can be found in [6, 14].

Since it is possible to construct weak solutions of the Euler equations with compact support by using convex integration, it is trivial to construct weak solutions on bounded domains. However, such solutions have a trivial 'interaction' with the boundary, since they are zero near the boundary. Because boundary effects play an important role in the understanding of turbulence [3, 22], it is worthwhile to try to understand the interaction of weak solutions with the boundary.

One essential element to that is the pressure. In this work, we consider the three-dimensional Euler equations of an ideal incompressible fluid in a C^3 bounded domain $\Omega \subset \mathbb{R}^3$, which are

$$\partial_t u + \nabla \cdot (u \otimes u) + \nabla p = 0, \quad \nabla \cdot u = 0, \tag{1.1}$$

where $u : \Omega \times (0,T) \to \mathbb{R}^3$ is the velocity field and $p : \Omega \times (0,T) \to \mathbb{R}$ is the pressure. In addition, we assume the following boundary condition

$$(u \cdot n)|_{\partial\Omega} = 0, \tag{1.2}$$

where n is the outward normal vector to the boundary.

In standard treatments of mathematical fluid mechanics, such as [33], the pressure is removed by using the Leray-Helmholtz decomposition. The pressure can then be recovered via the equation

$$-\Delta p = (\nabla \otimes \nabla) : (u \otimes u). \tag{1.3}$$

The goal of this paper is to prove that in a C^3 bounded domain $\Omega \subset \mathbb{R}^3$ for velocity fields with $u \in C^{0,\alpha}(\Omega)$ for $\alpha \in (0,1)$, the pressure $p \in C^{0,\alpha}(\Omega)$ with the same exponent α.

This type of pressure regularity problem, to the knowledge of the authors, was first considered in [10, 36]. It was proven in these papers that if $u \in C^{0,\alpha}(\mathbb{R}^n)$, then $p \in C^{0,2\alpha}(\mathbb{R}^n)$. This result was then extended to Besov spaces in [9] (see also [8]).

If equation (1.3) is considered in the presence of boundaries, formally one can take the normal component of the Euler equations in order to find

$$\partial_t (u \cdot n) + \big[\nabla \cdot (u \otimes u)\big] \cdot n + \partial_n p = 0.$$

Following [3], for a C^2 bounded domain the advective term can be written formally as follows (where we are using the Einstein summation convention)

$$\big[\nabla \cdot (u \otimes u)\big] \cdot n = u_i \partial_i u_j n_j = -\big(u \otimes u\big) : \nabla n + u \cdot \nabla (u \cdot n).$$

We now calculate the second term in the above expression. To fix ideas, we do the calculation and highlight the issue in two dimensions (but the three-dimensional case works the same way), see also [3]. It should be stressed that these computations are done in the context of smooth solutions. We will formally derive the boundary condition that will be rigorously justified in this paper. In what follows τ will denote the tangent vector to the boundary in the two-dimensional case, we compute that

$$\begin{aligned} u \cdot \nabla (u \cdot n) &= (u \cdot \tau) \partial_\tau (u \cdot n) + (u \cdot n) \partial_n (u \cdot n) \\ &= \frac{1}{2} \partial_n (u \cdot n)^2 + \partial_\tau \big((u \cdot \tau)(u \cdot n)\big) - (u \cdot n) \partial_\tau (u \cdot \tau) = \partial_n (u \cdot n)^2 + \partial_\tau \big((u \cdot \tau)(u \cdot n)\big), \end{aligned}$$

where for the last equality we have used the incompressibility of the velocity field, i.e. that $\partial_\tau(u\cdot\tau) = -\partial_n(u\cdot n)$. Therefore we can write the normal component of the Euler equations at the boundary as follows

$$\partial_n p = \big(u\otimes u\big) : \nabla n - \partial_n(u\cdot n)^2 - \partial_\tau\big((u\cdot\tau)(u\cdot n)\big) - \partial_t(u\cdot n).$$

Since $\partial_t(u\cdot n)|_{\partial\Omega}= 0$ and $\partial_\tau\big((u\cdot\tau)(u\cdot n)\big)|_{\partial\Omega}= 0$, we conclude that the boundary condition associated with equation (1.3) is

$$\partial_n\big(p + (u\cdot n)^2\big) = \big(u\otimes u\big) : \nabla n. \tag{1.4}$$

In particular, one can easily show that if $u\in C^{0,\alpha}(\Omega)$ with $\alpha > \frac{1}{2}$ then $\partial_n(u\cdot n)^2 = 0$. In this paper, however, we are also interested in the low regularity setting i.e. for $\alpha\in(0,\frac{1}{2}]$, so this equality, as we will show later, does not generally hold. In fact, in section 8 we will construct an example of a Hlder continuous incompressible velocity field $u\in C^{0,\alpha}(\Omega)$ satisfying the boundary condition $(u\cdot n)|_{\partial\Omega}= 0$ for which $\partial_n(u\cdot n)^2|_{\partial\Omega}\notin \mathcal{D}'(\partial\Omega)$.

We will study the problem given by equation (1.3) and Neumann boundary condition (1.4). We will refer to equation (1.4) as the very weak boundary condition for the pressure, in order to distinguish it from the usual weak boundary condition $\partial_n p = (u\otimes u) : \nabla n$. We will prove that if $u\in C^{0,\alpha}(\Omega)$, then $p\in C^{0,\alpha}(\Omega)$ for any $\alpha > 0$. Note that we will omit the time dependence of the pressure and the velocity field throughout this paper, as it does not play a role of significance here.

The reason we prove this regularity result in the setting of Hölder spaces is because these spaces play an essential role in the theory of turbulence. As was first pointed out by Onsager in [32], there is a relation between Hölder regularity of the velocity field of a fluid flow on the one hand and the loss of energy via anomalous dissipation on the other hand. This relation is referred to as Onsager's conjecture.

Onsager's conjecture was first proven on the torus in a series of works [7, 11, 20, 21]. Onsager's conjecture was proven on bounded domains with C^2 boundary in [1], after results for the half plane in [34]. Then in [2] the first half of the conjecture was proven under only an interior Hölder regularity assumption on the velocity field. For the proof in [1] the $C^{0,\alpha}$ regularity of the pressure was necessary. The purpose of this paper is to give a full proof of this statement in the three-dimensional case.

In the two-dimensional setting, in [3] the pressure regularity condition was addressed. In particular, it was shown that the velocity field and the pressure have the same Hölder regularity for a bounded domain Ω in two dimensions. The very weak formulation of the Neumann boundary condition for the pressure given in equation (1.4) was introduced in [3]. This was an essential part of the proof, as it allows to construct a trace formula which establishes that the normal derivative of $p+(u\cdot n)^2$ is continuous in the $H^{-2}(\partial\Omega)$ norm near the boundary. This is then applied in the elliptic estimates in order to establish the $C^{0,\alpha}(\Omega)$ regularity of the pressure.

The goal of this paper is to extend the approach in [3] to three dimensions. In order to go from two to three dimensions, several modifications of the proof are necessary. Instead of a global parametrisation of the boundary, we need a local parametrisation of the boundary.

While this work was being completed, the paper [18] came to our attention. In that paper the authors also prove a regularity result for the pressure, but with a different boundary condition. In particular, the authors use the boundary condition $\partial_n p = (u\otimes u) : \nabla n$ as opposed to the boundary condition (1.4). These two formulations of the boundary condition are equivalent say if $u \cdot \nabla(u \cdot n)|_{\partial\Omega} = 0$.

As we stressed above, this is true for classical solutions as well as for the case when $u \in C^{0,\alpha}(\Omega)$ with $\alpha > 1/2$, but in the case when $u \in C^{0,\alpha}(\Omega)$ for $0 < \alpha \leq \frac{1}{2}$ we will show in section 8 that $u \cdot \nabla(u \cdot n)|_{\partial\Omega}$ in some cases is not even an element of $\mathcal{D}'(\partial\Omega)$. For this reason, we prove the result with the weaker formulation of the boundary condition for the pressure (1.4), which holds for all Hölder continuous velocity fields. Moreover, our proof is more explicit because it relies on localisation arguments.

In [3] the proof relies on a global localisation which considers the velocity field near the boundary and away from the boundary. In this contribution, we modify this localisation, namely we introduce a partition of unity of the region near the boundary itself. The reason for doing so is that in two dimensions, the boundary can be parametrised globally, but in three dimensions this is not possible. That means that the near-boundary analysis has to be done in a local coordinate system and then extended globally. We expect that our proof is quite robust, i.e. it can be extended to higher dimensions without much effort and for other hydrodynamical systems.

We will first give an imprecise version of the result that we will prove (the precise version is stated in section 2).

Theorem 1.1. *Let $u \in C^{0,\alpha}(\Omega)$ be a velocity field on a open set $\Omega \subset \mathbb{R}^3$ with a C^3 boundary and let $\alpha \in (0,1)$. Moreover we assume that u is divergence-free and that $(u \cdot n)|_{\partial\Omega} = 0$ as the boundary condition. Then by introducing a new weak formulation of the boundary condition* (1.4)*, it holds that*

$$\|p\|_{C^{0,\alpha}(\Omega} \leq C\|u \otimes u\|_{C^{0,\alpha}(\Omega)}. \tag{1.5}$$

Now we outline the proof of this result. In section 2 we introduce a parametrisation of the boundary region (including the extension of the velocity field outside the domain Ω). In particular we define the local coordinate system and state the differential operators in these coordinates. The proof then proceeds in the following steps:

- We first mollify the velocity field, which is done in section 3. This is not as straightforward as in the torus or the whole space, as the mollified velocity u^ϵ has to satisfy the boundary condition (1.2). One needs to split the velocity field into an interior and boundary part. The parametrisation of the boundary region is then used to extend the velocity field over the boundary.

- Then it is possible to use standard Schauder theory, as can be found in [23, 24, 26, 29, 30]. This will give us a candidate for a near the boundary truncated and mollified pressure P^ϵ, of which we are going to take the limit $\epsilon \to 0$ at the end of the proof.

- We then prove that that the $C^{0,\alpha}(\Omega)$ norm of P^ϵ is bounded by the $C^{0,\alpha}(\Omega)$ norm of $u^\epsilon \otimes u^\epsilon$ uniformly in ϵ. In section 4 we derive the interior estimates, while in section 6 we obtain the boundary estimates. In order to deal with the boundary condition we establish a trace lemma in section 5. In section 7 we then combine the estimates from sections 4-6 and take the limit $\epsilon \to 0$ to establish the regularity estimate from Theorem 1.1.

Finally, in section 8 we provide an example that illustrates why the very weak boundary condition (1.4) is necessary. To be more precise, we will construct an example of a velocity field in $C^{0,\alpha}(\Omega)$ for $0 < \alpha \leq \frac{1}{2}$ with $(u \cdot n)|_{\partial\Omega} = 0$ for which $\partial_n (u \cdot n)^2|_{\partial\Omega} \notin \mathcal{D}'(\partial\Omega)$. Therefore one cannot consider the terms $\partial_n p$ or $\partial_n (u \cdot n)^2$ individually at the boundary (as they are ill-defined).

In particular, $\partial_n (u \cdot n)^2$ is not well-defined at the boundary and it is definitely not equal to zero in the case when the velocity field $u \in C^{0,\alpha}(\Omega)$ for $0 < \alpha \leq \frac{1}{2}$. As a result, it is necessary to consider (as was done in (1.4)) the sum $\partial_n (p + (u \cdot n)^2)$ together at the boundary to obtain a well-defined boundary condition.

In appendices A and B we establish some Schauder-type estimates that will be used throughout the paper. To conclude, in appendix C we will continue with the example given in section 8 and show that $\partial_n (u \cdot n)^2$ is not defined for a dense set of points away from the boundary.

2 Local parametrisation of the boundary

We introduce a coordinate system for the region near the boundary $\partial\Omega$. We will assume throughout that the domain Ω is simply connected.

We introduce $\phi : [0, \infty) \to [0, 1]$ to be a nonincreasing smooth function defined as follows (for some $\delta > 0$)

$$\phi(x) := \begin{cases} 1, & \text{if } \operatorname{dist}(x, \partial\Omega) \leq \delta, \\ 0, & \text{if } \operatorname{dist}(x, \partial\Omega) \geq 2\delta. \end{cases} \tag{2.1}$$

Now we introduce the sets (for a given open set $U \subset \partial\Omega$)

$$V_\delta := \{x \in \mathbb{R}^3 |\ d(x, \partial\Omega) < \delta\}, \tag{2.2}$$

$$V_{\delta,U} := \{x \in V_\delta |\ d(x, U) < \delta\}. \tag{2.3}$$

The fact that $\partial\Omega$ is C^3 means that around any point $x_0 \in \partial\Omega$ there exist Cartesian coordinates (x_1, x_2, x_3) and a C^3 function $a : \mathbb{R}^2 \to \mathbb{R}$ such that the surface $\partial\Omega$ is locally parametrised as $(x_1, x_2, a(x_1, x_2))$ on a subset $U_{x_0} \subset \partial\Omega$. Then by the compactness of the boundary $\partial\Omega$, we know that there exist finitely many sets $U_1, \ldots, U_m$ which cover the boundary (with corresponding points $x_1, \ldots, x_m$).

Locally on U_{x_0}, the inward normal vector to $\partial\Omega$ is given by

$$n = (n_1, n_2, n_3) = \frac{1}{\sqrt{1 + \left|\frac{\partial a}{\partial x_1}\right|^2 + \left|\frac{\partial a}{\partial x_2}\right|^2}} \left(\frac{\partial a}{\partial x_1}, \frac{\partial a}{\partial x_2}, -1 \right).$$

We then introduce the coordinate system (see [3] and [39, Theorem 2.12])

$$x_1 = \sigma_1 + s n_1(\sigma_1, \sigma_2), \tag{2.4}$$

$$x_2 = \sigma_2 + s n_2(\sigma_1, \sigma_2), \tag{2.5}$$

$$x_3 = a(\sigma_1, \sigma_2) + s n_3(\sigma_1, \sigma_2), \tag{2.6}$$

for $(\sigma_1, \sigma_2, s) \in [0, \delta]^3$. This transformation is C^2, as the normal vector n is C^2. Alternatively, equations (2.4)-(2.6) can be written as follows

$$x(\sigma_1, \sigma_2, s) = y(\sigma_1, \sigma_2) + s \cdot n(\sigma_1, \sigma_2), \tag{2.7}$$

where y moves on the local patch U_{x_0} of the surface $\partial\Omega$ and is given by $y(\sigma_1, \sigma_2) = (y_1, y_2, y_3) = (\sigma_1, \sigma_2, a(\sigma_1, \sigma_2))$. We introduce the following notation for the coordinate transformation

$$\phi_{x_0}(\sigma_1, \sigma_2, s) = x(\sigma_1, \sigma_2, s) = (x_1, x_2, x_3). \tag{2.8}$$

Taking the derivative of x in the normal coordinate s, we find that

$$\partial_s x = n(\sigma_1, \sigma_2) = \left(\frac{\partial a}{\partial \sigma_1}, \frac{\partial a}{\partial \sigma_2}, -1 \right).$$

Now we calculate the partial derivatives of x with respect to the tangential variables. We first calculate the partial derivatives of y to be

$$\partial_{\sigma_1} y = \left(1, 0, \frac{\partial a}{\partial \sigma_1} \right), \quad \partial_{\sigma_2} y = \left(0, 1, \frac{\partial a}{\partial \sigma_2} \right).$$

It is easy to see that these vectors are orthogonal to $n(\sigma_1, \sigma_2)$. We note that $\partial_{\sigma_1} n$ and $\partial_{\sigma_2} n$ are orthogonal to n by definition (as n has unit length). The tangent vectors at any point (σ_1, σ_2, s) are then given by

$$\begin{aligned} \tau_1(\sigma_1, \sigma_2, s) &= \partial_{\sigma_1} x = \partial_{\sigma_1} y + s \partial_{\sigma_1} n, \\ \tau_2(\sigma_1, \sigma_2, s) &= \partial_{\sigma_2} x = \partial_{\sigma_2} y + s \partial_{\sigma_2} n, \end{aligned}$$

which are orthogonal to n (as the component parts of the tangent vectors are). The vectors τ_1, τ_2 and n form a basis for $\mathbb{R}^3$ for every point in $V_{\delta, U_{x_0}}$. However, we observe that in general this coordinate system is not orthogonal.

Now we turn to computing the gradient, divergence and Laplacian in this new coordinate system. The Jacobian matrix of the coordinate transformation is given by

$$J := \frac{\partial x}{\partial(\sigma_1, \sigma_2, s)} = \begin{pmatrix} 1 + s\partial_{\sigma_1} n_1 & s\partial_{\sigma_2} n_1 & n_1 \\ s\partial_{\sigma_1} n_2 & 1 + s\partial_{\sigma_2} n_2 & n_2 \\ \partial_{\sigma_1} a + s\partial_{\sigma_1} n_3 & \partial_{\sigma_2} a + s\partial_{\sigma_2} n_3 & n_3 \end{pmatrix} = \big(\partial_{\sigma_1} y + s\partial_{\sigma_1} n, \partial_{\sigma_2} y + s\partial_{\sigma_2} n, n \big). \tag{2.9}$$

As shown in the proof of Theorem 2.12 in [39], as the Jacobian has nonzero determinant at x_0, it is locally invertible and a C^2 diffeomorphism.

We now introduce the following notation (for the sake of brevity)

$$a_{ij} := \big(J^{-1} (J^{-1})^T \big)_{ij} \quad \text{for } i, j = 1, 2, 3, \tag{2.10}$$

$$b := \sqrt{\det(J^T J)}. \tag{2.11}$$

It is easy to see that $(a_{ij})_{i,j=1}^3$ is a symmetric matrix, in fact it is the metric tensor that is associated with the coordinate system (2.4)-(2.6).

The gradient, the divergence and the Laplacian in the given coordinate system are given by (see equations 9.60, 9.69 and 9.70 in [25])

$$\nabla f = \frac{\partial f}{\partial \sigma_1}\big(a_{11}\tau_1 + a_{21}\tau_2 + a_{31}n\big) + \frac{\partial f}{\partial \sigma_2}\big(a_{12}\tau_1 + a_{22}\tau_2 + a_{32}n\big) + \frac{\partial f}{\partial s}\big(a_{13}\tau_1 + a_{23}\tau_2 + a_{33}n\big) \tag{2.12}$$

$$\nabla \cdot v = \frac{1}{b}\left[\frac{\partial}{\partial \sigma_1}(bv_1) + \frac{\partial}{\partial \sigma_2}(bv_2) + \frac{\partial}{\partial s}(bv_3)\right], \tag{2.13}$$

$$\begin{aligned}\Delta f = {}& \frac{1}{b}\left[\frac{\partial}{\partial \sigma_1}\left(b\left[a_{11}\frac{\partial f}{\partial \sigma_1} + a_{12}\frac{\partial f}{\partial \sigma_2} + a_{13}\frac{\partial f}{\partial s}\right]\right) + \frac{1}{b}\left[\frac{\partial}{\partial \sigma_2}\left(b\left[a_{21}\frac{\partial f}{\partial \sigma_1} + a_{22}\frac{\partial f}{\partial \sigma_2} + a_{23}\frac{\partial f}{\partial s}\right]\right)\right.\right. \\ & + \frac{1}{b}\left[\frac{\partial}{\partial s}\left(b\left[a_{31}\frac{\partial f}{\partial \sigma_1} + a_{32}\frac{\partial f}{\partial \sigma_2} + a_{33}\frac{\partial f}{\partial s}\right]\right)\right],\end{aligned} \tag{2.14}$$

where (v_1, v_2, v_3) are the components of the vector v in the coordinates (σ_1, σ_2, s).

We recall that we showed earlier that the normal vector n has unit length and is orthogonal to the other tangent vectors at every point in $V_{\delta,U_{x_0}}$. This means in particular that (as (a_{ij}) is the metric tensor)

$$a_{33} = 1, \quad a_{31} = a_{13} = a_{21} = a_{12} = 0. \tag{2.15}$$

We now introduce the operator Δ_τ to be

$$\Delta_\tau f := \frac{1}{b}\left[\frac{\partial}{\partial \sigma_1}\left(b\left[a_{11}\frac{\partial f}{\partial \sigma_1} + a_{12}\frac{\partial f}{\partial \sigma_2}\right]\right) + \frac{1}{b}\left[\frac{\partial}{\partial \sigma_2}\left(b\left[a_{21}\frac{\partial f}{\partial \sigma_1} + a_{22}\frac{\partial f}{\partial \sigma_2}\right]\right)\right.\right..$$

This allows us to rewrite the expression for the Laplacian as follows

$$\Delta f = \Delta_\tau f + \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial f}{\partial s} + \frac{\partial^2 f}{\partial s^2}. \tag{2.16}$$

Remark 2.1. We note that we could have derived the expression for the gradient, the divergence and the Laplacian directly. We observe that there is the following relation between the tangent vectors

$$\begin{aligned}\frac{\partial}{\partial x_1} &= (J^{-1})_{11}\frac{\partial}{\partial \sigma_1} + (J^{-1})_{21}\frac{\partial}{\partial \sigma_2} + (J^{-1})_{31}\frac{\partial}{\partial s}, \\ \frac{\partial}{\partial x_2} &= (J^{-1})_{12}\frac{\partial}{\partial \sigma_1} + (J^{-1})_{22}\frac{\partial}{\partial \sigma_2} + (J^{-1})_{32}\frac{\partial}{\partial s}, \\ \frac{\partial}{\partial x_3} &= (J^{-1})_{13}\frac{\partial}{\partial \sigma_1} + (J^{-1})_{23}\frac{\partial}{\partial \sigma_2} + (J^{-1})_{33}\frac{\partial}{\partial s},\end{aligned}$$

and similarly we have

$$\begin{aligned}\frac{\partial}{\partial \sigma_1} &= \left(1 + s\frac{\partial n_1}{\partial \sigma_1}(\sigma_1, \sigma_2)\right)\frac{\partial}{\partial x_1} + s\frac{\partial n_2}{\partial \sigma_1}(\sigma_1, \sigma_2)\frac{\partial}{\partial x_2} + \left(\frac{\partial a}{\partial \sigma_1} + s\frac{\partial n_3}{\partial \sigma_1}\right)\frac{\partial}{\partial x_3}, \\ \frac{\partial}{\partial \sigma_2} &= s\frac{\partial n_1}{\partial \sigma_2}(\sigma_1, \sigma_2)\frac{\partial}{\partial x_1} + \left(1 + s\frac{\partial n_2}{\partial \sigma_2}(\sigma_1, \sigma_2)\right)\frac{\partial}{\partial x_2} + \left(\frac{\partial a}{\partial \sigma_2} + s\frac{\partial n_3}{\partial \sigma_2}\right)\frac{\partial}{\partial x_3}, \\ \frac{\partial}{\partial s} &= n_1\frac{\partial}{\partial x_1} + n_2\frac{\partial}{\partial x_2} + n_3\frac{\partial}{\partial x_3} = n.\end{aligned}$$

These relations stipulate how the vector components transform between the different bases, this allows one to rewrite the differential operators in the coordinates from equations (2.4)-(2.6) by rewriting the standard expressions for these operators in Cartesian coordinates.

Remark 2.2. We note that it is straightforward to extend coordinate system (2.4)-(2.6) to higher dimensions. As in the three-dimensional case, one can rely on the compactness of the boundary to obtain a finite number of surface patches to cover the boundary. The formulae for the gradient, divergence and Laplacian operators have higher-dimensional generalisations of the same form.

Remark 2.3. In order to have a working proof we require that $\partial\Omega \in C^3$. The reason is that this implies that the normal vector n is a C^2 function of (σ_1, σ_2). The Jacobian matrix J involves tangential derivatives of n, which makes that J has C^1 regularity. Since the divergence and Laplacian involve first-order derivatives of J, we need J to be C^1 and therefore cannot lower the regularity requirement on the boundary.

We expect that by using the variational formulation of the equation with the same choice of coordinates will allow us to weaken the boundary regularity requirement to C^2 instead of C^3. Then one has to perform the Schauder-type estimates from section 6 in the weak formulation, see for example [23].

Throughout the paper, we will consider a modified pressure defined by

$$P := p + \phi(u \cdot n)^2, \tag{2.17}$$

where ϕ is a smooth cutoff function that was defined in (2.1). The result we will prove in this paper can now be stated more precisely as follows.

Theorem 2.4. *Let Ω be an open set in $\mathbb{R}^3$ with C^3 boundary and assume that $u \in C^{0,\alpha}(\Omega)$ for $\alpha \in (0,1)$ is a velocity field which is divergence-free and satisfies $(u \cdot n)|_{\partial\Omega} = 0$. Then there is a unique function $P \in C^{0,\alpha}(\Omega)$ with the following properties:*

1. *It satisfies the following estimate*

$$\|P\|_{C^{0,\alpha}} \leq C \|u \otimes u\|_{C^{0,\alpha}}, \tag{2.18}$$

the positive constant C depends only on Ω and α.

2. *In any region V_{δ,U_i} for $i = 1, \dots, m$, the map $s \mapsto \partial_s P(\cdot, s)$ lies in the space $C([0,\delta); H^{-2}(U_i))$ where $s \in [0,\delta)$ is the normal coordinate and U_i is the local patch of the boundary. By using a partition of unity over the patches $U_1, \dots, U_m$, the function $\partial_s P(\cdot, s)$ can be extended to a function in $C([0,\delta), H^{-2}(\partial\Omega))$, i.e. a function defined globally near the boundary.*

3. *The function P satisfies the following equation in Ω*

$$-\Delta P = (\nabla \otimes \nabla) : (u \otimes u) - \Delta(\phi(x)(u \cdot n)^2), \tag{2.19}$$

which is satisfied in the sense of distributions. In particular, it means that for test functions $\psi \in \mathcal{D}(\Omega)$

$$-\int_\Omega P \Delta\psi dx = \int_\Omega u_i u_j \partial_i \partial_j \psi dx - \int_\Omega \phi(x)(u \cdot n)^2 \Delta\psi dx. \tag{2.20}$$

Moreover, P satisfies the boundary condition

$$\partial_n P = \big(u \otimes u\big) : \nabla n \quad \text{on } \partial\Omega, \tag{2.21}$$

where ∂_n is the normal derivative. This equation holds in $H^{-2}(\partial\Omega)$. Moreover, the average of P satisfies

$$\int_\Omega P(x)dx = \int_\Omega \phi(x)(u \cdot n)^2 dx, \tag{2.22}$$

where $(u \cdot n)^2$ is defined locally on each patch V_{δ,U_i} and extended globally by using the partition of unity.

3 Mollification of the velocity field

Lemma 3.1. *Consider a velocity $u \in C^{0,\alpha}(\Omega)$ which is divergence-free and tangential to the boundary (so $(u\cdot n)|_{\partial\Omega}= 0$), there exists a family of divergence-free velocity fields $u^\epsilon \in C^\infty(\overline{\Omega})$ which converge to u in $C^{0,\beta}(\overline{\Omega})$ for $\beta \in (0, \alpha)$ as $\epsilon \to 0$. In addition, we have the estimate*

$$\|u^\epsilon\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{C^{0,\alpha}(\Omega)}. \tag{3.1}$$

Proof. We first introduce the three-dimensional vector stream function ψ to be the solution of the following elliptic boundary-value problem (cf. [3, 12, 37])

$$\begin{cases} -\Delta\psi = \nabla \times u & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.2}$$

which can be derived from $\nabla \times \psi = u, \nabla \cdot \psi = 0$ and $\psi|_{\partial\Omega}= 0$. This elliptic problem has a unique solution in $H^1_0(\Omega)$, while the equations hold in $H^{-1}(\Omega)$ and the boundary condition holds in the trace sense. We define

$$v := u - \nabla \times \psi.$$

It is easy to check that $\nabla \cdot v = 0$, $\nabla \times v = 0$ and $(v \cdot n)|_{\partial\Omega}= 0$ (as $((\nabla \times \psi) \cdot n)|_{\partial\Omega}= 0$ in $H^{-1/2}(\partial\Omega)$ by the generalised Stokes theorem), and as a result we have that

$$\Delta v = \Delta u - \nabla \times (\Delta\psi) = \nabla \times (-\nabla \times u + \Delta\psi) = 0.$$

This implies in particular that $v \in C^\infty(\Omega)$. Now since $\nabla \times v = 0$, there exists $q \in C^\infty(\Omega)$ such that $v = \nabla q$. Then we find that

$$\Delta q = 0 \text{ in } \Omega, \quad \partial_n q = 0 \text{ on } \partial\Omega. \tag{3.3}$$

Therefore we know that q is constant and hence v is equal to zero, and therefore $u = \nabla \times \psi$. This implies in particular that $\psi \in C^{1,\alpha}(\Omega)$. Now we move on to the localisation argument.

First we consider a function $\phi_1 \in C^2_c(\mathbb{R}^3)$ such that $\text{supp}(\phi_1) \subset \overline{V}_\delta$. We then introduce a partition of unity $\rho_1, \ldots, \rho_m$ of the sets $V_{\delta,U_1}, \ldots, V_{\delta,U_m}$ (which cover the region near the boundary). We define the following decompositions

$$\begin{aligned} \psi &= \psi_b + \psi_i := \phi_1\psi + (1 - \phi_1)\psi, \\ \psi_b &= \phi_1\rho_1\psi + \ldots + \phi_1\rho_m\psi =: \psi_1 + \ldots + \psi_m. \end{aligned}$$

We introduce a nonnegative radial mollifier φ with support in $B_0(1)$ and the property $\int_{\mathbb{R}^3} \varphi(x)dx = 1$. Moreover, we define

$$\varphi_\epsilon(x) := \frac{1}{\epsilon^3}\varphi\left(\frac{x}{\epsilon}\right).$$

Observe that $\psi_j \in C_c^{1,\alpha}(V_{\delta,U_j})$ for $j = 1, \dots, m$.

First we deal with the interior part of the velocity field, which we observe to have compact support in Ω. We define the function

$$\psi_i^\epsilon(x) := \varphi_\epsilon * \psi_i \in C_c^\infty(\Omega),$$

for ϵ suitably small. Note that ψ_i^ϵ converges to ψ_i in the $C^1(\overline{\Omega})$ norm as $\epsilon \to 0$ by standard mollification estimates. Moreover, it holds that $\|\psi_i^\epsilon\|_{C^{1,\alpha}(\Omega)} \leq C\|\psi_i\|_{C^{1,\alpha}(\Omega)}$. Therefore $u_i^\epsilon := \nabla \times \psi_i^\epsilon$ is the interior part of the mollified velocity and it satisfies the required properties.

Now we consider the boundary parts $\psi_1, \dots, \psi_m$. In particular, we need to define an extension for these functions in order to prove the mollification estimates. We consider an odd extension of the form

$$\widetilde{\psi}_j|_{V_{\delta,U_j}}(\sigma_1, \sigma_2, s) = \begin{cases} \psi_j(\sigma_1, \sigma_2, s) \text{ if } s \geq 0, \\ -\psi_j(\sigma_1, \sigma_2, -s) \text{ if } s \leq 0. \end{cases} \tag{3.4}$$

Recall that we assume the normal to point inward. Note that $\widetilde{\psi}_j \in C_c^{1,\alpha}(V_{\delta,U_j})$ and it can be extended by zero outside V_{δ,U_j}. We then observe that $\widetilde{\psi}_j^\epsilon \in C_c^\infty(V_{\delta,U_j})$ and also $\widetilde{\psi}_j(\sigma_1, \sigma_2, 0) = 0$. The odd extension ensures that $\widetilde{\psi}_j^\epsilon(\sigma_1, \sigma_2, 0) = 0$.

We now define the function

$$\widetilde{\psi}^\epsilon := \psi_i^\epsilon + \sum_{j=1}^m \widetilde{\psi}_j^\epsilon. \tag{3.5}$$

We prove this converges to ψ in the $C^1(\overline{\Omega})$ norm as $\epsilon \to 0$. It is easy to see that

$$\begin{aligned} \|\psi - \tilde{\psi}\|_{C^1(\overline{\Omega})} &\leq \|(1-\phi_1)\psi - \psi_i^\epsilon\|_{C^1(\overline{\Omega})} + \left\|\phi_1\psi - \sum_{j=1}^m \widetilde{\psi}_j^\epsilon\right\|_{C^1(\overline{\Omega})} \\ &\leq \|\psi_i - \psi_i^\epsilon\|_{C^1(\Omega)} + \sum_{j=1}^m \|\phi_1\rho_j\psi - \widetilde{\psi}_j^\epsilon\|_{C^1(\overline{V}_{\delta,U_j})} \xrightarrow{\epsilon\to 0} 0. \end{aligned}$$

In addition, it holds that $\widetilde{\psi}^\epsilon \to \psi$ in $C^{1,\alpha}(\Omega)$ and $\|\widetilde{\psi}\|_{C^{1,\alpha}(\Omega)} \leq C\|\psi\|_{C^{1,\alpha}}$ for some constant C. Now we take $\widetilde{u}^\epsilon := \nabla \times \widetilde{\psi}^\epsilon \in C_c^\infty(\mathbb{R}^3)$, which satisfies the divergence-free condition. We also get that $\tilde{u}^\epsilon \to u$ in $C^{0,\beta}(\overline{\Omega})$ for $\beta \in (0, \alpha)$ and moreover, it holds that

$$\|\tilde{u}^\epsilon\|_{C^{0,\alpha}(\Omega)} \leq C\|u\|_{C^{0,\alpha}(\Omega)}.$$

Since $\widetilde{\psi}^\epsilon|_{\partial\Omega} = 0$, simple calculations show that $(\nabla \times \widetilde{\psi}^\epsilon) \cdot n|_{\partial\Omega} = (u^\epsilon \cdot n)|_{\partial\Omega} = 0$. □

Remark 3.2. We remark that this result also holds if the boundary $\partial\Omega$ is C^2 instead of C^3.

Corollary 3.3. *Since u^ϵ is $C^{2,\alpha}(\Omega)$, by standard elliptic theory there exists a unique function $p^\epsilon \in C^{2,\alpha}(\Omega)$ such that*

$$-\Delta p^\epsilon = (\nabla \otimes \nabla) : (u^\epsilon \otimes u^\epsilon) \quad \textit{in } \Omega, \tag{3.6}$$

$$\partial_n p^\epsilon = u^\epsilon \otimes u^\epsilon : \nabla n, \quad \textit{on } \partial\Omega, \tag{3.7}$$

$$\int_\Omega p^\epsilon dx = 0. \tag{3.8}$$

The whole point of mollifying the velocity is that it allows to find a candidate mollified pressure p^ϵ by using standard Schauder theory. Since the mollified candidate pressure is smooth, it allows us to do many estimates more easily after which we can take the limit $\epsilon \to 0$.

Remark 3.4. As was mentioned before, we will not directly work with boundary condition (3.7), but we will introduce a weaker notion of boundary condition. We first observe that if $u \in C^{0,\alpha}(\Omega)$ with $\alpha > \frac{1}{2}$ and $(u \cdot n)|_{\partial\Omega} = 0$, then it holds that

$$\partial_n (u \cdot n)^2|_{\partial\Omega} = 0. \tag{3.9}$$

We will use this to modify the boundary condition (3.7) as follows

$$\partial_n \Big[p + \phi (u \cdot n)^2 \Big] = \big(u \otimes u \big) : \nabla n \quad \text{on } \partial\Omega. \tag{3.10}$$

This boundary condition will be referred to as the very weak boundary condition, in order to distinguish it from the usual boundary condition $\partial_n p = (u \otimes u) : \nabla n$. The two notions are equivalent when $\alpha > \frac{1}{2}$.

In section 5 we will show that $\partial_n (p + (u \cdot n)^2) \in C^{0,\alpha}([0,\delta); H^{-2}(\partial\Omega))$ for some $\delta > 0$. However, in section 8 we will give an example of a velocity field for which $\partial_n ((u \cdot n)^2) \notin \mathcal{D}'(\partial\Omega)$. This justifies the use of the very weak boundary condition, as for such a velocity field it is not possible to consider $\partial_n p|_{\partial\Omega}$ independently (even as a distribution at the boundary).

In appendix C we will even show more, namely that in the example of section 8 the normal derivative is not defined (as a distribution) on a dense set of points away from the boundary.

We note that the main difference of our approach compared to the one in [18] is that we use the very weak boundary condition (3.10), while the authors of [18] use the standard boundary condition.

4 Interior estimate

We will derive estimates for the pressure separately in the interior of the domain and in the region near the boundary. We first establish the interior estimate. We now introduce several cutoff functions. In order to separate the behaviour of the pressure near and far away from the boundary, we recall that in equation (2.1) we defined the function ϕ. We consider the parameters δ_1, δ_2 and δ_3 such that (for some small $\gamma < \delta$)

$$0 < \delta_1 < \delta_2 - \gamma < \delta_3 < \delta - 2\gamma.$$

We then introduce functions

$$\phi_i(s) = \begin{cases} 0 \text{ if } 0 \le s \le \delta_1, \\ 1 \text{ if } s \ge \delta_2 - \gamma, \end{cases} \qquad \phi_b(s) = \begin{cases} 1 \text{ if } 0 \le s < \delta_3 + \gamma, \\ 0 \text{ if } s \ge \delta - \gamma. \end{cases}$$

Observe that ϕ and ϕ_b are nonincreasing while ϕ_i is nondecreasing and that ϕ_i and ϕ_b are overlapping (but they are not a partition of unity). Once again, we write (for $x \in \Omega$ sufficiently close to the boundary)

$$\phi_i(x) \coloneqq \phi_i(d(x, \partial\Omega)), \quad \phi_b(x) \coloneqq \phi_b(d(x, \partial\Omega)).$$

Then we introduce the functions $\phi_{b,1}, \ldots, \phi_{b,m}$ through the definition

$$\phi_{b,j} \coloneqq \phi_b \rho_j.$$

We then define the functions

$$P^\epsilon(x) = p^\epsilon(x) + \phi(x)(u^\epsilon(x) \cdot n(x))^2, \tag{4.1}$$
$$P_i^\epsilon(x) = \phi_i(x) P^\epsilon(x) = \phi_i(x)(p^\epsilon(x) + \phi(x)(u^\epsilon(x) \cdot n(x))^2), \tag{4.2}$$
$$P_b^\epsilon(x) = \phi_b(x) P^\epsilon(x) = \phi_b(x)(p^\epsilon(x) + (u^\epsilon(x) \cdot n(x))^2), \tag{4.3}$$
$$P_{b,j}^\epsilon(x) = \rho_j(x)\phi_b(x) P^\epsilon(x) = \rho_j(x)\phi_b(x)(p^\epsilon(x) + (u^\epsilon(x) \cdot n(x))^2). \tag{4.4}$$

Note that we used that $\phi_b(x)\phi(x) = \phi_b(x)$ which holds by definition of the cutoff functions.

We first prove an estimate for the interior pressure P_i^ϵ.

Proposition 4.1. *Let P_i^ϵ be the function defined in equation* (4.2)*. The following estimate holds for the interior mollified pressure*

$$\|P_i^\epsilon\|_{C^{0,\alpha}(\Omega)} \le C_i \|u^\epsilon \otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)} + D_i \|P^\epsilon\|_{L^\infty(\Omega)}. \tag{4.5}$$

Note that the constants C_i and D_i are independent of ϵ.

Proof. We calculate that P_i^ϵ satisfies the equation

$$\begin{aligned} -\Delta P_i^\epsilon &= -(\Delta\phi_i) P^\epsilon - 2(\nabla\phi_i) \cdot \nabla P^\epsilon - \phi_i \Delta P^\epsilon \\ &= -(\Delta\phi_i) P^\epsilon - 2(\nabla\phi_i) \cdot \nabla P^\epsilon - \phi_i(\Delta(\phi(u^\epsilon \cdot n)^2) + \Delta p^\epsilon) \\ &= -(\Delta\phi_i) P^\epsilon - 2(\nabla\phi_i) \cdot \nabla P^\epsilon - \phi_i(\Delta(\phi(u^\epsilon \cdot n)^2) - (\nabla \otimes \nabla) : (u^\epsilon \otimes u^\epsilon)). \end{aligned}$$

We then decompose the interior pressure as $P_{i,1}^\epsilon$ and $P_{i,2}^\epsilon$ which satisfy that

$$\begin{aligned} -\Delta P_{i,1}^\epsilon &= -(\Delta\phi_i) P^\epsilon - 2(\nabla\phi_i) \cdot \nabla P^\epsilon, \\ -\Delta P_{i,2}^\epsilon &= -\phi_i(\Delta(\phi(u^\epsilon \cdot n)^2) - (\nabla \otimes \nabla) : (u^\epsilon \otimes u^\epsilon)). \end{aligned}$$

We establish estimate (4.5) separately for $P_{i,1}^\epsilon$ and $P_{i,2}^\epsilon$. We recall that the Green's function of the operator $-\Delta$ is given by

$$G(x) \coloneqq \frac{1}{4\pi|x|}.$$

This means that $P^\epsilon_{i,1}$ is given by

$$P^\epsilon_{i,1} = \frac{1}{4\pi|x|} * (-(\Delta\phi_i)P^\epsilon - 2(\nabla\phi_i)\cdot\nabla P^\epsilon).$$

which allows us to conclude estimate (4.5) for $P^\epsilon_{i,1}$.

By the Schauder estimate for the Dirichlet problem established in Theorem A.1 we know that $P^\epsilon_{i,2}$ also satisfies estimate (4.5), which concludes the proof. □

Now we move on to establishing the estimates for the boundary layer pressure.

5 Trace lemma

It follows that $P^\epsilon_{b,j}$ satisfies the following equation

$$-\Delta P^\epsilon_{b,j} = -(\Delta\phi_b\rho_j)P^\epsilon - 2(\nabla\phi_b\rho_j)\cdot\nabla P^\epsilon + \phi_b\rho_j\Big((\nabla\otimes\nabla):(u^\epsilon\otimes u^\epsilon) - \Delta\big((u^\epsilon\cdot n)^2\big)\Big). \quad (5.1)$$

Now we need to write this equation in terms of the local coordinate system in the region $\overline{V}_{\delta,U_j}$.

Lemma 5.1. *Assume that $x \in \overline{V}_{\delta,U_j}$, then we have the following expression of equation* (5.1) *(in terms of the local coordinates (σ_1,σ_2,s) defined in equations* (2.4)-(2.6)*)*

$$\begin{aligned}
&-\Delta_\tau P^\epsilon_{b,j} - \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial P^\epsilon_{b,j}}{\partial s} - \frac{\partial^2 P^\epsilon_{b,j}}{\partial s^2} = -(\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon - 2\partial_s(\phi_b\rho_j)\frac{\partial P^\epsilon}{\partial s}\\
&+\phi_b\rho_j\Bigg(\frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu^\epsilon_i u^\epsilon_j\big) + 2\sum_{i=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial s}\big(bu^\epsilon_i(u^\epsilon\cdot n)\big) + \frac{\partial^2}{\partial s^2}\big(b(u^\epsilon\cdot n)^2\big)\Bigg]\\
&-\Delta_\tau\big((u^\epsilon\cdot n)^2\big)\big) - \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial(u^\epsilon\cdot n)^2)}{\partial s} - \frac{\partial^2(u^\epsilon\cdot n)^2)}{\partial s^2}\Bigg),
\end{aligned} \quad (5.2)$$

where the specific form of the differential operator ∇_τ is given in the proof.

Proof. In the coordinate system (2.4)-(2.6) we can write the divergence as follows

$$\begin{aligned}
(\nabla\otimes\nabla):(u^\epsilon\otimes u^\epsilon) &= \frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu^\epsilon_i u^\epsilon_j\big)\\
&+2\sum_{i=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial s}\big(bu^\epsilon_i(u^\epsilon\cdot n)\big) + \frac{\partial^2}{\partial s^2}\big(b(u^\epsilon\cdot n)^2\big)\Bigg].
\end{aligned}$$

Moreover, we compute that (by using equation (2.16))

$$\begin{aligned}
\Delta P^\epsilon_{b,j} &= \Delta_\tau P^\epsilon_{b,j} + \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial P^\epsilon_{b,j}}{\partial s} + \frac{\partial^2 P^\epsilon_{b,j}}{\partial s^2},\\
\Delta\big((u^\epsilon\cdot n)^2\big)\big) &= \Delta_\tau\big((u^\epsilon\cdot n)^2\big)\big) + \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial(u^\epsilon\cdot n)^2)}{\partial s} + \frac{\partial^2(u^\epsilon\cdot n)^2)}{\partial s^2}.
\end{aligned}$$

We can now compute the gradient of the pressure to be (using (2.15))

$$\nabla P^\epsilon = \frac{\partial P^\epsilon}{\partial \sigma_1}\big(a_{11}\tau_1 + a_{21}\tau_2\big) + \frac{\partial P^\epsilon}{\partial \sigma_2}\big(a_{12}\tau_1 + a_{22}\tau_2\big) + \frac{\partial P^\epsilon}{\partial s} n,$$

and similarly we can compute the gradient of $\phi_b\rho_j$. By introducing the notation

$$\nabla_\tau f \coloneqq \frac{\partial f}{\partial \sigma_1}\big(a_{11}\tau_1 + a_{21}\tau_2\big) + \frac{\partial f}{\partial \sigma_2}\big(a_{12}\tau_1 + a_{22}\tau_2\big),$$

then we find that

$$(\nabla \phi_b\rho_j)\cdot \nabla P^\epsilon = \nabla_\tau(\phi_b\rho_j)\cdot \nabla_\tau P^\epsilon + \partial_s(\phi_b\rho_j)\frac{\partial P^\epsilon}{\partial s}.$$

These calculations allow us to express equation (5.1) as follows

$$\begin{aligned}
&-\Delta_\tau P^\epsilon_{b,j} - \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial P^\epsilon_{b,j}}{\partial s} - \frac{\partial^2 P^\epsilon_{b,j}}{\partial s^2} = -(\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon - 2\partial_s(\phi_b\rho_j)\frac{\partial P^\epsilon}{\partial s}\\
&+\phi_b\rho_j\Bigg(\frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu_i^\epsilon u_j^\epsilon\big) + 2\sum_{i=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial s}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big) + \frac{\partial^2}{\partial s^2}\big(b(u^\epsilon\cdot n)^2\big)\Bigg]\\
&-\Delta_\tau\big((u^\epsilon\cdot n)^2)\big) - \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial(u^\epsilon\cdot n)^2)}{\partial s} - \frac{\partial^2(u^\epsilon\cdot n)^2)}{\partial s^2}\Bigg).
\end{aligned}$$

□

It is crucial to observe that the definition of P^ϵ in equation (4.1) (i.e. combining the term $(u^\epsilon\cdot n)^2$ as part of P^ϵ) has the consequence that on the right-hand side of equation (5.2) there are no terms which have second order derivatives in s (after some further manipulations, which will be done in the proof of the next lemma). This makes it possible to establish the following trace lemma, which will be crucial for the proof of the final regularity result (Theorem 2.4).

Lemma 5.2. *The following equation holds for $\partial_s P^\epsilon_b$ for every region $\overline{V}_{\delta,U_j}$*

$$\partial_s P^\epsilon_{b,j}(\cdot,\cdot,s) = \Lambda^\epsilon_j(\cdot,\cdot,s) + \int_s^\delta \Theta^\epsilon_j(\cdot,\cdot,s')ds', \tag{5.3}$$

where Λ^ϵ and Θ^ϵ (which are specified in the proof) satisfy the local estimates

$$\|\Lambda^\epsilon_j\|_{C^{0,\alpha}([0,\delta],H^{-1}(U_j))} \leq C_b\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)} + D_b\|P^\epsilon\|_{L^\infty(\Omega)}, \tag{5.4}$$

$$\|\Theta^\epsilon\|_{C^{0,\alpha}([0,\delta],H^{-2}(U_j))} \leq C_b\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)} + D_b\|P^\epsilon\|_{L^\infty(\Omega)}. \tag{5.5}$$

These estimates can then be put together to yield a global estimate for Λ^ϵ and Θ^ϵ for the region near the boundary.

Proof. The proof will be done locally, i.e. for a given patch $V_{\delta,U_j}\subset V_\delta$, which can then be extended to the whole region near the boundary using the partition of unity of $U_1,\dots,U_m$. We start by rewriting equation (5.2) as follows

$$-\frac{\partial^2 P^\epsilon_{b,j}}{\partial s^2} = \Delta_\tau P^\epsilon_{b,j} + \frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial P^\epsilon_{b,j}}{\partial s} - (\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon - 2\partial_s(\phi_b\rho_j)\frac{\partial P^\epsilon}{\partial s}$$

$$+ \phi_b\rho_j\Bigg(\frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu_i^\epsilon u_j^\epsilon\big) + 2\sum_{i=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial s}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big) + \frac{\partial^2 b}{\partial s^2}(u^\epsilon\cdot n)^2 + \frac{\partial b}{\partial s}\frac{\partial(u^\epsilon\cdot n)}{\partial s}\Bigg]$$
$$- \Delta_\tau\big((u^\epsilon\cdot n)^2)\big)\Bigg).$$

By integrating over s we find that (and integrating by parts)

$$\partial_s P_{b,j}^\epsilon = -\frac{1}{b}\frac{\partial b}{\partial s}P_{b,j}^\epsilon + 2\partial_s(\phi_b\rho_j)P^\epsilon - \frac{2\phi_b\rho_j}{b}\sum_{i=1}^{2}\frac{\partial}{\partial\sigma_i}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big) - \frac{\phi_b\rho_j}{b}\partial_s b(u^\epsilon\cdot n)$$
$$+ \int_s^\delta \Bigg[\Delta_\tau P_{b,j}^\epsilon - (\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon + \phi_b\rho_j\Bigg(\frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu_i^\epsilon u_j^\epsilon\big) + \frac{\partial^2 b}{\partial s^2}(u^\epsilon\cdot n)^2\Bigg]$$
$$- \Delta_\tau\big((u^\epsilon\cdot n)^2)\big)\Bigg) - \frac{\partial}{\partial s}\left(\frac{1}{b}\frac{\partial b}{\partial s}\right)P_{b,j}^\epsilon + 2\partial_s^2(\phi_b\rho_j)P^\epsilon - 2\frac{\partial}{\partial s}\left(\frac{2\phi_b\rho_j}{b}\right)\sum_{i=1}^{2}\frac{\partial}{\partial\sigma_i}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big)$$
$$- \frac{\partial}{\partial s}\left(\frac{\phi_b\rho_j}{b}\partial_s b\right)(u^\epsilon\cdot n)\Bigg]ds'.$$

Now we introduce the notation

$$\Lambda^\epsilon := -\frac{1}{b}\frac{\partial b}{\partial s}P_{b,j}^\epsilon + 2\partial_s(\phi_b\rho_j)P^\epsilon - \frac{2\phi_b\rho_j}{b}\sum_{i=1}^{2}\frac{\partial}{\partial\sigma_i}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big) - \frac{\phi_b\rho_j}{b}\partial_s b(u^\epsilon\cdot n),$$
$$\Theta^\epsilon := \Delta_\tau P_{b,j}^\epsilon - (\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon + \phi_b\rho_j\Bigg(\frac{1}{b}\Bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu_i^\epsilon u_j^\epsilon\big) + \frac{\partial^2 b}{\partial s^2}(u^\epsilon\cdot n)^2\Bigg]$$
$$- \Delta_\tau\big((u^\epsilon\cdot n)^2)\big)\Bigg) - \frac{\partial}{\partial s}\left(\frac{1}{b}\frac{\partial b}{\partial s}\right)P_{b,j}^\epsilon + 2\partial_s^2(\phi_b\rho_j)P^\epsilon - 2\frac{\partial}{\partial s}\left(\frac{2\phi_b\rho_j}{b}\right)\sum_{i=1}^{2}\frac{\partial}{\partial\sigma_i}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big)$$
$$- \frac{\partial}{\partial s}\left(\frac{\phi_b\rho_j}{b}\partial_s b\right)(u^\epsilon\cdot n).$$

This yields equation (5.3).

Now we multiply the equation we derived for $\partial_s P_{b,j}^\epsilon$ by a test function $\phi(\sigma_1,\sigma_2)\in H^2(U_j)$. We then integrate with respect to σ_1 and σ_2 once or twice, dependent on the term (such that the terms P^ϵ and $u^\epsilon\otimes u^\epsilon$ no longer have any derivatives). From this we obtain estimates (5.4) and (5.5) for Λ^ϵ and Θ^ϵ .

One can see that Λ_j^ϵ only contains first-order derivatives of u^ϵ with respect to σ_1 and σ_2, which means that $\Lambda^\epsilon(\cdot,\cdot,s)\in H^{-1}(U_j)$. The terms in Θ^ϵ have at most second-order derivatives in σ_1 and σ_2, so Θ^ϵ lies in $H^{-2}(U_j)$. □

6 Estimate of the boundary layer pressure

We will now establish an estimate for the boundary layer pressure, analogous to estimate (4.1). As was calculated before, the local boundary layer pressure satisfies the problem

$$-\Delta P_{b,j}^\epsilon = -(\Delta\phi_b\rho_j)P^\epsilon - 2\nabla_\tau(\phi_b\rho_j)\cdot\nabla_\tau P^\epsilon - 2\partial_s(\phi_b\rho_j)\frac{\partial P^\epsilon}{\partial s}$$

$$
\begin{aligned}
&+\phi_b\rho_j\bigg(\frac{1}{b}\bigg[\sum_{i,j=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial\sigma_j}\big(bu_i^\epsilon u_j^\epsilon\big)+2\sum_{i=1}^{2}\frac{\partial^2}{\partial\sigma_i\partial s}\big(bu_i^\epsilon(u^\epsilon\cdot n)\big)+\frac{\partial^2}{\partial s^2}\big(b(u^\epsilon\cdot n)^2\big)\bigg]\\
&-\Delta_\tau\big((u^\epsilon\cdot n)^2)\big)-\frac{1}{b}\frac{\partial b}{\partial s}\frac{\partial(u^\epsilon\cdot n)^2)}{\partial s}-\frac{\partial^2(u^\epsilon\cdot n)^2)}{\partial s^2}\bigg).\text{ in } V_{\delta,U_j},\\
&\partial_n P_{b,j}^\epsilon=\rho_j\big(u\otimes u:\nabla n\big),\text{ on }\partial\Omega\cap U_j,\ P_{b,j}^\epsilon=0\text{ on }\partial V_{\delta,U_j}\backslash\partial\Omega.
\end{aligned}
$$

The local boundary Schauder-type estimate for $P_{b,j}^\epsilon$ is given in the next proposition (for each region V_{δ,U_j}), the local estimates can then be patched together to yield a global estimate for P_b^ϵ on V_δ.

Proposition 6.1. *The boundary layer pressure satisfies the following local estimate*

$$
\|P_{b,j}^\epsilon\|_{C^{0,\alpha}(V_{\delta,U_j})}\leq C\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(V_{\delta,U_j})}+D\|P_{b,j}^\epsilon\|_{L^\infty(V_{\delta,U_j})}.
$$

Proof. We first observe that there indeed exists a unique solution $P_{b,j}^\epsilon$ because of Theorem B.1. The estimate is derived in the proof of Theorem B.2. □

7 Taking the limit $\epsilon\to 0$

Now we see that by collecting the estimates from Propositions 4.1 and 6.1 we find that

$$
\|P^\epsilon\|_{C^{0,\alpha}(\Omega)}\leq\|P_i^\epsilon\|_{C^{0,\alpha}(\Omega)}+\|P_b^\epsilon\|_{C^{0,\alpha}(\Omega)}\leq C\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)}+D\|P^\epsilon\|_{L^\infty(\Omega)}. \tag{7.1}
$$

We will now show that the inequality still holds without the term $D\|P^\epsilon\|_{L^\infty(\Omega)}$, up to an enlarging of the constant C.

Proposition 7.1. *The following estimate holds for P^ϵ*

$$
\|P^\epsilon\|_{L^\infty(\Omega)}\leq C\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)}. \tag{7.2}
$$

Once again, the constant C does not depend on ϵ.

Proof. We argue by contradiction. If the inequality does not hold, there exists a subsequence (which we still call P^ϵ) such that

$$
\lim_{\epsilon\to 0}\frac{\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)}}{\|P^\epsilon\|_{L^\infty(\Omega)}}=0. \tag{7.3}
$$

Now we introduce the following functions

$$
\mathcal{G}^\epsilon:=\frac{P^\epsilon}{\|P^\epsilon\|_{L^\infty(\Omega)}}.
$$

These functions solve the following boundary-value problem

$$
-\Delta\mathcal{G}^\epsilon=\frac{1}{\|P^\epsilon\|_{L^\infty(\Omega)}}((\nabla\otimes\nabla):(u^\epsilon\otimes u^\epsilon)-\Delta(\phi(u^\epsilon\cdot n)^2)\quad\text{in }\Omega, \tag{7.4}
$$

$$\partial_n \mathcal{G}^\epsilon = \frac{1}{\|P^\epsilon\|_{L^\infty(\Omega)}}\big(u \otimes u : \nabla n\big) \quad \text{on } \partial\Omega, \tag{7.5}$$

$$\int_\Omega \mathcal{G}^\epsilon(x)dx = \frac{1}{\|P^\epsilon\|_{L^\infty(\Omega)}}\int_\Omega \phi(x)(u^\epsilon\cdot n)^2 dx. \tag{7.6}$$

The sequence $\|\mathcal{G}^\epsilon\|_{C^{0,\alpha}(\Omega)}$ is bounded. By using the Arzel-Ascoli theorem we know that there exists a subsequence, for which we also write $\mathcal{G}^\epsilon$, converging strongly to a given function $\mathcal{G}$ in $C^0(\overline{\Omega})$. Note that it also converges in any Hölder space with exponent less than α, which can be seen by using an interpolation inequality.

By assumption we know that $\|\mathcal{G}^\epsilon\|_{L^\infty(\Omega)} = \|\mathcal{G}\|_{L^\infty(\Omega)} = 1$. It follows by equation (7.3) that the right-hand sides of equations (7.5) and (7.6) of the boundary-value problem for $\mathcal{G}^\epsilon$ all go to zero as $\epsilon \to 0$ in the space $C^{0,\beta}(\Omega)$ for $\beta \in [0,\alpha)$. This means that $\mathcal{G}$ satisfies the equation

$$-\Delta\mathcal{G} = 0 \quad \text{in } \Omega.$$

Next we show that $\partial_n\mathcal{G}$ is well-defined and is equal to zero. Using Lemma 5.2, we know that

$$\partial_s \mathcal{G}^\epsilon_{b,j}(\cdot,\cdot,s) = \frac{\Lambda^\epsilon_j(\cdot,\cdot,s)}{\|P^\epsilon\|_{L^\infty}} + \int_s^\delta \frac{\Theta^\epsilon_j(\cdot,\cdot,s')}{\|P^\epsilon\|_{L^\infty}}ds'.$$

We first observe that the map $s \mapsto \partial_s\mathcal{G}^\epsilon_{b,j}(\cdot,\cdot,s)$ is a map from $[0,\delta)$ to $H^{-2}(U_j)$. By using estimates (5.4) and (5.5), we find that

$$\begin{aligned}
\left\|\frac{\Lambda^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}([0,\delta);H^{-1}(U_j)]} &\le C_b\left\|\frac{u^\epsilon\otimes u^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}(\Omega)} + D_b,\\
\left\|\frac{\Theta^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}([0,\delta);H^{-2}(U_j)]} &\le C_b\left\|\frac{u^\epsilon\otimes u^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}(\Omega)} + D_b,
\end{aligned}$$

which means that the sequence $\{\partial_s\mathcal{G}^\epsilon_{b,j}\}$ is equicontinuous in s. We next show that for every $s\in[0,\delta]$, the sequence $\{\partial_s\mathcal{G}^\epsilon(\cdot,\cdot,s)\}$ has a convergent subsequence. We have that the sequence $\left\{\frac{\Lambda^\epsilon(\cdot,\cdot,s)}{\|P^\epsilon\|_{L^\infty}}\right\}$ is bounded uniformly in ϵ in $H^{-1}(U_j)$ for fixed s, then by the compact embedding of $H^{-1}(U_j)$ into $H^{-2}(U_j)$ we have a strongly convergent subsequence in $H^{-2}(U_j)$.

Now by examining the expression for $\partial_s P^\epsilon_{b,j}$ in the proof of Lemma 5.2, it can be seen that estimate (5.5) can be improved to

$$\left\|\frac{\Theta^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}([0,\delta);H^{-2+\alpha}(U_j)]} \le C_b\left\|\frac{u^\epsilon\otimes u^\epsilon}{\|P^\epsilon\|_{L^\infty}}\right\|_{C^{0,\alpha}(\Omega)} + D_b,$$

where we have used Proposition 6.1 in bounding the term $\Delta_\tau P^\epsilon_{b,j}$ in the $H^{-2+\alpha}(U_j)$ norm (with respect to the variables σ_1 and σ_2).

This allows us to conclude that the sequence $\left\{\frac{\Theta^\epsilon(\cdot,\cdot,s)}{\|P^\epsilon\|_{L^\infty}}\right\}$ has a convergent subsequence in $H^{-2}(U_j)$ for fixed $s\in[0,\delta]$. Therefore the same holds for the sequence $\Big\{\partial_s\mathcal{G}^\epsilon(\cdot,\cdot,s)\Big\}$. By the Arzela-Ascoli theorem we therefore conclude that the sequence $\{\partial_s\mathcal{G}^\epsilon\}$ has a convergent subsequence in $C^{0,\alpha}([0,\delta]; H^{-2}(U_j))$, we will refer to the limit as $\partial_s \mathcal{G}$. Moreover, the right-hand side of equation (7.4) goes to zero in $H^{-2}(\partial\Omega)$. We conclude that (as $\partial_s \mathcal{G}$ is continuous in s)

$$\partial_s \mathcal{G}(\cdot,\cdot,0) = 0 \quad \text{in } H^{-2}(\partial\Omega). \tag{7.7}$$

Hence $\mathcal{G}$ satisfies the following boundary-value problem

$$-\Delta \mathcal{G} = 0, \quad \text{in } \Omega \tag{7.8}$$

$$\partial_n \mathcal{G}|_{\partial\Omega} = 0, \quad \int_\Omega \mathcal{G}(x)dx = 0. \tag{7.9}$$

The only solution to this boundary-value problem is $\mathcal{G} = 0$, this is in contradiction with the assumption $\|\mathcal{G}\|_{L^\infty(\Omega)} = 1$. Therefore inequality (7.2) must hold. □

Remark 7.2. We are providing the full details of the proof of Proposition 7.1 also in part to provide a corrected proof of Proposition 3.11 in [3]. In particular, equation (3.41) as given in [3] is not correct. The statement itself of Proposition 3.11 in [3] is correct and the given proof can be adapted by using the method outlined above.

Finally we are able to prove Theorem 2.4.

Proof of Theorem 2.4. By Lemma 3.1 we know that

$$\|u^\epsilon \otimes u^\epsilon\|_{C^{0,\alpha}(\Omega)} \leq \|u \otimes u\|_{C^{0,\alpha}(\Omega)}.$$

Moreover, we know that $u^\epsilon \otimes u^\epsilon$ converges to $u \otimes u$ in the $C^{0,\beta}(\Omega)$ norm for any $\beta \in [0,\alpha)$. This follows from standard mollification estimates and interpolation inequalities.

Then combining inequalities (7.1) and (7.2), we find that $\|P^\epsilon\|_{C^{0,\alpha}(\Omega)}$ is bounded. This means that we are able to take a subsequence, which we also denote by P^ϵ, which converges in the $C^0(\overline{\Omega})$ norm to the limit $P \in C^{0,\alpha}(\Omega)$.

By using Lemma 5.2, we find that

$$\begin{aligned} P(\sigma_1,\sigma_2,s) - P(\sigma_1,\sigma_2,0) &= \lim_{\epsilon\to 0}(P^\epsilon(\sigma_1,\sigma_2,s) - P^\epsilon(\sigma_1,\sigma_2,0)) \\ &= \int_0^s \sum_{j=1}^m \lim_{\epsilon\to 0}\left(\Lambda_j^\epsilon(\sigma_1,\sigma_2,s') + \int_{s'}^\delta \Theta_j^\epsilon(\sigma_1,\sigma_2,s'')ds''\right)ds'. \end{aligned} \tag{7.10}$$

By estimates (5.4) and (5.5), we know that the limits $\lim_{\epsilon\to 0}\Lambda^\epsilon$ and $\lim_{\epsilon\to 0}\Theta^\epsilon$ exist as elements of $C^{0,\alpha}([0,\delta]; H^{-2}(\partial\Omega))$. This implies that $P \in C^1([0,\delta_3+\epsilon], H^{-2}(\partial\Omega))$ near the boundary. Because it holds that $\partial_s P^\epsilon_{b,j} = \rho_j\big(u^\epsilon \otimes u^\epsilon : \nabla n\big)$ on $U_j \cap \partial\Omega$, then by the established convergence we conclude

$$\partial_s P(\sigma_1,\sigma_2,0) = u \otimes u : \nabla n,$$

which holds in $H^{-2}(U_j)$ (locally for every patch U_j and then extended globally by using a partition of unity). This allows us to conclude that $P \in C^{0,\alpha}(\Omega)$ solves the boundary-value problem stated in Theorem 2.4. □

8 The necessity of a very weak boundary condition

It has been argued before that we should consider the following weaker boundary condition

$$\frac{\partial}{\partial n}(p + (u \cdot n)^2) = \big(u \otimes u\big) : \nabla n,$$

when $u \in C^{0,\alpha}(\Omega)$ for $\alpha \in (0, \frac{1}{2}]$, rather than the standard weak formulation of the boundary condition $\partial_n p = (u \otimes u) : \nabla n$. We present an example of a Hölder continuous incompressible vector field $u \in C^{0,\alpha}(\Omega)$ for $\alpha \in (0, \frac{1}{2}]$ for which $\partial_n (u \cdot n)^2$ is not well-defined as a distribution on $\partial\Omega$ when $0 < \alpha < \frac{1}{2}$; moreover, when $\alpha = \frac{1}{2}$ it is not equal to zero whenever it makes sense as a distribution on $\partial\Omega$, while the velocity field satisfies the boundary condition $(u \cdot n)|_{\partial\Omega} = 0$.

Example 8.1. We consider the following stream function (for $0 < \alpha \leq \frac{1}{2}$)

$$\psi(x, y, t) = -\frac{1}{\pi} \sum_{k=0}^{\infty} 2^{-(\alpha+1)k} \sin(2^k \pi x) \sin(2^k \pi y), \tag{8.1}$$

in the two-dimensional periodic channel, i.e.

$$\Omega := \mathbb{T} \times [0, 1].$$

The velocity field corresponding to this stream function is given by

$$u_1(x, y, t) = -\sum_{k=0}^{\infty} 2^{-\alpha k} \sin(2^k \pi x) \cos(2^k \pi y), \tag{8.2}$$

$$u_2(x, y, t) = \sum_{k=0}^{\infty} 2^{-\alpha k} \cos(2^k \pi x) \sin(2^k \pi y). \tag{8.3}$$

We will refer to this velocity field as the Weierstrass flow.

We claim that the Weierstrass flow satisfies the following properties:

1. The velocity field $u = (u_1, u_2)$ belongs to $C^{0,\alpha}(\Omega)$ for every $0 < \alpha \leq 1$.
2. It satisfies the boundary condition $(u \cdot n)|_{\partial\Omega} = 0$.
3. It is divergence-free in the sense of distributions.
4. It holds that $\partial_n (u \cdot n)^2|_{\partial\Omega} \notin \mathcal{D}'(\partial\Omega)$.

We now present a proof of this claim.

Proof. 1) Let

$$u_1^N(x, y) = -\sum_{k=0}^{N} 2^{-\alpha k} \sin(2^k \pi x) \cos(2^k \pi y),$$

$$u_2^N(x, y) = \sum_{k=0}^{N} 2^{-\alpha k} \cos(2^k \pi x) \sin(2^k \pi y).$$

Observe that the partial sums u_1^N and u_2^N are smooth $C^\infty(\Omega)$ functions, which converge uniformly, as $N \to \infty$, to u_1 and u_2, respectively. Therefore the limit u_1 and u_2 are continuous.

Next, we prove that the Weierstrass flow is $C^{0,\alpha}(\Omega)$. We will only prove that u_2 is Hölder continuous, as the proof for u_1 is similar. Observe that

$$\begin{aligned} u_2(x+h_1, y+h_2) - u_2(x,y) &= \sum_{k=0}^{\infty} 2^{-\alpha k} \cos(2^k \pi (x+h_1)) \sin(2^k \pi (y+h_2)) \\ &\quad - \sum_{k=0}^{\infty} 2^{-\alpha k} \cos(2^k \pi x) \sin(2^k \pi y). \end{aligned}$$

We can rewrite this as follows

$$\begin{aligned} u_2(x+h_1, y+h_2) - u_2(x,y) &= \sum_{k=0}^{\infty} 2^{-\alpha k} \cos(2^k \pi (x+h_1)) \Big(\sin(2^k \pi (y+h_2)) - \sin(2^k \pi y) \Big) \\ &+ \sum_{k=0}^{\infty} 2^{-\alpha k} \Big(\cos(2^k \pi (x+h_1)) - \cos(2^k \pi x) \Big) \sin(2^k \pi y) \\ &= \sum_{k=0}^{\infty} 2^{-\alpha k+1} \cos(2^k \pi (x+h_2)) \cos(2^{k-1} \pi (2x+h_2)) \sin(2^{k-1} \pi h_2) \\ &+ \sum_{k=0}^{\infty} 2^{-\alpha k+1} \sin(2^{k-1} \pi (2x+h_1)) \sin(2^{k-1} \pi h_1) \sin(2^k \pi y). \end{aligned}$$

Splitting the above sum into parts $0 \le k \le p_1$, $k > p_1$ respectively $0 \le k \le p_2$ and $k > p_2$ for some positive integers p_1 and p_2 satisfying $2^{p_1-1}|h_1| \le 1 < 2^{p_1}|h_1|$ and $2^{p_2-1}|h_2| \le 1 < 2^{p_2}|h_2|$. This implies

$$\begin{aligned} |u_2(x+h_1, y+h_2) - u_2(x,y)| &\le \sum_{k=0}^{\infty} 2^{-\alpha k+1} |\cos(2^k \pi (x+h_1))| |\cos(2^{k-1} \pi (2y+h_2))| |\sin(2^{k-1} \pi h_2)| \\ &+ \sum_{k=0}^{\infty} 2^{-\alpha k+1} |\sin(2^{k-1} \pi (2x+h_1))| |\sin(2^{k-1} \pi h_1)| |\sin(2^k \pi y)| \le \sum_{k=0}^{\infty} 2^{-\alpha k+1} |\sin(2^{k-1} \pi h_2)| \\ &+ \sum_{k=0}^{\infty} 2^{-\alpha k+1} |\sin(2^{k-1} \pi h_1)| \le \sum_{k=0}^{p_2} \Big(2^{-\alpha k+1} \cdot 2^{k-1} \pi |h_2| \Big) + \sum_{k=p_2+1}^{\infty} 2^{-\alpha k+1} \\ &+ \sum_{k=0}^{p_1} \Big(2^{-\alpha k+1} \cdot 2^{k-1} \pi |h_1| \Big) + \sum_{k=p_1+1}^{\infty} 2^{-\alpha k+1} \\ &= (2^{-\alpha(p_1+1)} + 2^{-\alpha(p_2+1)}) \frac{2}{1-2^{-\alpha}} + \pi |h_2| \frac{1 - 2^{(1-\alpha)(p_2+1)}}{1 - 2^{1-\alpha}} + \pi |h_1| \frac{1 - 2^{(1-\alpha)(p_1+1)}}{1 - 2^{1-\alpha}} \\ &\le (2^{-\alpha p_1} + 2^{-\alpha p_2}) \frac{2^{1-\alpha}}{1-2^{-\alpha}} + \pi |h_2| \frac{2^{(1-\alpha)(p_2+1)}}{2^{1-\alpha} - 1} + \pi |h_1| \frac{2^{(1-\alpha)(p_1+1)}}{2^{1-\alpha} - 1}. \end{aligned}$$

In the above we have used the fact that $|\sin z| \le |z|$ that $1 < 2^{p_1}|h_1| \le 2$ and $1 < 2^{p_2}|h_2| \le 2$. In particular, this means that

$$2^{-\alpha p_1} \le |h_1|^\alpha, \quad 2^{-\alpha p_2} \le |h_2|^\alpha.$$

From this we are able to conclude that

$$
\begin{aligned}
|u_2(x+h_1,y+h_2)-u_2(x,y)| &\leq (|h_1|^\alpha+|h_2|^\alpha)\frac{2^{1-\alpha}}{1-2^{-\alpha}}+\pi\frac{2^{2-\alpha}}{2^{1-\alpha}-1}|h_2|^\alpha+\pi\frac{2^{2-\alpha}}{2^{1-\alpha}-1}|h_1|^\alpha \\
&\leq 2^{1-\alpha}\left(\frac{1}{1-2^{-\alpha}}+\frac{2\pi}{2^{1-\alpha}-1}\right)|h|^\alpha,
\end{aligned}
$$

where $h=(h_1,h_2)$. Therefore $u\in C^{0,\alpha}(\Omega)$.
2) We demonstrate that the velocity field satisfies the boundary condition $u_2=u\cdot n=0$ on $\partial\Omega$. Indeed, one can check that

$$
u_2(x,0)=\sum_{k=0}^{\infty}2^{-\alpha k}\cos(2^k\pi x)\sin(2^k\pi\cdot 0)=0,
$$

since the function is continuous and the series converges uniformly. Similarly, one can check that $u_2(x,1)=0$ and hence $(u\cdot n)|_{\partial\Omega}=0$.
3) We will now show that the velocity field is divergence-free in the sense of distributions. We can easily check that

$$
\partial_x u_1^N+\partial_y u_2^N=0.
$$

This means that the partial sums u^N are weakly divergence-free, i.e.

$$
\int_\Omega u^N\cdot\nabla\phi dx=0,\quad \forall\phi\in\mathcal{D}(\Omega;\mathbb{R}).
$$

Since u_1^N and u_2^N converge in $L^\infty(\Omega)$ to u_1 and u_2, therefore it follows that

$$
\int_\Omega u\cdot\nabla\phi dx=0.
$$

We conclude that u is divergence-free in the sense of distributions.
4) Now we focus on the case when $\alpha\in(0,\frac{1}{2})$ and show that $\partial_n(u\cdot n)^2|_{\partial\Omega}$ cannot be defined as an element of $\mathcal{D}'(\partial\Omega)$. In particular, this implies that $\partial_n(u\cdot n)^2\notin H^{-2}(\partial\Omega)$. In fact, in Appendix C we will show that away from the boundary $\partial_y u_2^2(\cdot,y)$ cannot be defined as an element of $\mathcal{D}'(\mathbb{T})$ for a dense set of points $y\in[0,1]$. It should be noted that $\partial_y u_2$ is perfectly well-defined as a distribution on the whole domain, but as we will show below $\partial_y u_2^2(\cdot,y)$ might not be a distribution.

In this section, we will consider the case $y=0$, as this concerns the boundary condition. More precisely first one observes that the function (for $\theta\in\mathcal{D}(\mathbb{T})$)

$$
U(y;\theta)\coloneqq\langle u_2^2(\cdot,y),\theta\rangle=\int_{\mathbb{T}}u_2^2(x,y)\theta(x)dx \tag{8.4}
$$

belongs to $C^{0,\alpha}(0,1)$ and is equal to 0 for $y=0,1$.

Hence the existence of the derivative on the boundary (i.e. for $y=0$) follows if the following limit exists

$$
\lim_{y\to 0+}\frac{1}{y}\int_{\mathbb{T}}u_2^2(x,y)\theta(x)dx.
$$

As already observed in [3] this limit exists and is equal to 0 as long as $\frac{1}{2} < \alpha \leq 1$. To explore the behaviour for the case $0 < \alpha \leq \frac{1}{2}$ we will consider the Weierstrass series defined in equations (8.2) and (8.3) and as a consequence, in this situation one has

$$v(x,y) := (u_2(x,y))^2 = \sum_{k_1,k_2=0}^{\infty} \left(2^{-\alpha(k_1+k_2)} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\sin(2^{k_1}y)\sin(2^{k_2}y) \right),$$

$$U(y;\theta) = \sum_{k_1,k_2=0}^{\infty} \int_{\mathbb{T}} v(x,y)\theta(x)dx \tag{8.5}$$

$$= \sum_{k_1,k_2=0}^{\infty} \left(\int_{\mathbb{T}} 2^{-\alpha(k_1+k_2)} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\theta(x)dx \right) \sin(2^{k_1}y)\sin(2^{k_2}y).$$

The purpose of this section is to consider the case $0 < \alpha \leq \frac{1}{2}$ and to prove the following proposition regarding the function $U(y;\theta)$ as defined in equation (8.5).

Proposition 8.2.

1. *Suppose $\theta \in \mathcal{D}(\mathbb{T})$ satisfies*
$$\int_{\mathbb{T}} \theta(x)dx = 0, \tag{8.6}$$
then the limit
$$\lim_{y\to 0+} \frac{1}{y} \int_{\mathbb{T}} v(x,y)\theta(x)dx$$
is well-defined and is equal to 0.

2. *Otherwise if $0 < \alpha < \frac{1}{2}$ and*
$$\int_{\mathbb{T}} \theta(x)dx \neq 0, \tag{8.7}$$
then it holds that
$$\liminf_{y\to 0+} \frac{1}{y} \left| \int_{\mathbb{T}} v(x,y)\theta(x)dx \right| = \infty.$$
As a consequence the function
$$U(y;\theta) = \int_{\mathbb{T}} v(x,y)\theta(x)dx \tag{8.8}$$
does not have a well-defined derivative at the point $y = 0$.

3. *If $\alpha = \frac{1}{2}$ and*
$$\int_{\mathbb{T}} \theta(x)dx \neq 0, \tag{8.9}$$
then we have that
$$\liminf_{y\to 0+} \frac{1}{y} \left| \int_{\mathbb{T}} v(x,y)\theta(x)dx \right| \geq 2. \tag{8.10}$$
Consequently, if the derivative of $U(y,\theta)$ exists it is not equal to zero.

Proof. For the proof one first eliminates the nonresonant terms (i.e., the terms involving $k_1 \neq k_2$) and then a comparison argument is used. As such the subscripts R and NR are used to denote the resonant and nonresonant parts of U, respectively.

Then one has the following.

Lemma 8.3. *The functions*

$$U_{NR}(y,\theta) := \sum_{k_1,k_2=0,k_1\neq k_2}^{\infty} \left(2^{-\alpha(k_1+k_2)} \int_{\mathbb{T}} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\theta(x)dx \sin(2^{k_1}y)\sin(2^{k_2}y)\right) \tag{8.11}$$

belong to $C^1([0,1])$*, moreover we have that* $\partial_y U_{NR}(y;\theta)|_{y=0} = 0$.

Proof. One first recalls the following trigonometric identity

$$\cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x) = \frac{1}{2}(\cos((2^{k_1}+2^{k_2})\pi x) + \cos((2^{k_1}-2^{k_2})\pi x)).$$

Then for $k_1 \neq k_2$ and $m > 0$ it holds that

$$\begin{aligned}\int_{\mathbb{T}} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\theta(x)dx &= \frac{(-1)^m}{2(\pi(2^{k_1}+2^{k_2}))^{2m}} \int_{\mathbb{T}} \cos((2^{k_1}+2^{k_2})\pi x)\frac{d^{2m}}{dx^{2m}}\theta(x)dx \\ &+ \frac{(-1)^m}{2(\pi(2^{k_1}-2^{k_2}))^{2m}} \int_{\mathbb{T}} \cos((2^{k_1}-2^{k_2})\pi x)\frac{d^{2m}}{dx^{2m}}\theta(x)dx,\end{aligned} \tag{8.12}$$

moreover we have

$$\frac{d}{dy}(\sin(2^{k_1}\pi y)\sin(2^{k_2}\pi y)) = 2^{k_1}\pi\cos(2^{k_1}\pi y)\sin(2^{k_2}\pi y)) + 2^{k_2}\pi\sin(2^{k_1}\pi y)\cos(2^{k_2}\pi y). \tag{8.13}$$

Combining equations (8.12) and (8.13) one obtains that

$$\begin{aligned}&\left|\frac{d}{dy}\left(\int_{\mathbb{T}} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\theta(x)dx\right)\sin(2^{k_1}y)\sin(2^{k_2}y)\right| \\ &\leq C\pi\left(\frac{1}{2(\pi(2^{k_1}+2^{k_2}))^{2m}} + \frac{1}{2((\pi(2^{k_1}-2^{k_2}))^{2m}}\right)(2^{k_1}+2^{k_2})\int_{\mathbb{T}}\left|\frac{d^{2m}}{dx^{2m}}\theta(x)\right|dx\end{aligned} \tag{8.14}$$

which for $m \geq 1$ constitute the terms of an absolutely converging series.

As a consequence the function

$$U_{NR}(y;\theta) = \sum_{k_1,k_2=0,k_1\neq k_2}^{\infty} \left(2^{-\alpha(k_1+k_2)} \int_{\mathbb{T}} \cos(2^{k_1}\pi x)\cos(2^{k_2}\pi x)\theta(x)dx \sin(2^{k_1}\pi y)\sin(2^{k_2}\pi y)\right)$$

belongs to the space $C^1([0,1])$ and satisfies the relation:

$$\partial_y U_{NR}(y;\theta)|_{y=0} = 0. \tag{8.15}$$

□

Therefore one only has to consider the resonant part of the Weierstrass series, which is

$$U_R(y;\theta) = \sum_{k=0}^{\infty} 2^{-2\alpha k}\Big(\int_{\mathbb{T}} (\cos(2^k\pi x))^2\theta(x)dx\Big)(\sin(2^k\pi y)^2. \tag{8.16}$$

By using the identity

$$(\cos(2^k\pi x))^2 = \frac{1}{2}\Big(1+\cos(2^{k+1}\pi x)\Big), \tag{8.17}$$

one has that

$$\begin{aligned} U_R(y;\theta) &= \frac{1}{2}\Big(\int_{\mathbb{T}}\theta(x)dx\Big)\sum_{k=0}^{\infty}\Big(2^{-2\alpha k}(\sin(2^k\pi y))^2\Big) \\ &+ \frac{1}{2}\sum_{k=0}^{\infty}\Big[2^{-2\alpha k}\Big(\int_{\mathbb{T}}\cos(2^{k+1}\pi x)\theta(x)dx\Big)(\sin(2^k\pi y))^2\Big]. \end{aligned}$$

Similarly as before, the functions

$$U_{RNR}(y;\theta) = \frac{1}{2}\sum_{k=0}^{\infty}2^{-2\alpha k}\Big(\int_{\mathbb{T}}\cos(2^{k+1}\pi x)\theta(x)dx\Big)(\sin(2^k\pi y))^2 \tag{8.18}$$

are the terms of a series converging in $C^1([0,1])$ with derivative equal to 0 for $y=0$. Hence the completion of the proof now relies only on the analysis of the behaviour of the term

$$\frac{1}{y}U_{RR}(y;\theta) = \frac{1}{2y}\int_{\mathbb{T}}\theta(x)dx\sum_{k=0}^{\infty}2^{-2\alpha k}(\sin(2^k\pi y)^2, \tag{8.19}$$

which is equal to 0 when

$$\int_{\mathbb{T}}\theta(x)dx = 0. \tag{8.20}$$

This proves point 1 of Proposition 8.2. To prove point 2 one introduces the sequence $y_n = 2^{-n}$ (which is converging to 0, as $n\to\infty$) and consider the expression

$$\frac{1}{y_n}U_{RR}(y_n;\theta) = 2^{n-1}\sum_{k=0}^{\infty}2^{-2\alpha k}(\sin(2^k\pi 2^{-n}))^2. \tag{8.21}$$

We first observe that $\sin(2^k\pi 2^{-n}) = 0$ for $k\geq n$. Therefore, the above sum is actually given by

$$\frac{1}{y_n}U_{RR}(y_n;\theta) = 2^{n-1}\sum_{k=0}^{n-1}2^{-2\alpha k}(\sin(2^{k-n}\pi))^2. \tag{8.22}$$

Now we observe that for $0\leq k\leq n-1$ we have that $0\leq 2^{k-n}\pi\leq\frac{\pi}{2}$. We recall that for $x\in\big[0,\frac{\pi}{2}\big]$ it holds that $\sin(x)\geq\frac{2}{\pi}x$. Applying this to the series above gives

$$\begin{aligned} \frac{1}{y_n}U_{RR}(y_n;\theta) &\geq 2^{n-1}\sum_{k=0}^{n-1}2^{-2\alpha k}\Big(\frac{2}{\pi}2^{k-n}\pi\Big)^2 = 2^{-n+1}\sum_{k=0}^{n-1}2^{2k(1-\alpha)} \\ &= 2^{-n+1}\frac{2^{2n(1-\alpha)}-1}{2^{2(1-\alpha)}-1} = \frac{2}{2^{2(1-\alpha)}-1}\big(2^{n(1-2\alpha)}-2^{-n}\big). \end{aligned}$$

From the above we conclude that

$$\liminf_{n\to\infty} \frac{1}{y_n} U_{RR}(y_n;\theta) \begin{cases} = \infty & \text{if } 0 < \alpha < \frac{1}{2}, \\ \geq 2 & \text{if } \alpha = \frac{1}{2}, \\ \geq 0 & \text{if } \alpha > \frac{1}{2}. \end{cases} \tag{8.23}$$

Observing that

$$U(y,\theta) = U_{NR}(y,\theta) + U_R(y,\theta) = U_{NR}(y,\theta) + U_{RNR}(y,\theta) + U_{RR}(y,\theta)$$

completes the proof of the point 2 of Proposition 8.2. Since for $\alpha > \frac{1}{2}$ we have that $\lim 2^{n(1-2\alpha)}$ goes to 0 with $n \to \infty$ point 2 is not in contradiction with point 1 of Proposition 8.2 □

□

Remark 8.4. The definition of $\partial_n (u \cdot n)^2$ turns out to be a subtle issue for solutions in $C^{0,\alpha}(\Omega)$. In the case $\alpha > \frac{1}{2}$ the definition is trivial, as was observed before. For $0 < \alpha \leq \frac{1}{2}$ it depends on the mean of the function $\theta(x)$. The trace of the $\partial_y u_2^2$ term remains well-defined as an element of the dual of test functions with mean value 0, on the other hand it is no longer defined when the mean value of the test functions is not 0 .

In two space variables in a domain Ω with a geodesic change of variable the same results hold when introducing the Weierstrass flow as in equations (8.2) and (8.3). Now x is the tangential variable and y the distance to $\partial\Omega$.

Remark 8.5. In fact the whole derivation which was detailed above is local in nature and could be considered on the boundary $\partial\Omega$ or on any hyper surface $\Sigma \subset \partial\Omega$. This leads to the following theorem.

Theorem 8.6. *Let $\Sigma \subset \overline{\Omega}$ be a hypersurface with local geodesic coordinates, tangential coordinate x and normal coordinate y. Suppose $F(u)$ is a C^1 function and consider the trace*

$$\partial_y F(u)(x,y)|_{\Sigma}.$$

Then:

1. *If $u \in C^{0,\alpha}(\overline{\Omega})$ with $\frac{1}{2} < \alpha$ and $(u \cdot n)|_{\partial\Omega} = 0$, then the trace of $\partial_y F(u)$ on Σ is well-defined.*

2. *Otherwise if $0 < \alpha < \frac{1}{2}$, the trace $\partial_y F(u) \neq 0$ on Σ is not well-defined even as an element of $y \mapsto \mathcal{D}'(\Sigma)$.*

3. *If $\alpha = \frac{1}{2}$, even if the trace $\partial_y F(u)$ is well-defined, it is nonzero.*

9 Conclusion

The result proven in this paper is a generalisation of the result proven in [3] from the 2D to the 3D incompressible Euler equations. We use the very weak boundary condition introduced in [3] and moreover show its necessity, namely by constructing the example of a velocity field for which $\partial_n (u \cdot n)^2|_{\partial\Omega} \notin \mathcal{D}'(\partial\Omega)$, as outlined in section 8. One key difference with the work [3] is that we now use local parametrisations of the boundary as opposed to a global parametrisation. Note that in principle our approach can be generalised easily without many problems to any dimension, by using the higher-dimensional analogue of the coordinate transformation (2.4)-(2.6).

This result constitutes the last part of the proof of the first half of the Onsager conjecture (the sufficient conditions for energy conservation of weak solutions) in the presence of boundaries, which was given in [1]. Because the phenomenon of anomalous dissipation is intimately related with low regularity weak solutions of the Euler equations, we expect the very weak boundary condition considered in this work to have connections with the dissipation anomaly and turbulence.

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A Schauder-type estimate for Dirichlet problem

In this appendix, we will prove a Schauder-type estimate that will be used in the main body of the paper. We will use the Einstein summation convention in what follows. The estimate is given in the following theorem.

Theorem A.1. *Let $v \in H^1_0(\Omega)$ (where $\Omega \subset \mathbb{R}^3$ is an open set with $C^{3,\alpha}$ boundary for $\alpha > 0$) be the unique solution of the following problem*

$$\begin{cases} \Delta v = \partial_i \partial_j (F_{ij}) \quad \text{in } \Omega, \\ v = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{A.1}$$

where $F_{ij} \in C^{2,\alpha}(\Omega)$. Then the following estimate holds

$$\|v\|_{C^{0,\alpha}(\Omega)} \leq C \|F\|_{C^{0,\alpha}(\Omega)} \tag{A.2}$$

Proof. We recall from [19, Theorem 1] that the Green's function G exists and it satisfies the following problem

$$\begin{cases} \Delta_y G(x,y) = -\delta(x-y) \quad \text{for } x \in \Omega, \\ G(x,y) = 0 \quad \text{for } y \in \partial\Omega. \end{cases} \tag{A.3}$$

Moreover, from from [19, Theorem 1] we know that it satisfies the following pointwise estimates (in the case of three dimensions)

$$|G(x,y)| \leq C|x-y|^{-1}, \tag{A.4}$$
$$|D^\beta G(x,y)| \leq C|x-y|^{-2}, \tag{A.5}$$
$$|D^\gamma G(x,y)| \leq C|x-y|^{-3}, \tag{A.6}$$
$$|D^\gamma G(x,y) - D^\gamma G(z,y)| \leq |x-z|^\alpha \max\{|x-y|^{-3-\alpha}, |z-y|^{-3-\alpha}\}, \tag{A.7}$$
$$|D^\gamma G(x,y) - D^\gamma G(z,y)| \leq |x-z| \max\{|x-y|^{-4}, |z-y|^{-4}\}, \tag{A.8}$$

where β is a multi-index of order 1 and γ is a multi-index of order 2. Now using the Green's function we can write the solution as (where the partial derivatives are with respect to y)

$$\begin{aligned} v(x) &= -\int_\Omega G(x,y)\partial_i\partial_j(F_{ij}(y))dy = \int_\Omega \partial_i G(x,y)\partial_j(F_{ij}(y) - F_{ij}(x))dy - \int_{\partial\Omega} G(x,y)\partial_j F_{ij} n_i dS \\ &= -\int_\Omega \partial_i\partial_j G(x,y)(F_{ij}(y) - F_{ij}(x))dy + \int_{\partial\Omega} (F_{ij}(y) - F_{ij}(x))\partial_i G(x,y) n_j dS. \end{aligned}$$

In the first line the last integral vanishes due to the properties of the Green's function. Now we derive the Schauder estimate (where we let $\overline{x} = (x_1+x_2)/2$ and $\delta = |x_1 - x_2|$, see [28] for a related derivation)

$$\begin{aligned} &v(x_1) - v(x_2) = -\int_\Omega \partial_i\partial_j G(x_1,y)(F_{ij}(y) - F_{ij}(x_1))dy + \int_{\partial\Omega} (F_{ij}(y) - F_{ij}(x_1))\partial_i G(x_1,y) n_j dy \\ &- \int_{\partial\Omega} (F_{ij}(y) - F_{ij}(x_2))\partial_i G(x_2,y) n_j dy + \int_\Omega \partial_i\partial_j G(x_2,y)(F_{ij}(y) - F_{ij}(x_2))dy \\ &= \int_{\Omega\setminus B_\delta(\overline{x})} (\partial_i\partial_j G(x_1,y) - \partial_i\partial_j G(x_2,y))(F_{ij}(x_2) - F_{ij}(y))dy \\ &+ (F_{ij}(x_1) - F_{ij}(x_2)) \int_{\Omega\setminus B_\delta(\overline{x})} \partial_i\partial_j G(x_1,y)dy - \int_{\Omega\cap B_\delta(\overline{x})} (\partial_i\partial_j G(x_1,y)(F_{ij}(y) - F_{ij}(x_1))dy \\ &+ \int_{\Omega\cap B_\delta(\overline{x})} (\partial_i\partial_j G(x_2,y)(F_{ij}(y) - F_{ij}(x_2))dy - (F_{ij}(x_1) - F_{ij}(x_2)) \int_{\partial\Omega\setminus B_\delta(\overline{x})} \partial_i G(x_1,y)dy \\ &+ \int_{\partial\Omega\setminus B_\delta(\overline{x})} \big[\partial_i G(x_2,y) - \partial_i G(x_1,y)\big]\big[F_{ij}(x_2) - F_{ij}(y)\big] n_j dy \\ &+ \int_{\partial\Omega\cap B_\delta(\overline{x})} (F_{ij}(y) - F_{ij}(x_1))\partial_i G(x_1,y) n_j dy - \int_{\partial\Omega\cap B_\delta(\overline{x})} (F_{ij}(y) - F_{ij}(x_2))\partial_i G(x_2,y) n_j dy \\ &\leq C\delta^\alpha = C|x_1 - x_2|^\alpha, \end{aligned}$$

which follows from the following bounds (which rely on the pointwise estimates on the Green's function)

$$\begin{aligned} &\int_{\Omega\setminus B_\delta(\overline{x})} (\partial_i\partial_j G(x_1,y) - \partial_i\partial_j G(x_2,y))(F_{ij}(x_2) - F_{ij}(y))dy \leq \int_{\Omega\setminus B_\delta(\overline{x})} |x_1 - x_2||x_2 - y|^\alpha \\ &\cdot \max\{|x_1 - y|^{-4}, |x_2 - y|^{-4}\}dy \lesssim \delta^\alpha, \\ &(F_{ij}(x_1) - F_{ij}(x_2)) \int_{\Omega\setminus B_\delta(\overline{x})} \partial_i\partial_j G(x_1,y)dy = (F_{ij}(x_1) - F_{ij}(x_2)) \int_{\partial(\Omega\setminus B_\delta(\overline{x}))} \partial_j G(x_1,y) n_i dy \end{aligned}$$

$$\leq |x_1 - x_2|^\alpha \int_{\partial(\Omega \backslash B_\delta(\overline{x}))} |x_1 - y|^{-2} dy \lesssim \delta^\alpha,$$

$$\int_{\Omega \cap B_\delta(\overline{x})} (\partial_i \partial_j G(x_1, y)(F_{ij}(y) - F_{ij}(x_1))dy \leq \int_{\Omega \cap B_\delta(\overline{x})} |x_1 - y|^{-3+\alpha} dy \lesssim \delta^\alpha,$$

$$(F_{ij}(x_1) - F_{ij}(x_2)) \int_{\partial\Omega \backslash B_\delta(\overline{x})} \partial_i G(x_1, y) dy \leq |x_1 - x_2|^\alpha \int_{\partial\Omega \backslash B_\delta(\overline{x})} |x_1 - y|^{-2} dy \lesssim \delta^\alpha,$$

$$\int_{\partial\Omega \backslash B_\delta(\overline{x})} \left[\partial_i G(x_2, y) - \partial_i G(x_1, y)\right] \left[F_{ij}(x_2) - F_{ij}(y)\right] n_j dy$$

$$\leq \int_{\partial\Omega \backslash B_\delta(\overline{x})} |x_2 - x_1| \max\{|x_1 - y|^{-3}, |x_2 - y|^{-3}\} |x_2 - y|^\alpha \lesssim \delta^\alpha,$$

$$\int_{\partial\Omega \cap B_\delta(\overline{x})} (F_{ij}(y) - F_{ij}(x_1)) \partial_i G(x_1, y) n_j dy \leq \int_{\partial\Omega \cap B_\delta(\overline{x})} |x_1 - y|^{-2+\alpha} dy \lesssim \delta^\alpha,$$

the other bounds follow in a similar fashion. In the above we have used that $|y - x_1|^{-1}, |y - x_2|^{-1} \leq \frac{2}{\delta}$ on $\Omega \backslash B_\delta(\overline{x})$ (which can be seen by using the reverse triangle inequality).

This concludes the proof of the Schauder estimate. □

Remark A.2. This proof can be easily adapted to the case where terms of the form $c(x)v$ appear on the right-hand side.

B Schauder-type estimate for Dirichlet-Neumann problem

In section 6 we consider the pressure near the boundary. The resulting functions satisfy a Dirichlet-Neumann problem of the following type

$$\begin{cases} -\Delta v = f & \text{in } \Omega, \\ v = 0 & \text{in } \Gamma_D, \\ \partial_n v = g & \text{in } \Gamma_N. \end{cases} \tag{B.1}$$

Here Ω is still the domain, while Γ_D and Γ_N are open sets in $\partial\Omega$ such that $\Gamma_N = \partial\Omega \backslash \overline{\Gamma_D}$. We will first prove existence and uniqueness of solutions to this problem in the space $H^1_{0,\Gamma_D}(\Omega)$, which consists of the $H^1(\Omega)$ functions with zero trace on Γ_D. The weak formulation of the problem is (as can be found in [35, p. 516])

$$\int_\Omega \nabla v \cdot \nabla \psi dx = \int_\Omega f \psi dx + \int_{\Gamma_N} g \psi d\sigma, \tag{B.2}$$

where ψ is an arbitrary test function in $H^1_{0,\Gamma_D}(\Omega)$. We will first prove existence and uniqueness, based on the method outlined in [35, Chapter 8].

Theorem B.1. *The Dirichlet-Neumann problem* (B.1) *has a unique solution in* $H^1_{0,\Gamma_D}(\Omega)$.

Proof. By Theorem 7.91 in [35], we know that the Poincar inequality holds for functions in $H^1_{0,\Gamma_D}(\Omega)$. This means in particular that we may take the following norm

$$\|u\|_{H^1_{0,\Gamma_D}} := \|\nabla u\|_{L^2}. \tag{B.3}$$

Now we prove the existence and uniqueness for the Dirichlet-Neumann problem by using the Lax-Milgram theorem. We introduce the bilinear form $B : H^1_{0,\Gamma_D}(\Omega) \times H^1_{0,\Gamma_D}(\Omega) \to \mathbb{R}$ which is given by

$$B[v_1, v_2] := \int_\Omega \nabla v_1 \cdot \nabla v_2 dx.$$

It is easy to verify that B is coercive and continuous in the space H^1_{0,Γ_D}. Moreover, since $f \in L^2(\Omega)$ and $g \in L^2(\partial\Omega)$ it is easy to check that the map $\psi \mapsto \int_\Omega f\psi dx + \int_{\Gamma_N} g\psi d\sigma$ is in $H^{-1}_{0,\Gamma_D}(\Omega)$. Therefore by the Lax-Milgram theorem there exists a unique solution in $H^1_{0,\Gamma_D}(\Omega)$ for the Dirichlet-Neumann problem given in equation (B.1). □

Now for the sake of completeness we would like to establish a Schauder estimate for the Dirichlet-Neumann problem, as it does not seem to be stated in standard references such as [23, 24, 29]. In order to do so, we will rely on the approach given in [31]. We will prove the following result.

Theorem B.2. *Let $\partial\Omega \cap U_j$ be a patch of $\partial\Omega$ given by the localisation. Let $P^\epsilon_{b,j}$ be a solution to the following problem*

$$\begin{cases} -\Delta P^\epsilon_{b,j} = -(\Delta\phi_b\rho_j)P^\epsilon - 2(\nabla\phi_b\rho_j)\cdot\nabla P^\epsilon + \phi_b\rho_j\Big((\nabla\otimes\nabla) : (u^\epsilon\otimes u^\epsilon) - \Delta\big((u^\epsilon\cdot n)^2\big)\Big) \text{ in } V_{\delta,U_j}, \\ P^\epsilon = 0 \quad \text{on } \partial V_{\delta,U_j}\backslash(U_j\cap\partial\Omega), \quad \partial_n P^\epsilon = \rho_j\big(u\otimes u : \nabla n\big) \quad \text{on } \partial\Omega\cap U_j. \end{cases} \tag{B.4}$$

Then $P^\epsilon_{b,j}$ satisfies the following Schauder-type estimate

$$\|P^\epsilon_{b,j}\|_{C^{0,\alpha}(V_{\delta,U_j})} \le C\|u^\epsilon\otimes u^\epsilon\|_{C^{0,\alpha}(V_{\delta,U_j})} + D\|P^\epsilon_{b,j}\|_{L^\infty(V_{\delta,U_j})}. \tag{B.5}$$

Proof. By Theorem B.1 we know that problem (B.4) has a unique solution in $H^1_{0,\partial V_{\delta,U_j}\backslash(U_j\cap\partial\Omega)}(\Omega)$.

Now because $\partial\Omega \in C^2$ we can map $U_j\cap\Omega$ by a C^2 mapping ψ such that $\psi(\partial\Omega\cap U_j)$ is flat (or alternatively, it is characterised by the second coordinate being 0) and $\psi(V_{\delta,U_j})$ is the upper half of an open ball. If this is not possible, we can restrict to subsets of U_j such that the part of the boundary which intersects with $\partial\Omega$ can be mapped to a flat set, by compactness there are finitely many such sets. We can therefore transform the problem to (see [29, Section 6.2] for concrete computations)

$$\begin{cases} -a^{ij}(x)\partial_i\partial_j p^\epsilon + b_i(x)\partial_i p^\epsilon(x) = -(\Delta\phi_b\rho_j)P^\epsilon - 2(\nabla\phi_b\rho_j)\cdot\nabla P^\epsilon + \phi_b\rho_j\Big((\nabla\otimes\nabla) : (u^\epsilon\otimes u^\epsilon) \\ -\Delta\big((u^\epsilon\cdot n)^2\big)\Big) =: F' \quad \text{in } \psi(V_{\delta,U_j}), \\ p^\epsilon = 0 \quad \text{in } \psi\big(\partial V_{\delta,U_j}\backslash(\partial\Omega\cap U_j)\big)\big), \quad \partial_n p^\epsilon = \rho_j\big(u\otimes u : \nabla n\big) \quad \text{if } x_n = 0. \end{cases} \tag{B.6}$$

We have replaced P^ϵ by p^ϵ, in order to distinguish the 'unknown' from the source terms. In addition, the uniform ellipticity is preserved by Lemma 6.2.1 in [29] (this is why the C^2

regularity assumption on the boundary is crucial). Now we want to homogenise the Neumann boundary condition. One can find a function $G \in C^\infty(\partial\Omega)$ such that

$$\frac{\partial G}{\partial x_n} = \rho_j\big(u \otimes u : \nabla n\big) \quad \text{if } x_n = 0.$$

Then by taking (where F' is the forcing on the right-hand side of the equation in problem B.6)

$$\begin{aligned} F_1 &= F' + a^{ij}\partial_i\partial_j G - b_i\partial_i G, \\ F_2 &= -G, \end{aligned}$$

we end up with the problem

$$\begin{cases} -a^{ij}(x)\partial_i\partial_j\partial_j p^\epsilon + b_i(x)\partial_i p^\epsilon(x) = F_1 \quad \text{in } \psi(V_{\delta,U_j}), \\ p^\epsilon = F_2 \quad \text{in } \psi\big(\partial V_{\delta,U_j}\backslash(\partial\Omega \cap U_j))\big), \\ \partial_n p^\epsilon = 0 \quad \text{if } x_n = 0. \end{cases} \tag{B.7}$$

Then we take an even extension of F and ψ with respect to x_n. We write $\widehat{\psi(V_{\delta,U_j})}$ for the even extension of $\psi(V_{\delta,U_j})$ through $x_n = 0$. This then leads to the problem

$$\begin{cases} -a^{ij}(x)\partial_i\partial_j p^\epsilon + b_i(x)\partial_i p^\epsilon(x) = F_1 \quad \text{in } \widehat{\psi(V_{\delta,U_j})}, \\ p^\epsilon = F_2 \quad \text{on } \partial\widehat{\psi(V_{\delta,U_j})}. \end{cases} \tag{B.8}$$

Now we will prove the Schauder estimate by the continuity method, see [23, Section 5.5.1] for example. We first define the following operator

$$L \coloneqq -a^{ij}(x)\partial_i\partial_j + b_i(x)\partial_i,$$

which is the differential operator of the stated Dirichlet problem (B.8). Then we introduce the continuous family of operators

$$L_t \coloneqq (1-t)\Delta + tL.$$

For all $t \in [0,1]$ the problem associated with L_t and the data F_1 and F_2 has a unique solution. Now we need to prove that it satisfies the Schauder estimate from Theorem A.1. We define the subset $\Sigma \subset [0,1]$ such that for all $t \in \Sigma$ the estimate is satisfied. Now we will prove that Σ is both open and closed and hence $\Sigma = [0,1]$. We first observe that for $t = 0$ by Theorems 4.1 and A.1 estimate (B.5) holds.

We will first show that Σ is closed. Suppose we take a sequence $t_k \to t$, such that for all t_k estimate B.5 holds. Then by compactness we know that there exists a subsequence of solutions to the problems with operators L_{t_k} (which we label with v_{t_k}) converging uniformly to v_t (i.e. strong convergence in $C^0(\Omega)$). This means that the Schauder estimate also holds for v_t.

Now we have to show that Σ is open. We take an arbitrary $t_0 \in \Sigma$. We denote by $u_w = T_t w$ the solution to the problem

$$L_{t_0}u_w = (L_{t_0} - L_t)w + F_1 \text{ in } \widehat{\psi(V_{\delta,U_j})}, \quad u_w = F_2 \text{ on } \partial\widehat{\psi(V_{\delta,U_j})}.$$

Then we observe that $L_{t_0} - L_t = (t - t_0)\Delta + (t_0 - t)L$. By the assumed Schauder estimate we get that (by an adaption of the proof of Theorem A.1)

$$\|T_t w_1 - T_t w_2\|_{C^{0,\alpha}} \leq c|t - t_0|\|w_1 - w_2\|_{C^{0,\alpha}}. \tag{B.9}$$

Then if $|t - t_0|$ is sufficiently small, the mapping T_t is a contraction and hence the fixed point solves the Dirichlet problem with the operator L_t. Therefore the Schauder estimate holds for a small neighbourhood around t and hence Σ is open and therefore $\Sigma = [0, 1]$ (as Σ cannot be the empty set). In particular, estimate (B.5) is true for the case $t = 1$, which is what we wanted to show. □

C The normal derivative of the Weierstrass flow away from the boundary

We recall that in Example 8.1 we constructed a flow (given in equations (8.2) and (8.3)) such that $\partial_n(u \cdot n)^2|_{\partial\Omega} \notin \mathcal{D}'(\partial\Omega)$. In this appendix, we will show more, namely that actually $\partial_n(u\cdot n)^2(\cdot, y) \notin \mathcal{D}'(\mathbb{T})$ for any $y = \frac{j}{2^m}$, where $j = 1, 2, \ldots, 2^m - 1$ and $m \geq 1$. By reexamining the proofs in section 8, one can check that for any $y > 0$ $U_{NR}(\cdot; \theta)$ and $U_{RNR}(\cdot; \theta)$ are C^1 functions.

We recall that U_{RR} was defined by

$$U_{RR}(y; \theta) = \frac{1}{2} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{\infty} 2^{-2\alpha k}(\sin(2^k \pi y)^2, \tag{C.1}$$

Now we will consider the following difference quotient for some $y_1 = \frac{j}{2^m}$ (for some $m \geq 2$ and $j = 1, 2, \ldots, 2^m$)

$$\frac{U_{RR}(y_1 + h; \theta) - U_{RR}(y_1; \theta)}{|h|} = \frac{1}{2|h|} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{\infty} 2^{-2\alpha k} \Big[(\sin(2^k \pi(y_1 + h))^2 - (\sin(2^k \pi y_1))^2\Big].$$

We will first rewrite the difference quotient as

$$\frac{U_{RR}(y_1 + h; \theta) - U_{RR}(y_1; \theta)}{|h|} = \frac{1}{4|h|} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{\infty} 2^{-2\alpha k} \Big[\cos(2^{k+1}\pi y_1) - \cos(2^{k+1}\pi(y_1 + h))\Big]$$
$$= \frac{1}{2|h|} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{\infty} 2^{-2\alpha k} \sin(2^k \pi(2y_1 + h)) \sin(2^k \pi h).$$

Now once again we select the sequence $h_n = 2^{-n}$, as we did in Example 8.1. Once again, we notice that $\sin(2^{k-n}\pi) = 0$ for $k \geq n$, so we end up with the partial sum

$$\frac{U_{RR}(y_1 + h_n; \theta) - U_{RR}(y_1; \theta)}{|h_n|} = 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{n-1} 2^{-2\alpha k} \sin(2^k \pi(2y_1 + 2^{-n})) \sin(2^{k-n}\pi)$$
$$= 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{n-1} 2^{-2\alpha k} \Big[\sin(2^{k+1}\pi y_1) \cos(2^{k-n}\pi) + \cos(2^{k+1}\pi y_1) \sin(2^{k-n}\pi)\Big] \sin(2^{k-n}\pi).$$

Now we substitute our choice for y_1. Since we are interested in the behaviour of U_{RR} as $n \to \infty$ we may assume without loss of generality that $m \leq n$. Then we obtain that

$$\frac{U_{RR}(y_1 + h_n;\theta) - U_{RR}(y_1;\theta)}{|h_n|} = 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{n-1} 2^{-2\alpha k} \Big[\sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)$$

$$+ \cos(2^{k+1-m}\pi j)\sin(2^{k-n}\pi) \Big] \sin(2^{k-n}\pi)$$

$$= 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-2} \Big[2^{-2\alpha k} \sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)\sin(2^{k-n}\pi) \Big] \tag{C.2}$$

$$+ 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-1} \Big[2^{-2\alpha k} \cos(2^{k+1-m}\pi j)(\sin(2^{k-n}\pi))^2 \Big] \tag{C.3}$$

$$+ 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=m}^{n-1} \Big[2^{-2\alpha k}(\sin(2^{k-n}\pi))^2 \Big]. \tag{C.4}$$

We first investigate the third sum in line (C.4), we estimate

$$2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=m}^{n-1} 2^{-2\alpha k}(\sin(2^{k-n}\pi))^2 \geq 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=m}^{n-1} 2^{-2\alpha k} \cdot 2^{2k-2n+2}$$

$$= 2^{-n+1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=m}^{n-1} 2^{2(1-\alpha)k} = 2^{-n+1} \int_{\mathbb{T}} \theta(x)dx \left[\frac{2^{2(1-\alpha)n} - 1}{2^{2(1-\alpha)} - 1} - \frac{2^{2(1-\alpha)m} - 1}{2^{2(1-\alpha)} - 1} \right]$$

$$= \int_{\mathbb{T}} \theta(x)dx \cdot \frac{2^{(1-2\alpha)n+1}}{2^{2(1-\alpha)} - 1} + 2^{-n+1} \int_{\mathbb{T}} \theta(x)dx \left[-\frac{1}{2^{2(1-\alpha)} - 1} - \frac{2^{2(1-\alpha)m} - 1}{2^{2(1-\alpha)} - 1} \right].$$

Now we need to show that the other two sums remain bounded, we observe that for $x \in \left[0, \frac{\pi}{2}\right]$ we have that $\frac{2}{\pi}x \leq \sin(x) \leq x$, as a consequence we obtain

$$2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-1} \Big[2^{-2\alpha k} \cos(2^{k+1-m}\pi j)(\sin(2^{k-n}\pi))^2 \Big]$$

$$\geq 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-1} \Big[2^{-2\alpha k} \min\{\cos(2^{k+1-m}\pi j)2^{2k-2n+2}, 2^{-2\alpha k} \min\{\cos(2^{k+1-m}\pi j)2^{2k-2n}\pi^2\} \Big]$$

$$= 2^{-n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-1} \Big[2^{-2\alpha k} \min\{\cos(2^{k+1-m}\pi j)2^{2k+2}, 2^{-2\alpha k} \min\{\cos(2^{k+1-m}\pi j)2^{2k}\pi^2\} \Big],$$

$$2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-2} \Big[2^{-2\alpha k} \sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)\sin(2^{k-n}\pi) \Big]$$

$$\geq 2^{n-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-2} \Big[2^{-2\alpha k} \min\{\sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)2^{k-n+1}, \sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)2^{k-n}\pi\} \Big]$$

$$= 2^{-1} \int_{\mathbb{T}} \theta(x)dx \sum_{k=0}^{m-2} \Big[2^{-2\alpha k} \min\{\sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)2^{k+1}, \sin(2^{k+1-m}\pi j)\cos(2^{k-n}\pi)2^{k}\pi\} \Big].$$

Therefore both sums can be either bounded from below independent of n or they go to zero as $n \to \infty$. Because the sum in equation (C.4) is going to infinity, we conclude that if $\alpha < \frac{1}{2}$ we have that

$$\liminf_{n\to\infty} \left| \frac{U_{RR}(y_1 + h_n; \theta) - U_{RR}(y_1; \theta)}{h_n} \right| = \infty, \tag{C.5}$$

for points y_1 of the form $y_1 = \frac{j}{2^m}$ for $j = 1, \ldots, 2^{m-1}$ and $m \geq 1$. We conclude as a result that $\partial_y u_2^2(\cdot, y)$ cannot be defined as a distribution (i.e. as an element of $\mathcal{D}'(\mathbb{T})$) for a dense set of points $y \in [0, 1]$.

In the case $\alpha = \frac{1}{2}$ we have that

$$\liminf_{n\to\infty} \left| \frac{U_{RR}(y_1 + h_n; \theta) - U_{RR}(y_1; \theta)}{h_n} \right| \geq \left| \int_{\mathbb{T}} \theta(x) dx \right|$$
$$\cdot \left| 2 + \sum_{k=0}^{m-2} \left[2^{-2\alpha k - 1} \min \left\{ \sin(2^{k+1-m}\pi j) \cos(2^k \pi) 2^{k+1}, \sin(2^{k+1-m}\pi j) \cos(2^k \pi) 2^k \pi \right\} \right] \right|,$$

it therefore depends on the values of j and m whether this lower bound is nonzero.

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