On the incompressible limit of a strongly stratified heat conducting fluid

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Abstract

A compressible, viscous and heat conducting fluid is confined between two parallel plates maintained at a constant temperature and subject to a strong stratification due to the gravitational force. We consider the asymptotic limit, where the Mach number and the Froude number are of the same order proportional to a small parameter. We show the limit problem can be identified with Majda's model of layered "stack-of-pancake" flow.

Keywords: Navier–Stokes–Fourier system, stratified fluid, incompressible limit, Majda's model

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1 Introduction

Consider the motion of a compressible viscous and heat conducting fluid confined between two parallel plates. For simplicity, we suppose the motion is space–periodic with respect to the horizontal variable. Consequently, the spatial domain Ω may be identified with

$$\Omega = \mathbb{T}^2 \times (0,1),\ \mathbb{T}^2 = \left([-1,1]\Big|_{\{-1,1\}} \right)^2 .$$

The time evolution of the fluid mass density $\varrho = \varrho(t,x)$, the absolute temperature $\vartheta = \vartheta(t,x)$, and the velocity $\mathbf{u} = \mathbf{u}(t,x)$ is governed by the *Navier–Stokes–Fourier (NSF) system*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon^2}\nabla_x p(\varrho,\vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) + \frac{1}{\varepsilon^2}\varrho \nabla_x G, \tag{1.2}$$

$$\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) = \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)\cdot \nabla_x \vartheta}{\vartheta} \right), \tag{1.3}$$

supplemented with the Dirichlet boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0, \tag{1.4}$$

$$\vartheta|_{\partial\Omega} = \vartheta_B. \tag{1.5}$$

The viscous stress tensor is given by Newton's rheological law

$$\mathbb{S}(\vartheta, \nabla_x \mathbf{u}) = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \lambda(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{1.6}$$

and the internal energy flux by Fourier's law

$$\mathbf{q}(\vartheta, \nabla_x \vartheta) = -\kappa(\vartheta)\nabla_x \vartheta. \tag{1.7}$$

The quantity $s = s(\varrho,\vartheta)$ in (1.3) is the entropy of the system, related to the pressure $p = p(\varrho,\vartheta)$ and the internal energy $e = e(\varrho,\vartheta)$ through Gibbs' equation

$$\vartheta D s = De + pD\left(\frac{1}{\varrho}\right). \tag{1.8}$$

The potential G represents the effect of gravitation. The Mach number $\mathrm{Ma} = \varepsilon$ and the Froude number $\mathrm{Fr} = \varepsilon$ are both proportional to a small parameter. If $\varepsilon > 0$ is small, the fluid is almost incompressible and strongly stratified, cf. Klein et al. [11]. Our goal is to identify the limit problem for $\varepsilon \to 0$.

1.1 Asymptotic limit

In accordance with the scaling of (1.2), (1.3), the zero–th order terms in the asymptotic limit are determined by the stationary (static) problem

$$\nabla_x p(\varrho, \vartheta) = \varrho \nabla_x G. \tag{1.9}$$

Applying **curl** operator to identity (1.9), we successively deduce

$$\nabla_x \varrho \times \nabla_x G = 0,$$

and

$$\frac{\partial p(\varrho,\vartheta)}{\partial \varrho} \nabla_x \varrho + \frac{\partial p(\varrho,\vartheta)}{\partial \vartheta} \nabla_x \vartheta = \varrho \nabla_x G \ \Rightarrow \ \nabla_x \vartheta \times \nabla_x G = 0,$$

where we have anticipated that the pressure also depends non-trivially on the temperature ϑ and is such that $\frac{\partial p(\varrho,\vartheta)}{\partial \vartheta} \neq 0$. Thus for the static problem (1.9) to be solvable, both $\nabla_x \varrho$ and $\nabla_x \vartheta$ must be parallel to $\nabla_x G$. This fact imposes certain restrictions on the distribution of the boundary temperature ϑ_B. In particular, the motion in an inclined layer studied by Daniels et al. [5] does not admit any static solution. Accordingly, we focus on the particular case

$$G = -g x_3, \ \vartheta_B = \left\{ \begin{array}{l} \Theta_{\rm up} \ \mbox{if} \ x_3 = 1, \\ \Theta_{\rm bott} \ \mbox{if} \ x_3 = 0, \end{array} \right.$$
$$\mbox{where } g > 0, \ \Theta_{\rm up} > 0, \ \Theta_{\rm bott} > 0 \mbox{ are constant.} \tag{1.10}$$

Fixing the temperature profile $\vartheta_B = \Theta(x_3)$ to comply with the boundary conditions (1.10), we may recover $\varrho = r(x_3)$ as a solution of the ODE

$$\frac{\partial p(r,\Theta)}{\partial \varrho} \partial_{x_3} r + \frac{\partial p(r,\Theta)}{\partial \vartheta} \partial_{x_3} \Theta = -rg. \tag{1.11}$$

Needless to say, such a problem may admit infinitely many solutions.

To simplify, we focus on the case $\Theta_{\rm bott} = \Theta_{\rm up} > 0$. Accordingly, we consider the reference temperature profile $\Theta = \Theta_{\rm bott} = \Theta_{\rm up}$ - a positive constant. Then it follows from (1.11) that the static density profile $r = r(x_3)$ must be non–constant as long as $g \neq 0$. Anticipating the asymptotic limit

$$\varrho_\varepsilon \to r, \ \vartheta_\varepsilon \to \Theta, \ \mathbf{u}_\varepsilon \to \mathbf{U} \ \mbox{(in some sense)}$$

we deduce from the equation of continuity (1.1)

$${\rm div}_x (r \mathbf{U}) = 0. \tag{1.12}$$

Applying (formally) the same argument to the entropy balance (1.3) we get

$${\rm div}_x (r s(r,\Theta) \mathbf{U}) = 0. \tag{1.13}$$

Equations (1.12), (1.13) are compatible only if

$$\nabla_x r \cdot \mathbf{U} = 0.$$

As r depends only on the vertical x_3-variable, this yields

$$U_3 \equiv 0. \tag{1.14}$$

In view of the previous arguments, the limit fluid motion exhibits the "stack of pancakes structure" described in Chapter 6 of Majda's book [14]. Specifically, $\mathbf{U} = [\mathbf{U}_h, 0]$, and

$$\frac{\partial p(r,\Theta)}{\partial \varrho} \partial_{x_3} r = -rg, \tag{1.15}$$

$$\mathrm{div}_h \mathbf{U}_h = 0, \tag{1.16}$$

$$r\Big(\partial_t \mathbf{U}_h + \mathbf{U}_h \cdot \nabla_h \mathbf{U}_h\Big) + \nabla_h \Pi = \mu(\Theta)\Delta_h \mathbf{U}_h + \mu(\Theta)\partial^2_{x_3,x_3}\mathbf{U}_h. \tag{1.17}$$

Here and hereafter, the subscript h refers to the horizontal variable $x_h = (x_1, x_2)$, $\nabla_h = [\partial_{x_1}, \partial_{x_2}]$, $\mathrm{div}_h \mathbf{v} = \nabla_h \cdot \mathbf{v}$, $\Delta_h = \mathrm{div}_h \nabla_h$. The fluid motion is purely horizontal, the coupling between different layers only through the vertical component of the viscous stress.

To the best of our knowledge, there is no rigorous justification of the system (1.15)–(1.17) available in the literature except the inviscid case discussed in [7]. It is worth noting that a similar problem for the barotropic Navier–Stokes system gives rise to a *different* limit, namely the so–called anelastic approximation, see Masmoudi [15] or Feireisl et al. [8]. Furthermore, as observed in [3], the related case of a *low stratification* with $\mathrm{Ma} = \varepsilon$ and $\mathrm{Fr} = \sqrt{\varepsilon}$ leads to a limiting system of Oberbeck-Boussinesq type with non-local boundary conditions for the temperature.

1.2 The strategy of the convergence proof

We start with the concept of *weak* solutions for the NSF system with Dirichlet boundary conditions introduced in [4]. In particular, we recall the ballistic energy and the associated relative energy inequality in Section 2. Next, we introduce the concept of *strong* solutions to Majda's system in Section 3. In Section 4, we state our main result.

The strategy is of type "weak" → "strong", meaning the strong solution of the target system is used as a "test function" in the relative energy inequality associated to the primitive system. In Section 5, we derive the basic energy estimates that control the amplitude of the fluid velocity as well as the distance of the density and temperature profiles from their limit values independent of the scaling parameter ε. In Section 6, we show convergence to the target system (1.15)–(1.17) anticipating the latter admits a regular solution. This formal argument is made rigorous in Section 7, where global existence for Majda's model is established. The last result may be of independent interest.

2 Weak solutions to the primitive NSF system

Our analysis is based on the concept of weak solutions to the NSF system introduced in [4], cf. also [10].

Definition 2.1 (Weak solution to the NSF system). We say that a trio $(\varrho, \vartheta, \mathbf{u})$ is a weak solution of the NSF system (1.1)–(1.7), with the initial data

$$\varrho(0,\cdot) = \varrho_0,\ \ \varrho\mathbf{u}(0,\cdot) = \varrho_0\mathbf{u}_0,\ \ \varrho s(0,\cdot) = \varrho_0 s(\varrho_0, \vartheta_0),$$

if the following holds:

- The solution belongs to the **regularity class**:

$$\begin{aligned} \varrho &\in L^\infty(0,T;L^\gamma(\Omega)) \text{ for some } \gamma > 1,\ \ \varrho \geq 0 \text{ a.a. in } (0,T)\times\Omega, \\ \mathbf{u} &\in L^2(0,T;W^{1,2}_0(\Omega;R^3)), \\ \vartheta^{\beta/2},\ \log(\vartheta) &\in L^2(0,T;W^{1,2}(\Omega)) \text{ for some } \beta \geq 2,\ \ \vartheta > 0 \text{ a.a. in } (0,T)\times\Omega, \\ (\vartheta - \vartheta_B) &\in L^2(0,T;W^{1,2}_0(\Omega)), \end{aligned} \tag{2.1}$$

  where ϑ_B is an extension of the boundary data to the whole Ω.

- The **equation of continuity** (1.1) is satisfied in the sense of distributions,

$$\int_0^T \int_\Omega \Big[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi \Big] \ \mathrm{d}x\,\mathrm{d}t = -\int_\Omega \varrho(0)\varphi(0,\cdot)\ \ \mathrm{d}x \tag{2.2}$$

  for any $\varphi \in C^1_c([0,T)\times\overline{\Omega})$.

- The **momentum equation** (1.2) is satisfied in the sense of distributions,

$$\begin{aligned} &\int_0^T \int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\boldsymbol{\varphi} + \frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathrm{div}_x\boldsymbol{\varphi} \right] \ \mathrm{d}x\,\mathrm{d}t \\ &\quad = \int_0^T \int_\Omega \left[\mathbb{S}(\vartheta, \nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} - \frac{1}{\varepsilon^2}\varrho\nabla_x G\cdot\boldsymbol{\varphi} \right] \ \mathrm{d}x\,\mathrm{d}t - \int_\Omega \varrho_0\mathbf{u}_0\cdot\boldsymbol{\varphi}(0,\cdot)\ \ \mathrm{d}x \end{aligned} \tag{2.3}$$

  for any $\boldsymbol{\varphi} \in C^1_c([0,T)\times\Omega;R^3)$.

- The **entropy balance** (1.3) is replaced by the inequality

$$\begin{aligned} &-\int_0^T \int_\Omega \left[\varrho s(\varrho,\vartheta)\partial_t\varphi + \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\varphi + \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\varphi \right] \ \mathrm{d}x\,\mathrm{d}t \\ &\quad \geq \int_0^T \int_\Omega \frac{\varphi}{\vartheta}\left[\varepsilon^2\mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta} \right] \ \mathrm{d}x\,\mathrm{d}t + \int_\Omega \varrho_0 s(\varrho_0,\vartheta_0)\varphi(0,\cdot)\ \ \mathrm{d}x \end{aligned} \tag{2.4}$$

  for any $\varphi \in C^1_c([0,T)\times\Omega)$, $\varphi \geq 0$, where $\mathbb{D}_x\mathbf{u} = \frac{1}{2}(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u})$ is the symmetric gradient.

- The **ballistic energy balance**

$$
\begin{aligned}
&-\int_0^T \partial_t\psi \int_\Omega \left[\varepsilon^2 \frac{1}{2}\varrho|\mathbf{u}|^2 + \varrho e(\varrho,\vartheta) - \tilde{\vartheta}\varrho s(\varrho,\vartheta)\right] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \int_0^T \psi \int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left[\varepsilon^2 \mathbb{S}(\vartheta,\nabla_x\mathbf{u}) : \mathbb{D}_x\mathbf{u} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\nabla_x\vartheta}{\vartheta}\right] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \psi \int_\Omega \left[\varrho\mathbf{u}\cdot\nabla_x G - \varrho s(\varrho,\vartheta)\partial_t\tilde{\vartheta} - \varrho s(\varrho,\vartheta)\mathbf{u}\cdot\nabla_x\tilde{\vartheta} - \frac{\mathbf{q}(\vartheta,\nabla_x\vartheta)}{\vartheta}\cdot\nabla_x\tilde{\vartheta}\right] \ \mathrm{d}x\,\mathrm{d}t \\
&\quad + \psi(0)\int_\Omega \left[\frac{1}{2}\varepsilon^2\varrho_0|\mathbf{u}_0|^2 + \varrho_0 e(\varrho_0,\vartheta_0) - \tilde{\vartheta}(0,\cdot)\varrho_0 s(\varrho_0,\vartheta_0)\right] \ \mathrm{d}x \qquad (2.5)
\end{aligned}
$$

holds for any $\psi \in C^1_c([0,T))$, $\psi \ge 0$, and any $\tilde{\vartheta} \in C^1([0,T)\times\overline{\Omega})$,

$$\tilde{\vartheta} > 0, \ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B.$$

2.1 Relative energy inequality

In addition to Gibbs' equation (1.8), we impose the hypothesis of thermodynamic stability written in the form

$$\frac{\partial p(\varrho,\vartheta)}{\partial\varrho} > 0, \ \frac{\partial e(\varrho,\vartheta)}{\partial\vartheta} > 0 \text{ for all } \varrho,\vartheta > 0. \tag{2.6}$$

Next, following [4], we introduce the scaled *relative energy*

$$
\begin{aligned}
&E_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}\right) \\
&= \frac{1}{2}\varrho|\mathbf{u}-\tilde{\mathbf{u}}|^2 + \frac{1}{\varepsilon^2}\left[\varrho e - \tilde{\vartheta}\Big(\varrho s - \tilde{\varrho}s(\tilde{\varrho},\tilde{\vartheta})\Big) - \left(e(\tilde{\varrho},\tilde{\vartheta}) - \tilde{\vartheta}s(\tilde{\varrho},\tilde{\vartheta}) + \frac{p(\tilde{\varrho},\tilde{\vartheta})}{\tilde{\varrho}}\right)(\varrho-\tilde{\varrho}) - \tilde{\varrho}e(\tilde{\varrho},\tilde{\vartheta})\right].
\end{aligned}
$$

Now, the hypothesis of thermodynamic stability (2.6) can be equivalently rephrased as (strict) convexity of the total energy expressed with respect to the conservative entropy variables

$$E_\varepsilon\Big(\varrho, S = \varrho s(\varrho,\vartheta), \mathbf{m} = \varrho\mathbf{u}\Big) \equiv \frac{1}{2}\frac{|\mathbf{m}|^2}{\varrho} + \frac{1}{\varepsilon^2}\varrho e(\varrho,S),$$

whereas the relative energy can be written as

$$E_\varepsilon\left(\varrho,S,\mathbf{m}\middle|\tilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}}\right) = E_\varepsilon(\varrho,S,\mathbf{m}) - \left\langle \partial_{\varrho,S,\mathbf{m}}E_\varepsilon(\tilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}}); (\varrho-\tilde{\varrho}, S-\widetilde{S}, \mathbf{m}-\widetilde{\mathbf{m}})\right\rangle - E_\varepsilon(\tilde{\varrho},\widetilde{S},\widetilde{\mathbf{m}}).$$

Finally, as observed in [4], any weak solution in the sense of Definition 2.1 satisfies the *relative energy inequality*

$$\left[\int_\Omega E_\varepsilon\left(\varrho,\vartheta,\mathbf{u}\middle|\tilde{\varrho},\tilde{\vartheta},\tilde{\mathbf{u}}\right) \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$

$$
\begin{aligned}
&+\int_0^\tau \int_\Omega \frac{\tilde{\vartheta}}{\vartheta}\left(\mathbb{S}(\vartheta,\nabla_x\mathbf{u}):\mathbb{D}_x\mathbf{u}+\frac{1}{\varepsilon^2}\frac{\kappa(\vartheta)|\nabla_x\vartheta|^2}{\vartheta}\right)\ \mathrm{d}x\,\mathrm{d}t\\
&\leq -\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\left(\varrho(s-s(\tilde{\varrho},\tilde{\vartheta}))\partial_t\tilde{\vartheta}+\varrho(s-s(\tilde{\varrho},\tilde{\vartheta}))\mathbf{u}\cdot\nabla_x\tilde{\vartheta}-\left(\frac{\kappa(\vartheta)\nabla_x\vartheta}{\vartheta}\right)\cdot\nabla_x\tilde{\vartheta}\right)\ \mathrm{d}x\,\mathrm{d}t\\
&-\int_0^\tau\int_\Omega\left[\varrho(\mathbf{u}-\tilde{\mathbf{u}})\otimes(\mathbf{u}-\tilde{\mathbf{u}})+\frac{1}{\varepsilon^2}p(\varrho,\vartheta)\mathbb{I}-\mathbb{S}(\vartheta,\nabla_x\mathbf{u})\right]:\mathbb{D}_x\tilde{\mathbf{u}}\ \mathrm{d}x\,\mathrm{d}t\\
&+\int_0^\tau\int_\Omega\varrho\left[\frac{1}{\varepsilon^2}\nabla_x G-\partial_t\tilde{\mathbf{u}}-(\tilde{\mathbf{u}}\cdot\nabla_x)\tilde{\mathbf{u}}\right]\cdot(\mathbf{u}-\tilde{\mathbf{u}})\ \mathrm{d}x\,\mathrm{d}t\\
&+\frac{1}{\varepsilon^2}\int_0^\tau\int_\Omega\left[\left(1-\frac{\varrho}{\tilde{\varrho}}\right)\partial_t p(\tilde{\varrho},\tilde{\vartheta})-\frac{\varrho}{\tilde{\varrho}}\mathbf{u}\cdot\nabla_x p(\tilde{\varrho},\tilde{\vartheta})\right]\ \mathrm{d}x\,\mathrm{d}t
\end{aligned}
\tag{2.7}
$$

for a.a. $\tau > 0$ and any trio of continuously differentiable functions $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}})$ satisfying

$$
\tilde{\varrho} > 0,\ \tilde{\vartheta} > 0,\ \tilde{\vartheta}|_{\partial\Omega} = \vartheta_B,\ \tilde{\mathbf{u}}|_{\partial\Omega} = 0. \tag{2.8}
$$

2.2 Constitutive relations

The existence theory developed in [4] is conditioned by certain restrictions imposed on the constitutive relations (state equations) similar to those introduced in the monograph [9, Chapters 1,2]. Specifically, the equation of state reads

$$
p(\varrho,\vartheta) = p_{\mathrm{m}}(\varrho,\vartheta) + p_{\mathrm{rad}}(\vartheta),
$$

where p_{m} is the pressure of a general *monoatomic* gas,

$$
p_{\mathrm{m}}(\varrho,\vartheta) = \frac{2}{3}\varrho e_{\mathrm{m}}(\varrho,\vartheta), \tag{2.9}
$$

enhanced by the radiation pressure

$$
p_{\mathrm{rad}}(\vartheta) = \frac{a}{3}\vartheta^4,\ a > 0.
$$

Accordingly, the internal energy reads

$$
e(\varrho,\vartheta) = e_{\mathrm{m}}(\varrho,\vartheta) + e_{\mathrm{rad}}(\varrho,\vartheta),\ e_{\mathrm{rad}}(\varrho,\vartheta) = \frac{a}{\varrho}\vartheta^4.
$$

Moreover, using several physical principles it was shown in [9, Chapter 1]:

- **Gibbs' relation** together with (2.9) yield

$$
p_{\mathrm{m}}(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right)
$$

  for a certain $P \in C^1[0,\infty)$. Consequently,

$$
p(\varrho,\vartheta) = \vartheta^{\frac{5}{2}} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{3}\vartheta^4,\ e(\varrho,\vartheta) = \frac{3}{2}\frac{\vartheta^{\frac{5}{2}}}{\varrho} P\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right) + \frac{a}{\varrho}\vartheta^4,\ a > 0. \tag{2.10}
$$

- **Hypothesis of thermodynamic stability** (2.6) expressed in terms of P gives rise to

$$P(0)=0,\ P'(Z)>0 \text{ for } Z\geq 0,\ 0<\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z}\leq c \text{ for } Z>0. \tag{2.11}$$

In particular, the function $Z\mapsto P(Z)/Z^{\frac{5}{3}}$ is decreasing, and we suppose

$$\lim_{Z\to\infty}\frac{P(Z)}{Z^{\frac{5}{3}}}=p_\infty>0. \tag{2.12}$$

- Accordingly, the associated entropy takes the form

$$s(\varrho,\vartheta)=s_{\rm m}(\varrho,\vartheta)+s_{\rm rad}(\varrho,\vartheta),\ s_{\rm m}(\varrho,\vartheta)=\mathcal{S}\left(\frac{\varrho}{\vartheta^{\frac{3}{2}}}\right),\ s_{\rm rad}(\varrho,\vartheta)=\frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.13}$$

where

$$\mathcal{S}'(Z)=-\frac{3}{2}\frac{\frac{5}{3}P(Z)-P'(Z)Z}{Z^2}<0. \tag{2.14}$$

In addition, we impose the **Third law of thermodynamics**, cf. Belgiorno [1], [2], requiring the entropy to vanish when the absolute temperature approaches zero,

$$\lim_{Z\to\infty}\mathcal{S}(Z)=0. \tag{2.15}$$

Finally, we suppose the transport coefficients are continuously differentiable functions satisfying

$$\begin{aligned}0<\underline{\mu}(1+\vartheta)\leq\mu(\vartheta),\ |\mu'(\vartheta)|\leq\overline{\mu},\\ 0\leq\eta(\vartheta)\leq\overline{\eta}(1+\vartheta),\\ 0<\underline{\kappa}(1+\vartheta^\beta)\leq\kappa(\vartheta)\leq\overline{\kappa}(1+\vartheta^\beta),\ \text{where } \beta>6.\end{aligned} \tag{2.16}$$

As a consequence of the above hypotheses, we get the following estimates:

$$\varrho^{\frac{5}{3}}+\vartheta^4\lesssim\varrho e(\varrho,\vartheta)\lesssim 1+\varrho^{\frac{5}{3}}+\vartheta^4, \tag{2.17}$$

$$s_{\rm m}(\varrho,\vartheta)\lesssim\left(1+|\log(\varrho)|+[\log(\vartheta)]^+\right), \tag{2.18}$$

see [9, Chapter 3, Section 3.2].

3 Strong solutions to Majda's system

Problem (1.16)–(1.17) shares many common features with the $2d-$incompressible Navier–Stokes system solved in the celebrated work by Ladyženskaja [12], [13]. Indeed we show that problem (1.16)–(1.17), endowed with the boundary conditions

$$\mathbf{U}_h|_{\partial\Omega}=0,\ \Omega=\mathbb{T}^2\times(0,1),\ \mathbb{T}^2=\left([-1,1]\Big|_{\{-1,1\}}\right)^2, \tag{3.1}$$

is globally well posed in the framework of Sobolev spaces $W^{2,p}$ with $p>1$ large enough. We report the following result that may be of independent interest.

Theorem 3.1 (Global existence for Majda's system). *Let $\Theta > 0$ be given. Suppose that*

$$r \in C^1([0,1]),\ 0 < \underline{r} \le r(x_3) \text{ for all } x_3 \in [0,1]. \tag{3.2}$$

Let the initial data $\mathbf{U}_{0,h}$ belong to the class

$$\mathbf{U}_{0,h} \in W^{3,q} \cap W_0^{1,q}(\Omega; R^2),\ \mathrm{div}_h \mathbf{U}_{0,h} = 0 \tag{3.3}$$

for all $1 \le q < \infty$.

Then the system (1.16)–(1.17)*, with the boundary conditions* (3.1) *and the initial condition* (3.3)*, admits a strong solution $\mathbf{U}_h$ in $(0,T) \times \Omega$, unique in the class*

$$\partial_t \mathbf{U}_h \in L^p(0,T; L^p(\Omega; R^2)),\ (\mathbf{U}_h, \nabla_h \mathbf{U}_h) \in L^p(0,T; W^{2,p}(\Omega; R^2) \times W^{2,p}(\Omega; R^{2\times 2})) \tag{3.4}$$

for any $1 \le p < \infty$.

Remark 3.2. To avoid any misunderstanding we emphasize that by

$$\mathbf{U}_{0,h} \in W^{3,q} \cap W_0^{1,q}(\Omega; R^2),\ \mathrm{div}_h \mathbf{U}_{0,h} = 0$$

for all $1 \le q < \infty$ we mean

$$\mathbf{U}_{0,h} \in \bigcap_{q \ge 1} W^{3,q} \cap W_0^{1,q}(\Omega; R^2),\ \mathrm{div}_h \mathbf{U}_{0,h} = 0.$$

Similarly,

$$\partial_t \mathbf{U}_h \in L^p(0,T; L^p(\Omega; R^2)),\ (\mathbf{U}_h, \nabla_h \mathbf{U}_h) \in L^p(0,T; W^{2,p}(\Omega; R^2) \times W^{2,p}(\Omega; R^{2\times 2}))$$

for all finite $1 \le p < \infty$ means

$$\partial_t \mathbf{U}_h \in \bigcap_{p \ge 1} L^p(0,T; L^p(\Omega; R^2)),\ (\mathbf{U}_h, \nabla_h \mathbf{U}_h) \in \bigcap_{p \ge 1} L^p(0,T; W^{2,p}(\Omega; R^2) \times W^{2,p}(\Omega; R^{2\times 2})).$$

The proof of Theorem 3.1 is postponed to Section 7.

4 Main result

Having collected the necessary preliminary material, we are ready to state our main result.

Theorem 4.1 (Singular limit). *Let the thermodynamic functions p, e, and s as well as the transport coefficients μ, λ, and κ comply with the structural hypotheses specified in Section 2.2. Let*

$$G = -gx_3,\ g > 0,\ \Theta_{\rm up} = \Theta_{\rm bott} = \Theta > 0, \tag{4.1}$$

and let

$$r \in C^1([0,1]),\ 0 < \underline{r} \le r,\ \frac{\partial p(r,\Theta)}{\partial \varrho}\partial_{x_3} r = -rg. \tag{4.2}$$

Let $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ be a family of weak solutions of the scaled NSF system in the sense of Definition 2.1 emanating from the initial data

$$\varrho_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon},\ \varrho_\varepsilon \mathbf{u}_\varepsilon(0,\cdot) = \varrho_{0,\varepsilon}\mathbf{u}_{0,\varepsilon},\ \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0,\cdot) = \varrho_{0,\varepsilon} s(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}),$$

where

$$\int_\Omega E_\varepsilon \left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \ \middle|\ r, \Theta, [\mathbf{U}_{0,h}, 0] \right) \ {\rm d}x \to 0 \ \text{as } \varepsilon \to 0, \tag{4.3}$$

and $\mathbf{U}_{0,h}$ belongs to the class (3.3)*.*

Then

$$\operatorname*{ess\,sup}_{\tau\in(0,T)} \int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \ \middle|\ r, \Theta, [\mathbf{U}_h, 0] \right)(\tau,\cdot) \ {\rm d}x \to 0 \ \text{as } \varepsilon \to 0, \tag{4.4}$$

where $\mathbf{U}_h$ is the unique solution of Majda's system, the existence of which is guaranteed by Theorem 3.1.

Hypothesis (4.3) corresponds to *well–prepared* initial data. In view of the coercivity properties of the relative energy stated in (5.1), (5.2) below, relation (4.4) implies, in particular,

$$\begin{aligned} \varrho_\varepsilon &\to r && \text{in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)), \\ \vartheta_\varepsilon &\to \Theta && \text{in } L^\infty(0,T; L^2(\Omega)), \\ \varrho_\varepsilon \mathbf{u}_\varepsilon &\to r[\mathbf{U}_h, 0] && \text{in } L^\infty(0,T; L^1(\Omega; R^3)) \end{aligned}$$

as $\varepsilon \to 0$.

The next two sections are devoted to the proof of Theorem 4.1.

5 Uniform bounds

In order to perform the singular limit in the NSF system we need the associated sequence of weak solutions $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon>0}$ to be bounded at least in the energy space. First, we introduce the notation of [9] to distinguish between the "essential" and "residual" range of the thermostatic variables (ϱ, ϑ). Specifically, given a compact set

$$K \subset \left\{ (\varrho, \vartheta) \in R^2 \ \Big| \ \varrho > 0, \vartheta > 0 \right\}$$

we introduce

$$g_{\rm ess} = g \mathbb{1}_{(\varrho,\vartheta) \in K}, \ g_{\rm res} = g - g_{\rm ess} = g \mathbb{1}_{(\varrho,\vartheta) \in R^2 \setminus K}.$$

As shown in [9, Chapter 5, Lemma 5.1], the relative energy enjoys the following coercivity properties:

$$E_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \geq E_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right)_{\rm ess} \geq C \left(\frac{|\varrho - \tilde{\varrho}|^2}{\varepsilon^2} + \frac{|\vartheta - \tilde{\vartheta}|^2}{\varepsilon^2} + |\mathbf{u} - \tilde{\mathbf{u}}|^2 \right)_{\rm ess} \tag{5.1}$$

$$E_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right) \geq E_\varepsilon \left(\varrho, \vartheta, \mathbf{u} \Big| \tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}} \right)_{\rm res} \geq C \left(\frac{1}{\varepsilon^2} + \frac{1}{\varepsilon^2} \varrho e(\varrho, \vartheta) + \frac{1}{\varepsilon^2} \varrho |s(\varrho, \vartheta)| + \varrho |\mathbf{u}|^2 \right)_{\rm res} \tag{5.2}$$

whenever $(\tilde{\varrho}, \tilde{\vartheta}) \in {\rm int}[K]$, where the constant C depends on K and the distance

$$\mathrm{dist} \left[(\tilde{\varrho}, \tilde{\vartheta}); \partial K \right].$$

5.1 Energy estimates for ill–prepared data

We examine a slightly more general situation than in Theorem 4.1. Let $\Theta > 0$ be constant and r the solution of the static problem

$$\frac{\partial p(r, \Theta)}{\partial \varrho} \partial_{x_3} r = -r g. \tag{5.3}$$

Next, we consider a family $(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon)_{\varepsilon > 0}$ emanating from *ill–prepared* data $(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon})_{\varepsilon > 0}$,

$$\int_\Omega E_\varepsilon \left(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \mathbf{u}_{0,\varepsilon} \Big| r, \Theta, 0 \right) \ {\rm d}x \stackrel{<}{\sim} 1 \text{ independently of } \varepsilon \to 0. \tag{5.4}$$

The relative energy inequality (2.7) yields

$$\begin{aligned} &\left[\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \Theta, 0 \right) \ {\rm d}x \right]_{t=0}^{t=\tau} \\ &+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right) \ {\rm d}x \, {\rm d}t \\ &\leq \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \frac{\varrho_\varepsilon}{r} \left(r \nabla_x G - \nabla_x p(r, \Theta) \right) \cdot \mathbf{u}_\varepsilon \ {\rm d}x \, {\rm d}t. \end{aligned} \tag{5.5}$$

Moreover, in view of (4.1) and (4.2), we deduce the stationary equation

$$\nabla_x p(r, \Theta) = r \nabla_x G; \tag{5.6}$$

hence (5.5) reduces to

$$\left[\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \Theta, 0\right) \ \mathrm{d}x\right]_{t=0}^{t=\tau}$$
$$+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta_\varepsilon)|\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right) \ \mathrm{d}x \, \mathrm{d}t \leq 0. \tag{5.7}$$

5.2 Conclusion, uniform bounds for ill-prepared data

In view of the estimates obtained in the previous section, we deduce from (5.7) for ill–prepared initial data satisfying (5.4) the following bounds independent of the scaling parameter $\varepsilon \to 0$:

$$\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \Theta, 0\right) \ \mathrm{d}x \lesssim 1, \tag{5.8}$$

$$\int_0^T \|\mathbf{u}_\varepsilon\|^2_{W_0^{1,2}(\Omega;R^3)} \, \mathrm{d}t \lesssim 1, \tag{5.9}$$

$$\frac{1}{\varepsilon^2} \int_0^T \left(\|\nabla_x \log(\vartheta_\varepsilon)\|^2_{L^2(\Omega;R^3)} + \|\nabla_x \vartheta_\varepsilon^{\frac{\beta}{2}}\|^2_{L^2(\Omega;R^3)} \right) \lesssim 1. \tag{5.10}$$

Next, it follows from (5.8) that the measure of the residual set shrinks to zero, specifically

$$\frac{1}{\varepsilon^2} \operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega [1]_{\rm res} \ \mathrm{d}x \lesssim 1. \tag{5.11}$$

In addition, we get from (5.8):

$$\begin{aligned}
\operatorname*{ess\,sup}_{t \in (0,T)} \int_\Omega \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 \ \mathrm{d}x &\lesssim 1, \\
\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\varrho_\varepsilon - r}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} &\lesssim 1, \\
\operatorname*{ess\,sup}_{t \in (0,T)} \left\| \left[\frac{\vartheta_\varepsilon - \Theta}{\varepsilon} \right]_{\rm ess} \right\|_{L^2(\Omega)} &\lesssim 1, \\
\frac{1}{\varepsilon^2} \operatorname*{ess\,sup}_{t \in (0,T)} \|[\varrho_\varepsilon]_{\rm res}\|^{\frac{5}{3}}_{L^{\frac{5}{3}}(\Omega)} + \frac{1}{\varepsilon^2} \operatorname*{ess\,sup}_{t \in (0,T)} \|[\vartheta_\varepsilon]_{\rm res}\|^4_{L^4(\Omega)} &\lesssim 1.
\end{aligned} \tag{5.12}$$

Combining (5.10), (5.11), and (5.12), we conclude

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\Theta)}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \mathrm{d}t + \int_0^T \left\| \frac{\vartheta_\varepsilon - \Theta}{\varepsilon} \right\|^2_{W^{1,2}(\Omega)} \mathrm{d}t \lesssim 1. \tag{5.13}$$

Finally, we claim the bound on the entropy flux

$$\int_0^T \left\| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right\|^q_{L^q(\Omega;R^3)} \ {\rm d}t \stackrel{<}{\sim} 1 \text{ for some } q > 1. \tag{5.14}$$

Indeed we have

$$\left| \left[\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \right]_{\rm res} \frac{\nabla_x \vartheta_\varepsilon}{\varepsilon} \right| \stackrel{<}{\sim} \frac{1}{\varepsilon} \left| \nabla_x \log(\vartheta_\varepsilon) \right| + \frac{1}{\varepsilon} \left| \left[\vartheta_\varepsilon^{\frac{\beta}{2}} \nabla_x \vartheta_\varepsilon^{\frac{\beta}{2}} \right]_{\rm res} \right|,$$

where the former term on the right–hand side is controlled via (5.13). As for the latter, we deduce from (5.10) that

$$\left\| \frac{1}{\varepsilon} \nabla_x \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{L^2((0,T)\times\Omega;R^3)} \stackrel{<}{\sim} 1;$$

hence it is enough to check

$$\left\| \left[\vartheta_\varepsilon^{\frac{\beta}{2}} \right]_{\rm res} \right\|_{L^r((0,T)\times\Omega)} \stackrel{<}{\sim} 1 \text{ for some } r > 2. \tag{5.15}$$

To see (5.15) first observe that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \| [\vartheta_\varepsilon]_{\rm res} \|_{L^4(\Omega)} \stackrel{<}{\sim} 1, \tag{5.16}$$

and, in view of (5.10) and Poincaré inequality,

$$\left\| \vartheta_\varepsilon^{\frac{\beta}{2}} \right\|_{L^2(0,T;L^6(\Omega))} \stackrel{<}{\sim} 1.$$

Consequently, (5.15) follows by interpolation.

Of course, the above uniform bound remain valid also for the well-prepared initial data considered in Theorem 4.1.

6 Convergence to the target system

We show convergence to the regular solution $\mathbf{U}_h$ in Majda's system claimed in Theorem 4.1. To get a lean notation, we will identify the two-dimensional velocity $\mathbf{U}_h$ with its three-dimensional counterpart $[\mathbf{U}_h, 0]$. The ansatz $(\tilde{\varrho}, \tilde{\vartheta}, \tilde{\mathbf{u}}) = (r, \Theta, \mathbf{U}_h)$ in the relative energy inequality (2.7) yields

$$\begin{aligned} &\left[\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \Theta, \mathbf{U}_h \right) \ {\rm d}x \right]_{t=0}^{t=\tau} \\ &+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right) \ {\rm d}x\, {\rm d}t \\ &\leq - \int_0^\tau \int_\Omega \left[\varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_h) \otimes (\mathbf{u}_\varepsilon - \mathbf{U}_h) + \frac{1}{\varepsilon^2} p(\varrho_\varepsilon, \vartheta_\varepsilon) \mathbb{I} - \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) \right] : \mathbb{D}_x \mathbf{U}_h \ {\rm d}x\, {\rm d}t \end{aligned}$$

$$+\int_0^\tau \int_\Omega \varrho_\varepsilon \left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x) \mathbf{U}_h \right] \cdot (\mathbf{U}_h - \mathbf{u}_\varepsilon) \ {\rm d}x \, {\rm d}t - \frac{1}{\varepsilon^2} \int_0^\tau \int_\Omega \varrho_\varepsilon \nabla_x G \cdot \mathbf{U}_h \ {\rm d}x \, {\rm d}t, \tag{6.1}$$

where we have used the stationary equation

$$\nabla_x p(r, \Theta) = r \nabla_x G.$$

Next, seeing that

$$\mathrm{div}_x \mathbf{U}_h = 0, \ \nabla_x G \cdot \mathbf{U}_h = 0,$$

we deduce

$$\begin{aligned}
&\left[\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \Theta, \mathbf{U}_h \right) \ {\rm d}x \right]_{t=0}^{t=\tau} \\
&+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon} \left(\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2} \frac{\kappa(\vartheta_\varepsilon) |\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon} \right) \ {\rm d}x \, {\rm d}t \\
&\leq - \int_0^\tau \int_\Omega \Big[\varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_h) \otimes (\mathbf{u}_\varepsilon - \mathbf{U}_h) - \mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) \Big] : \mathbb{D}_x \mathbf{U}_h \ {\rm d}x \, {\rm d}t \\
&+ \int_0^\tau \int_\Omega \varrho_\varepsilon \left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x) \mathbf{U}_h \right] \cdot (\mathbf{U}_h - \mathbf{u}_\varepsilon) \ {\rm d}x \, {\rm d}t.
\end{aligned} \tag{6.2}$$

Now, in view of the uniform bounds (5.9), (5.12),

$$\begin{aligned}
&\int_0^\tau \int_\Omega \varrho_\varepsilon \left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x) \mathbf{U}_h \right] \cdot (\mathbf{U}_h - \mathbf{u}_\varepsilon) \ {\rm d}x \, {\rm d}t \\
&\quad = \int_0^\tau \int_\Omega r \left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x) \mathbf{U}_h \right] \cdot (\mathbf{U}_h - \mathbf{u}_\varepsilon) \ {\rm d}x \, {\rm d}t + \mathcal{Q}(\varepsilon),
\end{aligned}$$

where $\mathcal{Q}(\varepsilon)$ denotes a generic function with the property $\mathcal{Q}(\varepsilon) \to 0$ as $\varepsilon \to 0$.

Next, in view of (5.9), (5.12), we may assume

$$\varrho_\varepsilon \to r \text{ in } L^\infty(0,T; L^{\frac{5}{3}}(\Omega)), \ \mathbf{u}_\varepsilon \to \mathbf{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega)),$$

up to a suitable subsequence, where

$$\mathrm{div}_x (r \mathbf{u}) = 0. \tag{6.3}$$

Similarly, using the bounds (5.12), (5.13) we may perform the limit in the entropy inequality (2.4) obtaining

$$\mathrm{div}_x (r s(r, \Theta) \mathbf{u}) \geq 0.$$

However, thanks to the no–slip boundary conditions,

$$\int_\Omega \mathrm{div}_x (r s(r, \Theta) \mathbf{u}) \ {\rm d}x = 0;$$

therefore

$$\operatorname{div}_x(rs(r,\Theta)\mathbf{u}) = 0. \tag{6.4}$$

Combining (6.3), (6.4) we may infer

$$r\frac{\partial s(r,\Theta)}{\partial \varrho}\nabla_x r \cdot \mathbf{u} = 0.$$

As entropy is given by the constitutive equation (2.13), (2.14),

$$\frac{\partial s(r,\Theta)}{\partial \varrho} < 0,$$

and we conclude

$$u_3 = 0,\ \operatorname{div}_h \mathbf{u} = 0. \tag{6.5}$$

Now,

$$\begin{aligned}
&\int_0^\tau \int_\Omega \varrho_\varepsilon \left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x)\mathbf{U}_h\right] \cdot (\mathbf{U}_h - \mathbf{u}_\varepsilon)\ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega r\left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x)\mathbf{U}_h\right] \cdot (\mathbf{U}_h - \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t + \mathcal{Q}(\varepsilon).
\end{aligned}$$

In addition, since $\mathbf{U}_h$, $\mathbf{u}$ satisfy (1.17), (6.5), respectively, we obtain

$$\begin{aligned}
&\int_0^\tau \int_\Omega r\left[\partial_t \mathbf{U}_h + (\mathbf{U}_h \cdot \nabla_x)\mathbf{U}_h\right] \cdot (\mathbf{U}_h - \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t \\
&\quad = \int_0^\tau \int_\Omega \mu(\Theta)\left[\Delta_h \mathbf{U}_h + \partial^2_{x_3,x_3}\mathbf{U}_h\right] \cdot (\mathbf{U}_h - \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t \\
&\quad = -\int_0^\tau \int_\Omega \mathbb{S}(\Theta, \nabla_x \mathbf{U}_h) : \mathbb{D}_x(\mathbf{U}_h - \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t.
\end{aligned} \tag{6.6}$$

Going back to (6.2), we deduce

$$\begin{aligned}
&\left[\int_\Omega E_\varepsilon\left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \middle| r, \Theta, \mathbf{U}_h\right)\ \mathrm{d}x\right]_{t=0}^{t=\tau} \\
&+ \int_0^\tau \int_\Omega \frac{\Theta}{\vartheta_\varepsilon}\left(\mathbb{S}(\vartheta_\varepsilon, \nabla_x \mathbf{u}_\varepsilon) : \mathbb{D}_x \mathbf{u}_\varepsilon + \frac{1}{\varepsilon^2}\frac{\kappa(\vartheta_\varepsilon)|\nabla_x \vartheta_\varepsilon|^2}{\vartheta_\varepsilon}\right)\ \mathrm{d}x\,\mathrm{d}t \\
&\leq -\int_0^\tau \int_\Omega \Big[\varrho_\varepsilon(\mathbf{u}_\varepsilon - \mathbf{U}_h) \otimes (\mathbf{u}_\varepsilon - \mathbf{U}_h) - \mathbb{S}(\Theta, \nabla_x \mathbf{u})\Big] : \mathbb{D}_x \mathbf{U}_h\ \mathrm{d}x\,\mathrm{d}t \\
&- \int_0^\tau \int_\Omega \mathbb{S}(\Theta, \nabla_x \mathbf{U}_h) : \mathbb{D}_x(\mathbf{U}_h - \mathbf{u})\ \mathrm{d}x\,\mathrm{d}t + \mathcal{Q}(\varepsilon).
\end{aligned} \tag{6.7}$$

Finally, exploiting weak lower semi–continuity of convex functions, we conclude

$$\left[\int_\Omega E_\varepsilon \left(\varrho_\varepsilon, \vartheta_\varepsilon, \mathbf{u}_\varepsilon \Big| r, \Theta, \mathbf{U}_h \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau}$$
$$+ \int_0^\tau \int_\Omega (\mathbb{S}(\Theta, \nabla_x \mathbf{u}) - \mathbb{S}(\Theta, \nabla_x \mathbf{U}_h) : (\mathbb{D}_x \mathbf{u} - \mathbb{D}_x \mathbf{U}_h) \ \mathrm{d}x \, \mathrm{d}t$$
$$\leq - \int_0^\tau \int_\Omega \Big[\varrho_\varepsilon (\mathbf{u}_\varepsilon - \mathbf{U}_h) \otimes (\mathbf{u}_\varepsilon - \mathbf{U}_h) \Big] : \mathbb{D}_x \mathbf{U}_h \ \mathrm{d}x \, \mathrm{d}t + \mathcal{Q}(\varepsilon), \tag{6.8}$$

which, applying the standard Grönwall argument, yields the desired convergence as well as $\mathbf{u} = \mathbf{U}_h$.

We have proved Theorem 4.1.

7 Global existence for Majda's problem

Our ultimate goal is to show global existence of strong solutions to Majda's model claimed in Theorem 3.1. To this end, it is more convenient to consider the (horizontal) vorticity formulation of (1.16), (1.17). With a slight abuse of notation in the definition of $\mathbf{U}_h$, this formulation reads

$$\partial_t \omega + \mathbf{U}_h \cdot \nabla_x \omega = \nu \Delta_x \omega, \tag{7.1}$$
$$\mathbf{U}_h = \left[\nabla_h^\perp \Delta_h^{-1} [\omega], 0 \right], \tag{7.2}$$
$$\nu = \nu(x_3), \tag{7.3}$$

with the boundary conditions

$$\omega|_{\partial\Omega} = 0, \tag{7.4}$$

and the initial condition

$$\omega(0, \cdot) = \omega_0. \tag{7.5}$$

Here, $\nu = \frac{\mu(\Theta)}{r}$, and

$$\omega = \mathrm{curl}_h \mathbf{U}_h, \ \ \mathrm{curl}_h[\mathbf{v}] = \partial_{x_1} v_2 - \partial_{x_2} v_1. \tag{7.6}$$

For given ω, the velocity field $\mathbf{U}_h$ can be recovered via Biot-Savart law:

$$\mathbf{U}_h = \left[\nabla_h^\perp \Delta_h^{-1} [\omega], 0 \right], \ \ \nabla_h^\perp = [-\partial_{x_2}, \partial_{x_1}]. \tag{7.7}$$

Remark 7.1. Strictly speaking, the velocity $\mathbf{U}_h$ is determined by (7.7) up to its horizontal average

$$\overline{\mathbf{U}}_h = \int_{\mathbb{T}^2} \mathbf{U}_h \ \mathrm{d}x_h$$

that can be recovered as the unique solution of the parabolic problem

$$r \partial_t \overline{\mathbf{U}}_h = \mu(\Theta) \partial^2_{x_3, x_3} \overline{\mathbf{U}}_h \text{ in } (0, T) \times (0, 1),$$
$$\overline{\mathbf{U}}_h|_{x_3 = 0,1} = 0,$$
$$\overline{\mathbf{U}}_h(0, \cdot) = \int_{\mathbb{T}^2} \mathbf{U}_{0,h} \ \mathrm{d}x_h.$$

7.1 Construction via a fixed point argument

The desired solution ω to (7.1)–(7.5) can be constructed via a simple fixed point argument. Consider the set

$$X_M = \left\{ \widetilde{\omega} \in C([0,T] \times \overline{\Omega}) \ \middle| \ \widetilde{\omega}|_{\partial\Omega} = 0,\ \widetilde{\omega}(0,\cdot) = \mathrm{curl}_h \mathbf{U}_{0,h}\ ,\ \|\widetilde{\omega}\|_{C([0,T]\times\overline{\Omega})} \leq M \right\}.$$

As the initial velocity $\mathbf{U}_{0,h}$ belongs to the class (3.3), the set X_M is a bounded closed convex subset of the Banach space $C([0,T] \times \overline{\Omega})$. Moreover, X_M is non-empty as long as M is large enough to accommodate the initial condition.

We define a mapping $\mathcal{T}[\widetilde{\omega}] = \omega$, where ω is the unique solution of the problem

$$\partial_t \omega + b_L(\widetilde{\mathbf{U}}_h) \cdot \nabla_x \omega = \nu \Delta_x \omega, \tag{7.8}$$

$$\widetilde{\mathbf{U}}_h = \left[\nabla_h^\perp \Delta_h^{-1} [\widetilde{\omega}], 0 \right], \tag{7.9}$$

$$\omega|_{\partial\Omega} = 0, \tag{7.10}$$

$$\omega(0,\cdot) = \mathrm{curl}_h \mathbf{U}_{0,h}, \tag{7.11}$$

for some cut–off function b_L. Specifically,

$$b_L(\widetilde{\mathbf{U}}_h) = [b_L(\widetilde{U}_h^1), b_L(\widetilde{U}_h^2), 0],$$

where

$$b_L \in L^\infty(R) \cap C^\infty(R),\ b_L(Z) = Z \text{ whenever } |Z| \leq L.$$

7.1.1 Maximum principle

Applying the standard maximum principle, we deduce

$$\sup_{t\in[0,T]} \|\mathcal{T}[\widetilde{\omega}](t,\cdot)\|_{C(\overline{\Omega})} = \sup_{t\in[0,T]} \|\omega(t,\cdot)\|_{C(\overline{\Omega})} = \|\omega(0,\cdot)\|_{C(\overline{\Omega})} \lesssim \|\mathbf{U}_{0,h}\|_{W^{2,q}(\Omega;R^2)} \text{ as long as } q > 3. \tag{7.12}$$

Note carefully that the bound (7.12) depends solely on the initial data. In particular, it is independent of the specific form of the cut–off function b_L.

7.1.2 Maximal $L^p - L^q$ regularity

In view of hypothesis (4.2),

$$\nu \in C^1([0,1]),\ 0 < \underline{\nu} \leq \nu(x_3) \text{ for any } x_3 \in [0,1].$$

Consequently, we can apply the maximal $L^p - L^q$ regularity estimates, see, e.g., Denk, Hieber, and Prüss [6], to obtain

$$\|\partial_t \omega\|_{L^p(0,T;L^q(\Omega))} + \|\omega\|_{L^p(0,T;W^{2,q}(\Omega))} \leq c(p,q) \left(\|\omega(0,\cdot)\|_{W^{2,q}\cap W_0^{1,q}(\Omega)} + \|b_L(\widetilde{\mathbf{U}}_h) \cdot \nabla_x \omega\|_{L^p(0,T;L^q(\Omega))} \right),$$

$$1 < p, q < \infty. \tag{7.13}$$

Here

$$\|\omega(0,\cdot)\|_{W^{2,q}\cap W_0^{1,q}(\Omega)} \stackrel{<}{\sim} \|\mathbf{U}_{0,h}\|_{W^{3,q}(\Omega;R^2)},$$

while, by interpolation and (7.12),

$$\begin{aligned}\|b_L(\widetilde{\mathbf{U}}_h)\cdot\nabla_x\omega\|_{L^q(\Omega)} \le L\|\nabla_x\omega\|_{L^q(\Omega;R^3)} &\le L\|\omega\|^{\lambda}_{W^{2,q}(\Omega)}\|\omega\|^{1-\lambda}_{L^q(\Omega)}\\ &\le Lc(q)\|\mathbf{U}_{0,h}\|^{1-\lambda}_{W^{3,q}(\Omega;R^2)}\|\omega\|^{\lambda}_{W^{2,q}(\Omega)},\ t\in(0,T)\end{aligned}$$

for some $0 < \lambda < 1$. Consequently, it follows from (7.13) and our hypotheses imposed on the initial data that

$$\|\partial_t\mathcal{T}[\widetilde{\omega}]\|_{L^p(0,T;L^q(\Omega))} + \|\mathcal{T}[\widetilde{\omega}]\|_{L^p(0,T;W^{2,q}(\Omega))} \le c\left(p,q,\|\mathbf{U}_{0,h}\|_{W^{3,q}(\Omega;R^2)}\right)(1+L) \tag{7.14}$$

for all finite p, q.

7.2 Fixed point

It follows from the estimates (7.12), (7.14) that $\mathcal{T}$ is a compact (continuous) mapping of X_M into X_M provided M is large enough, therefore, by means of Tikhonov–Schauder fixed point Theorem, there is a fixed point $\omega \in X_M$ satisfying

$$\begin{aligned}\partial_t\omega + b_L(\mathbf{U}_h)\cdot\nabla_x\omega &= \nu\Delta_x\omega,\\ \mathbf{U}_h &= \left[\nabla_h^{\perp}\Delta_h^{-1}[\omega],0\right]\\ \omega|_{\partial\Omega} &= 0,\\ \omega(0,\cdot) &= \mathrm{curl}_h\mathbf{U}_{0,h}.\end{aligned}$$

Finally, as $\mathbf{U}_h$ is given by the Biot–Savart law, we get

$$\sup_{x_3\in(0,1)}\|\nabla_h\mathbf{U}_h\|_{L^q(\mathbb{T}^2;R^{2\times 2})} \le c(q)\|\omega(0,\cdot)\|_{L^\infty(\Omega)} \text{ uniformly for } t\in(0,T) \text{ for any } 1<q<\infty,$$

in particular

$$\|\mathbf{U}_h\|_{L^\infty((0,T)\times\Omega;R^2)} \stackrel{<}{\sim} \|\omega(0,\cdot)\|_{L^\infty(\Omega)} \stackrel{<}{\sim} \|\mathbf{U}_{0,h}\|_{W^{2,q}(\Omega;R^2)} \text{ as soon as } q>3.$$

Since this bound is independent of L, we may choose L large enough so that $b_L(\mathbf{U}_h) = \mathbf{U}_h$ to get the desired conclusion

$$\begin{aligned}\partial_t\omega + \mathbf{U}_h\cdot\nabla_x\omega &= \nu\Delta_x\omega,\\ \mathbf{U}_h &= \left[\nabla_h^{\perp}\Delta_h^{-1}[\omega],0\right]\\ \omega|_{\partial\Omega} &= 0,\end{aligned}$$

$$\omega(0,\cdot) = \mathrm{curl}_h \mathbf{U}_{0,h}.$$

Finally, it is easy to check that the solution is unique in the regularity class (3.4). As a matter of fact, a more general *weak–strong* uniqueness holds that could be shown adapting the above arguments based on the relative energy inequality.

We have proved Theorem 3.1.

References

[1] F. Belgiorno. Notes on the third law of thermodynamics, I. *J. Phys. A*, **36**:8165–8193, 2003.

[2] F. Belgiorno. Notes on the third law of thermodynamics, II. *J. Phys. A*, **36**:8195–8221, 2003.

[3] P. Bella, E. Feireisl, and F. Oschmann. Rigorous derivation of the Oberbeck-Boussinesq approximation revealing unexpected term. *preprint available at* `http://www.math.cas.cz/fichier/preprints/IM_20220721120958_85.pdf`

[4] N. Chaudhuri and E. Feireisl. Navier-Stokes-Fourier system with Dirichlet boundary conditions. *Appl. Anal.*, **101**(12):4076–4094, 2022.

[5] K. E. Daniels, O. Brausch, W. Pesch, and E. Bodenschatz. Competition and bistability of ordered undulations and undulation chaos in inclined layer convection. *J. Fluid Mech.*, **597**:261–282, 2008.

[6] R. Denk, M. Hieber, and J. Prüss. Optimal $L^p - L^q$-estimates for parabolic boundary value problems with inhomogenous data. *Math. Z.*, **257**:193–224, 2007.

[7] E. Feireisl, C. Klingenberg, O. Kreml, and S. Markfelder. On oscillatory solutions to the complete Euler system. *J. Differential Equations*, **269**(2):1521–1543, 2020.

[8] E. Feireisl, J. Málek, A. Novotný, and I. Straškraba. Anelastic approximation as a singular limit of the compressible Navier-Stokes system. *Commun. Partial Differential Equations*, **33**:157–176, 2007.

[9] E. Feireisl and A. Novotný. *Singular Limits in Thermodynamics of Viscous Fluids*. Advances in Mathematical Fluid Mechanics. Birkhäuser/Springer, Cham, 2017. Second edition.

[10] E. Feireisl and A. Novotný. *Mathematics of Open Fluid Systems*. Birkhäuser–Verlag, Basel, 2022.

[11] R. Klein, N. Botta, T. Schneider, C.D. Munz, S. Roller, A. Meister, L. Hoffmann, and T. Sonar. Asymptotic adaptive methods for multi-scale problems in fluid mechanics. *J. Engrg. Math.*, **39**:261–343, 2001.

[12] O. A. Ladyženskaja. Solution "in the large" to the boundary-value problem for the Navier-Stokes equations in two space variables. *Soviet Physics. Dokl.*, **123** (3):1128–1131 (427–429 Dokl. Akad. Nauk SSSR), 1958.

[13] O. A. Ladyženskaja. Solution "in the large" of the nonstationary boundary value problem for the Navier-Stokes system with two space variables. *Comm. Pure Appl. Math.*, **12**:427–433, 1959.

[14] A. Majda. *Introduction to PDEs and Waves for the Atmosphere and Ocean*. Volume 9 of *Courant Lecture Notes in Mathematics*. New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 2003.

[15] N. Masmoudi. Rigorous derivation of the anelastic approximation. *J. Math. Pures Appl.*, **88**:230–240, 2007.