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On the algebra and groups of incompressible vortex dynamics

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Abstract

An algebraic representation for 2D and 3D incompressible, inviscid fluid motion based on the continuous Nambu representation of Helmholtz vorticity equation is introduced. The Nambu brackets of conserved quantities generate a Lie algebra. Physically, we introduce matrix representations for the components of the linear momentum (2D and 3D), the circulation (2D) and the total flux of vorticity (3D). These quantities form the basis of the vortex-Heisenberg Lie algebra. Applying the matrix commutator to the basis matrices leads to the same physical relations as the Nambu bracket for this quantities expressed classically as functionals. Using the matrix representation of the Lie algebra we derive the matrix and vector representations for the nilpotent vortex-Heisenberg groups that we denote by $VH(2)$ and $VH(3)$. It turns out that $VH(2)$ is a covering group of the classical Heisenberg group for mass point dynamics. $VH(3)$ can be seen as central extension of the abelian group of translations. We further introduce the Helmholtz vortex group $V(3)$, where additionally the angular momentum is included. Regarding application-oriented aspects, the novel matrix representation might be useful for numerical investigations of the group, whereas the vector representation of the group might provide a better process-related understanding of vortex flows.

Keywords: Nambu mechanics, fluid dynamics, vorticity equation, algebra

(Some figures may appear in colour only in the online journal)

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1. Introduction

In a seminal paper published in German in 1858, and in an English translation by P G Tait a few years later, Helmholtz set down the basic laws of vortex dynamics (Helmholtz 1858). Since that time a vast amount of basic research results has been achieved in this field. An important recapitulation of the early work in this area is given in the book of Truesdell (1954). A fundamental generalization of Helmholtz work in the field of vortex dynamics was the incorporation of the thermodynamic degree of freedom. This approach is deeply related to the work of Hans Ertel (1942), who established the modern version of atmospheric vortex dynamics. Ertel generalized the famous Helmholtz vortex theorems for atmospheric vortices and derived the conservation of potential vorticity (PV) for a compressible and adiabatic fluid. In this context of atmospheric dynamics it is appropriate to mention the work of Charney, Fjortoft, and von Neumann who established in 1950 the first successful numerical weather prediction by the integration of the barotropic non-divergent 2D vorticity equation (Charney *et al* 1950).

One early fundamental development, still pioneered by Helmholtz himself, is the theory of point vortex dynamics. It is a discrete version of 2D non-divergent vorticity dynamics, describing the temporal evolution of the position of n vortices in terms of ordinary differential equations. Nearly all conservation laws can be transferred from the continuous case to the discrete approximation. Moreover, Kirchhoff (1876) showed that these equations of motion have a Hamiltonian structure. Here we can derive a canonical Poisson bracket, where the positions of the vortices in the plane serve as canonical conjugate variables (Aref 1983, Newton 2001). This is a first and early hint that algebraic aspects are deeply related to vorticity dynamics. Furthermore, in his textbook Sommerfeld (1964) discusses the essence of the dynamics of point vortices. He came to the conclusion that point vortices behave fundamentally different compared to the dynamics of mass points. His deep physical intuition led him to the explanation that the point vortices create a distinguished frame of reference in the fluid, so that Newton's first and second law have to be interpreted in a completely different way compared to the well-known interpretation regarding the dynamics of mass points. In contrast to mass point dynamics, for vortex dynamics, the conservation of the momentum plays a central role, as also shown by Saffman (1995). Recently, Smith (2011) satisfies Newton's second law for force-free motion in an integral sense for the fluid around a vortex. Not only because of these reasons, it is challenging to discover the algebra and related group structure of vortex dynamics.

For the algebraic representation, Hamiltonian mechanics is of great importance in classical mechanics and quantum mechanics based on the energy as central quantity. The question arose of a corresponding concept in fluid dynamics. Unexpectedly, it turned out to be difficult to answer, especially for the Eulerian representation. However, as expected, the Lagrangian representation was easier to handle. It leads to a canonical Poisson representation regarding the momentum and the location of the continuum of particles (also called air parcels), analogously to discrete mass point dynamics. The Clebsch representation of fluid dynamics can be seen as a local field representation, where the so-called Clebsch potentials are introduced. This representation provides a canonical form of the equation of motion from the field perspective with the total information of the locations and momentum (Marsden and Weinstein 1983).

Regarding practical applications, such as in meteorology, the Eulerian view is suitable for the representation of fluid motion. In a first step of integrating the Hamiltonian theory into the Eulerian view Morrison and Greene (1980), analyse a non-canonical Poisson bracket as Lie algebra for the compressible equations of motions, see also the detailed review article of Salmon (1988). Shepherd (1990) also investigates non-canonical Hamiltonian mechanics. He discusses the singularity of the non-canonical Poisson differential operator and shows that the singularity is related to Casimir functions that commute with all other functionals. The

Casimir functionals correspond to the vortex topology, such as the helicity and the PV (Shepherd 1990b). In the next step Névir and Blender (1993), use the Casimir functions to introduce the Nambu formulation for fluid dynamics, where a Casimir function acts as second quantity and has equal status as the energy. They show that the Nambu field representation of incompressible, continuous vortex dynamics is based on the enstrophy in 2D and the helicity in 3D. An important feature of this Nambu formalism is the non-degeneracy of its brackets. Whereas in Hamiltonian mechanics these vorticity conserved quantities are the Casimir invariants of the singular Poisson bracket, using the trilinear antisymmetric Nambu bracket it is possible to put the enstrophy (in 2D) and helicity (in 3D) on the same fundamental level as the energy. Thus the nonlinear temporal evolution of vortices in an incompressible fluid is just given by these constitutive conserved quantities. Salmon (2005) used the Nambu representation for the spatial discretization of fluid dynamical systems. Salmon shows that the Arakawa Jacobian (Arakawa 1966) can be derived from Nambu mechanics, where he uses the advantage of the Nambu formulation that the conservation of two conserved quantities, the energy and the enstrophy is given, which is, in general, important for numerical simulations (see, e.g. Sommer 2010). Two years later Salmon (2007), applies Nambu mechanics to the shallow water equations. Névir and Sommer (2009) analyse in detail the Nambu bracket for compressible, non-hydrostatic, adiabatic fluids, for the quasi-geostrophic and the shallow water equations.

Gardner (1971) considers the algebra for fluid dynamics from an alternative perspective. He investigates the Poisson brackets for fluid systems based on the Korteweg–de Vries equations and shows that the associated Poisson bracket can be regarded as Lie algebra. The Korteweg–de Vries equations provide an integrable system related to a partial differential equation in one spatial dimension, where infinitely many conserved quantities can be derived (Lax 1968). Today this representation is known as Bi-Hamiltonian system (Fuchssteiner and Oevel 1982, Kupershmidt and Wilson 1980).

Holm and Kupershmidt (1983) investigate the non-canonical Poisson structure for ideal magnetohydrodynamics. The authors view the algebra from the physical perspective and identify the structure with a differential Lie algebra based on the Hamiltonian, given by the sum of the kinetic energy, the thermal energy, and magnetic energy of the fluid. They show that the algebra based on the density of the momentum, the mass and the entropy form a realization of the Lie algebra of the Galilean group. See also the book of Holm *et al* (2009), where the authors focus on the geometric aspect on different physical dynamical systems. Furthermore, Arnold shows that the Poisson structure can also be formulated for the vorticity equation (Arnold 1969a, 1969b, Arnold and Khesin 1992). See also the more recent work of Thiffeault and Morrison (2000) or Marsden and Ratiu (2013) who underline that one can express the Hamiltonian structure in terms of the vorticity as basic dynamic variable.

The non-canonical form of the incompressible vorticity equation can be transferred and represented in terms of Nambu mechanics (Névir and Blender 1993), which provides a novel algebraic structure that we will represent in this paper. In the following section 2 we will repeat and explain the basic results of the Nambu field representation of the 2D and 3D Helmholtz vorticity equation and introduce the trilinear antisymmetric Nambu bracket. The Nambu bracket is based on globally conserved quantities that constrain the vorticity dynamics in two and three dimensions. An important requirement here is that all conserved quantities have to be expressed as functionals of the vorticity vector in 3D, respectively as functionals of the vorticity in 2D. As we will discuss in section 3 the Nambu bracket with respect to the enstrophy generates an algebra for incompressible 2D vortex dynamics. Of course, this had been done for two-dimensional discrete point vortex dynamics in the Hamiltonian context

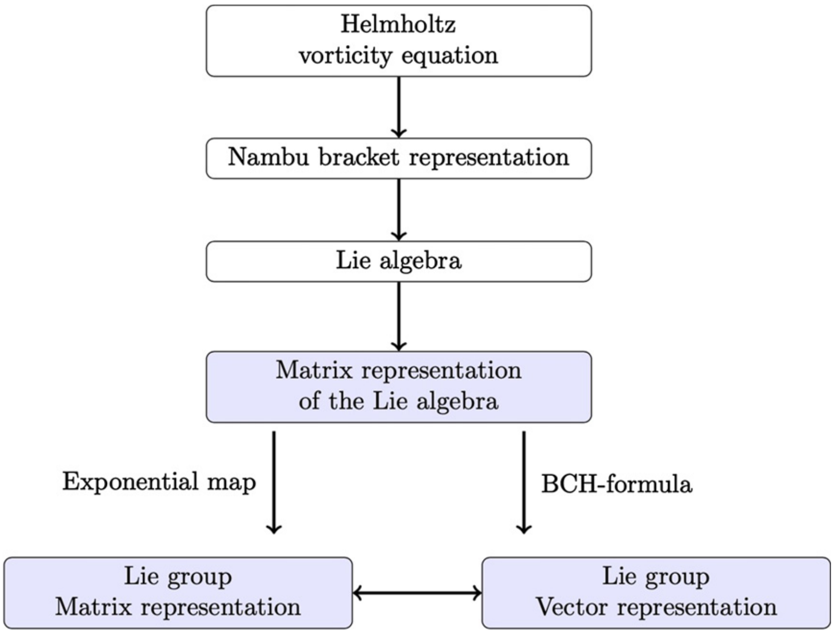

Figure 1. The steps are sketched, how we will derive the vortex groups for two-dimensional, inviscid vortex dynamics. The blue boxes indicate the novelties introduced in this paper.

(see e.g. the book of Newton 2001) but not in the case of the partial differential Helmholtz equation using the Nambu bracket representation. A central result in the framework of discrete 2D point vortex dynamics is that the brackets of the two components of the linear momentum do *not* commute in case of non-vanishing total circulation (Aref 1983). Although this property plays a special role from a physical point of view, it is only rarely discussed. We will show that this is a characteristic behaviour which can be obtained also in the continuous case of vortex dynamics in two as well as in three dimensions. In two dimensions, the algebra is generated by the Nambu bracket of the 2D linear momentum and the circulation with respect to the enstrophy. We will introduce a matrix Lie algebra and the derive a matrix and a vector representation of the Lie group for two-dimensional incompressible vortex dynamics. Comparing this novel group to classical mass point dynamics, in section 4 we find a similar mathematical structure, but with different physical meaning. In section 5 we will show that the Nambu bracket of the 3D momentum and the total flux of vorticity with respect to the helicity yields an algebra for 3D vortex dynamics. As for 2D vortex dynamics, we will represent a matrix Lie algebra and a matrix Lie group representation. The steps for the derivation of the Lie group are sketched in figure 1. Finally, in section 7 the main results are summarized.

2. Nambu representation of incompressible vortex dynamics

In this section, we will recall the Nambu representation for 3D and 2D vortex dynamics, following Névir and Blender (1993) and Névir (1998).

2.1. Nambu field representation of 3D vortex dynamics

The classical incompressible Euler equations are the starting point to formulate the 3D vorticity equations for inviscid flows from an algebraic point of view. The set of equations is given by:

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{1}{\rho_0} \nabla p \tag{1}$$

$$\nabla \cdot \mathbf{v} = 0 \tag{2}$$

with the 3D velocity vector $\mathbf{v}$, constant density ρ_0 and pressure p. Applying the operation $\nabla\times$ to the former equation, leads to the set of the 3D Helmholtz equation for the time evolution of the 3D vorticity $\boldsymbol{\xi}$, today called vorticity equation, and (2):

$$\frac{\partial \xi}{\partial t} = -\nabla \times (\xi \times \mathbf{v}) = \boldsymbol{\xi} \cdot \nabla \mathbf{v} - \mathbf{v} \cdot \nabla \boldsymbol{\xi}, \quad \nabla \cdot \mathbf{v} = 0. \tag{3}$$

The first term on the right-hand side is called twisting term and the second term on the right-hand side represents the advection of the vorticity.

To derive an algebra of vorticity dynamics all fluid dynamical conserved quantities that constrain the fluid dynamical system are considered as volume integrals. These conserved quantities are formulated as functionals of the vorticity vector $\boldsymbol{\xi}$ with respect to the infinitesimal volume element $\mathrm{d}\tau$ and volume V:

$$\mathcal{H}[\boldsymbol{\xi}] = -\frac{1}{2}\int_V \mathrm{d}\tau\ \mathbf{A} \cdot \boldsymbol{\xi} \quad \text{(kinetic energy)} \tag{4}$$

$$\mathscr{P}[\boldsymbol{\xi}] = \frac{1}{2}\int_V \mathrm{d}\tau\ (\mathbf{r} \times \boldsymbol{\xi}) \quad \text{(linear momentum)} \tag{5}$$

$$\mathscr{L}[\boldsymbol{\xi}] = -\frac{1}{2}\int_V \mathrm{d}\tau\ r^2 \boldsymbol{\xi} \quad \text{(angular momentum)} \tag{6}$$

$$\mathscr{Z}[\boldsymbol{\xi}] = \int_V \mathrm{d}\tau\ \boldsymbol{\xi} \quad \text{(total flux of vorticity)} \tag{7}$$

$$h[\boldsymbol{\xi}] = \frac{1}{2}\int_V \mathrm{d}\tau\ \mathbf{v} \cdot \boldsymbol{\xi} \quad \text{(helicity)} \tag{8}$$

(See also Majda and Bertozzi 2002, p 25). In the definition of the energy, $\mathbf{A}$ denotes the vector potential $\mathbf{v} = -\nabla \times \mathbf{A}$, as solution of $\nabla \cdot \mathbf{v} = 0$. The latter equation is a central condition for the Nambu representation incorporating the conservation of mass in the algebraic framework. The kinetic energy, the linear momentum $\mathscr{P} = (\mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z)$ and the angular momentum $\mathscr{L} = (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z)$ are classical conservation laws of dynamical systems. In the context of vortex dynamics these quantities are expressed with respect to the vorticity vector. These important laws are related to the invariance of space translations and spatial rotations.

The Lagrangian formulation of fluid mechanics is classically expressed via the energy (see e.g. Holm *et al* 1998). The energy plays an important role in the description of vortex dynamics. But vortices have an additional important characterization: the strength and direction of spin-like rotations. The latter two conserved quantities, the total flux of vorticity $\mathscr{Z}$ and the helicity h are special characteristics for 3D vorticity dynamics, because they depict the spin-like rotation. Their sign is explicitly related to the direction of rotation. Moreover, the magnitude of $\mathcal{Z}$ denote the strength of the rotation and h the helical shape of the vortex. First works related to the concept of helicity can be ascribed to Ertel and Rossby (1949). The helicity itself was introduced by Betchov (1961) and Moreau (1961); Moffatt (1969) considers helicity as a degree of linkage of the vortex lines of a flow. Since then especially the helicity dissipation has been investigated in terms of turbulence theory, see e.g. Kraichnan (1973), Brissaud *et al* (1973), Chen *et al* (2003), Pouquet and Mininni (2010) or Dallas and Tobias (2016). Biferale *et al* (2013) point out the importance of the helicity, its sign and its relation to the energy cascade. Regarding atmospheric applications Lilly (1986), characterizes supercell thunderstorms by high helicity and Davies-Jones *et al* (1990) used the helicity concept to classify tornadoes. Further atmospheric applications are for example investigated by Zhemin and Rongsheng (1994), who shows the importance of the sign of the helicity in the boundary layer, and that the amplitude of the helicity can help to describe frontogenesis and the development of frontal structures.

Based on the concept of Nambu (1973), Névir and Blender (1993) introduced a formulation of the vorticity equation, where the kinetic energy *and* the helicity have equal status. They invented the so-called Nambu bracket for incompressible flows by writing the vorticity equation (3) in terms of the functional derivative of the helicity and energy:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = -\nabla \times \left(\left(\nabla \times \frac{\delta h}{\delta \boldsymbol{\xi}} \right) \times \left(\nabla \times \frac{\delta \mathcal{H}}{\delta \boldsymbol{\xi}} \right) \right). \tag{9}$$

To simplify the notation, a constant bilinear and antisymmetric differential operator $\mathbf{K}$ can be defined:

$$\mathbf{K}(\mathbf{x}, \mathbf{y}) := -\nabla \times ((\nabla \times \mathbf{x}) \times (\nabla \times \mathbf{y})) \tag{10}$$

for $\mathbf{x}$ and $\mathbf{y}$ are vectors in $\mathbb{R}^3$. This bilinear operator can be seen analogously to the Jacobi-operator representation of the vorticity equation in two dimensions as shown in the next section. Combining (9) with (10) the continuous Nambu representation of the three-dimensional vorticity equation for incompressible, inviscid flows reads as:

$$\frac{\partial \boldsymbol{\xi}}{\partial t} = \mathbf{K}\left(\frac{\delta h}{\delta \boldsymbol{\xi}}, \frac{\delta \mathcal{H}}{\delta \boldsymbol{\xi}} \right). \tag{11}$$

Furthermore, the trilinear Nambu bracket can be defined as:

$$\frac{\partial \mathcal{F}}{\partial t} = \{\mathcal{F}, h, \mathcal{H}\} := \int_V \mathrm{d}\tau \frac{\delta \mathcal{F}}{\delta \boldsymbol{\xi}} \cdot \frac{\partial \boldsymbol{\xi}}{\partial t} = \int_V \mathrm{d}\tau \frac{\delta \mathcal{F}}{\delta \boldsymbol{\xi}} \cdot \mathbf{K}\left(\frac{\delta h}{\delta \boldsymbol{\xi}}, \frac{\delta \mathcal{H}}{\delta \boldsymbol{\xi}} \right), \tag{12}$$

respectively,

$$\{\mathcal{F}, h, \mathcal{G}\} = -\int_V \mathrm{d}\tau \left[\left(\nabla \times \frac{\delta \mathcal{F}}{\delta \boldsymbol{\xi}} \right) \times \left(\nabla \times \frac{\delta h}{\delta \xi} \right) \cdot \left(\nabla \times \frac{\delta \mathcal{G}}{\delta \boldsymbol{\xi}} \right) \right]. \tag{13}$$

See Névir and Blender (1993) and Névir (1998) for more details and the proofs of the algebraic properties of the Nambu bracket for fluid dynamics. The helicity can have positive as well as negative sign. To respect the sign in the algebraic representations, we write:

$$\{\mathcal{F}, h, \mathcal{G}\} = -\,\mathrm{sign}(h) \int_V \mathrm{d}\tau \left[\left(\nabla \times \frac{\delta \mathcal{F}}{\delta \boldsymbol{\xi}} \right) \times \left(\nabla \times \frac{\delta |h|}{\delta \xi} \right) \cdot \left(\nabla \times \frac{\delta \mathcal{G}}{\delta \boldsymbol{\xi}} \right) \right] \tag{14}$$

with $\mathrm{sign}(h) = +1$ for positive helicity and $\mathrm{sign}(h) = -1$ for negative helicity. The Nambu bracket is antisymmetric in all arguments, which follows from the triple and cross products. It is also multilinear. Moreover, keeping the helicity h, the middle argument, fixed, it can be reduced to a Poisson bracket that satisfies the Jacobi identity.

Determining the Nambu brackets (14) of the conserved quantities for incompressible flows, given in (4), leads to the following bracket relations:

$$\begin{aligned}
&\{\mathcal{P}_\alpha, h, \mathcal{P}_\beta\} = \varepsilon_{\alpha\beta\gamma} \mathcal{Z}_\gamma, && \{\mathcal{L}_\alpha, h, \mathcal{L}_\beta\} = \varepsilon_{\alpha\beta\gamma} \mathcal{L}_\gamma \\
&\{\mathcal{P}_\alpha, h, \mathcal{Z}_\beta\} = 0, && \{\mathcal{L}_\alpha, h, \mathcal{P}_\beta\} = \varepsilon_{\alpha\beta\gamma} \mathcal{P}_\gamma \\
&\{\mathcal{Z}_\alpha, h, \mathcal{Z}_\beta\} = 0 && \\
&&& \\
&\{\mathcal{P}_\alpha, h, \mathcal{H}\} = 0, && \{\mathcal{L}_\alpha, h, \mathcal{Z}_\beta\} = 0 \\
&\{\mathcal{Z}_\alpha, h, \mathcal{H}\} = 0, && \{\mathcal{L}_\alpha, h, \mathcal{H}\} = 0
\end{aligned} \tag{15}$$

with the Levi-Civita symbol $\varepsilon_{\alpha\beta\gamma}$ and $\alpha, \beta, \gamma \in \{1, 2, 3\}$ (see Névir 1998). Consider the upper block on the left-hand side, where the Nambu bracket of the components of the total flux of vorticity $\mathcal{Z}_\alpha$ and the momentum $\mathcal{P}_\alpha$, $\alpha = 1, 2, 3$, with respect to the helicity (4) is calculated. Comparing this bracket to the Poisson bracket for mass point dynamics, we notice that the brackets concerning the components of the linear momentum of mass points commute generating an abelian algebra. In contrast, regarding fluid dynamics, the Nambu bracket of the components of the momentum is the total flux of vorticity generating a nilpotent Lie algebra, as it will discuss in section 5. Therefore, this Nambu bracket is a central extension of the translation algebra.

2.2. Nambu field representation of 2D vortex dynamics

The vorticity equation for two-dimensional incompressible, inviscid fluids can be derived by neglecting the twisting term of the 3D vorticity equation (3) leading to:

$$\frac{\partial \zeta}{\partial t} = -\mathbf{v} \cdot \nabla \zeta, \qquad \nabla \cdot \mathbf{v} = 0. \tag{16}$$

The 2D vorticity equation (16) can also be represented in terms of the stream function and the vorticity. The idea is to use the condition of a solenoidal vector field

$$\nabla \cdot \mathbf{v} = 0 \quad \text{with solution} \quad \mathbf{v} = \mathbf{e}_3 \times \nabla \psi(x, y) \tag{17}$$

with $\mathbf{e}_3 = (0, 0, 1)$ to express the vorticity equation in terms of the Jacobian J of the streamfunction ψ and the vorticity:

$$\frac{\partial \zeta}{\partial t} = -J(\psi, \zeta) \tag{18}$$

as for example discussed in Majda *et al* (2002).

To calculate the Nambu bracket for two-dimensional, incompressible, inviscid fluid dynamics, the conserved quantities for such flows are formulated as functionals of the vorticity with respect to the infinitesimal area element $\mathrm{d}f$ and the material surface F. Formulating the conserved quantities in terms the vorticity allows for the Nambu representation of vortex dynamics:

$$\mathcal{H}[\zeta] = -\frac{1}{2}\int_F \mathrm{d}f\,\psi\zeta \quad \text{(kinetic energy)} \tag{19}$$

$$\mathcal{P}_x[\zeta] = +\int_F \mathrm{d}f\,y\zeta \quad \text{(linear momentum component in } x-\text{direction)} \tag{20}$$

$$\mathcal{P}_y[\zeta] = -\int_F \mathrm{d}f\,x\zeta \quad \text{(linear momentum component in } y-\text{direction)} \tag{21}$$

$$\mathcal{L}_z[\zeta] = -\frac{1}{2}\int_F \mathrm{d}f\,r^2\zeta \quad \text{(angular momentum component in z} - \text{direction)} \tag{22}$$

$$\mathcal{Z}[\zeta] = \int_F \mathrm{d}f\,\zeta \quad \text{(circulation)} \tag{23}$$

$$\mathcal{E}[\zeta] = \frac{1}{2}\int_F \mathrm{d}f\,\zeta^2 \quad \text{(enstrophy).} \tag{24}$$

The Nambu bracket for two-dimensional incompressible fluids, introduced by Névir and Blender (1993) is calculated with respect to the enstrophy $\mathcal{E}$ as positive definite constitutive vortex conservation quantity. Physically, the enstrophy can be seen as an integral measure of the vortical degree of a 2D flow. The kinetic energy and the enstrophy are both quadratic, and therefore positive definite, conserved quantities for two-dimensional incompressible fluid dynamics. The enstrophy is defined as integral over the squared vorticity and hence a natural choice to investigate and describe turbulent eddies, e.g. in the atmosphere (see e.g. Chemke *et al* 2016). The dissipation of the enstrophy and the energy generate two distinct cascades: the energy flows towards larger scales and the enstrophy towards small scales, see for example Kraichnan (1967), Kraichnan and Montgomery (1980), Boffetta and Ecke (2012), or Mininni and Pouquet (2013).

For two arbitrary functions $\mathcal{F}$ and $\mathcal{G}$ the Nambu bracket reads as:

$$\{\mathcal{F}, \mathcal{E}, \mathcal{G}\} = -\int_V \mathrm{d}f\,\frac{\delta\mathcal{F}}{\delta\zeta} J\left(\frac{\delta\mathcal{E}}{\delta\zeta}, \frac{\delta\mathcal{G}}{\delta\zeta}\right), \tag{25}$$

where $J(.,.)$ defines the classical Jacobi-determinant with respect to the x- and y-coordinates. Again, we notice that this representation written in terms of the Jacobi-operator can be compared to the Nambu representation in three spatial dimensions in terms of the **K**-operator in (12). According to Noether, the energy, here the kinetic energy, can be used to express the time evolution of the vorticity. Inserting for $\mathcal{F}$ the vorticity ζ and for $\mathcal{G}$ the energy $\mathcal{H}$, i.e. $\mathcal{F} = \zeta$ and $\mathcal{G} = \mathcal{H}$, the vorticity equation (18) can be formulated in terms of the Nambu bracket:

$$\frac{\partial\zeta}{\partial t} = -J(\psi, \zeta) = \{\zeta, \mathcal{E}, \mathcal{H}\}. \tag{26}$$

Thus, regarding the Nambu representation for a two-dimensional incompressible flow we have to express the stream function and the 2D vorticity as functional derivatives of the kinetic energy and the enstrophy. This differs from the three-dimensional flow, where the 3D vector potential and explicitly the velocity are expressed as functional derivatives of the kinetic energy and the helicity. Moreover, the conservation of helicity in the 3D case can only be derived by using both the original equation of motion and the related 3D vorticity equation. In the 2D case we obtain the conservation of enstrophy by applying only the 2D vorticity equation.

In the latter equation, the temporal evolution is expressed via the enstrophy and the kinetic energy in the Nambu bracket. We note that the 2D Nambu bracket as well the 3D Nambu bracket are calculated with respect to the spatial integrated quantities, the enstrophy and the helicity. While the kinetic energy as argument in the Nambu bracket (see the right block in (27)) gives us information about the temporal evolution, the Nambu bracket with respect to the enstrophy (2D) or the helicity (3D) gives rise to spatial changes of the structure. To explore the spatial behaviour of vortex dynamics in the following sections, the Nambu bracket with respect to the conserved quantities (19)–(24) are summarized (Névir 1998):

$$\begin{aligned}
&\{\mathcal{P}_x,\mathcal{E},\mathcal{P}_y\}=\mathcal{Z}, &&\{\mathcal{L}_z,\mathcal{E},\mathcal{P}_x\}=+\mathcal{P}_y, &&\{\mathcal{P}_x,\mathcal{E},\mathcal{H}\}=0\\
&\{\mathcal{P}_x,\mathcal{E},\mathcal{Z}\}=0, &&\{\mathcal{L}_z,\mathcal{E},\mathcal{P}_y\}=-\mathcal{P}_x, &&\{\mathcal{P}_y,\mathcal{E},\mathcal{H}\}=0\\
&\{\mathcal{P}_y,\mathcal{E},\mathcal{Z}\}=0, &&\{\mathcal{L}_z,\mathcal{E},\mathcal{Z}\}=0, &&\{\mathcal{L}_z,\mathcal{E},\mathcal{H}\}=0.
\end{aligned} \tag{27}$$

We note that the enstrophy $\mathcal{E}$ is positive definite. This is a main difference to the Nambu bracket for 3D vortex dynamics (14) that is calculated with respect to the helicity that can have positive or negative sign.

3. Algebra and group representation of 2D incompressible vortex dynamics

The tangent space to a linear Lie group at the identity[1] has special properties. It is equipped with a multiplication operation—the Lie bracket—such that it can be defined as a Lie algebra. Because Lie algebras are linear spaces, they can be easier applied to many problems than Lie groups. We cite Stillwell (2008) that 'the miracle of Lie theory is that a curved object, a Lie group, can be almost completely captured by a flat one, the tangent space of the Lie group at the identity'. Mathematically, a Lie algebra is defined as:

Definition 1. Lie algebra

A *Lie algebra* is a vector space g over some field F together with a binary operation $[\cdot,\cdot]_L : g\times g\to g$ called the *Lie bracket*. For $\forall\, a,b\in F, \forall\, g_1,g_2,g_3\in g$ the Lie bracket satisfies the following axioms:

(a) $[ag_1+bg_2,g_3]_L=a[g_1,g_3]_L+b[g_2,g_3]_L,\ [g_3,ag_1+bg_2]_L=a[g_3,g_1]_L+b[g_3,g_2]_L$ (bilinearity)

(b) $[g_1,g_1]_L=0$ (alternativity)

(c) $[g_1,[g_2,g_3]_L]_L+[g_3,[g_1,y]_L]_L+[g_2,[g_3,g_1]_L]_L=0$ (Jacobi identity)

Here, we use the subscript L to differ between the general Lie bracket and the matrix commutator that we will introduce later in this section. We remark that the classical definition of a Lie algebra is formulated with respect to vector spaces and not explicitly with respect to functionals. A vector space is a nonempty set with a scalar multiplication and a vector addition. This definition can be extended such that the definition of a Lie algebra holds for functionals locally, because a nonempty set of arbitrary functions $X\to V$ with a locally defined scalar multiplication and vector addition also forms a vector space (Akcoglu *et al* 2009).

In order to derive the matrix representation of an algebra for two-dimensional incompressible and inviscid fluids consider in (27) the block on the left-hand side:

[1] A closed subgroup $G\subseteq GL(n;\mathbb{K})$ is called linear group.

$$\{\mathcal{P}_x, \mathcal{E}, \mathcal{P}_y\} = \mathcal{Z}, \quad \{\mathcal{Z}, \mathcal{E}, \mathcal{P}_x\} = 0, \quad \{\mathcal{Z}, \mathcal{E}, \mathcal{P}_y\} = 0. \tag{28}$$

We note that the components of the linear momentum $\mathcal{P}_x$ and $\mathcal{P}_y$ and the circulation $\mathcal{Z}$ are functionals of the vorticity ζ, see the definitions (19)–(24). The set of the elements $\mathcal{P}_x, \mathcal{P}_y$ and $\mathcal{Z}$ together with the Nambu bracket (24) satisfies all conditions of the Lie algebra definition 1. It is multi-linear and antisymmetric. Moreover, keeping the enstrophy fixed, the Jacobi-identity is satisfied

$$\{\mathcal{Z}, \mathcal{E}, \{\mathcal{P}_\alpha, \mathcal{E}, \mathcal{P}_\beta\}\} + \{\mathcal{P}_\beta, \mathcal{E}, \{\mathcal{Z}, \mathcal{E}, \mathcal{P}_\alpha\}\} + \{\mathcal{P}_\alpha, \mathcal{E}, \{\mathcal{P}_\beta, \mathcal{E}, \mathcal{Z}\}\} = 0 \tag{29}$$

for $\alpha, \beta \in \{x, y\}$, because each summand vanishes. Therefore, the Nambu bracket generates a Lie algebra for incompressible, inviscid flows. As an interesting property, the Lie algebra is nilpotent, because the lower central series terminates. Therefore, this Lie algebra is solvable. We will call this second-step nilpotent Lie algebra V*ortex-Heisenberg Lie algebra* and denote it $vh(2)$.

Névir (1998) shows that the Nambu bracket of these global, continuous conserved quantities with respect to the enstrophy $\mathcal{E}$ (28) yields the same relations as for the discrete point vortex model, where the discrete conservation quantities are regarded. The 2D Nambu bracket results into the same relations as the Lie–Poisson bracket for discrete point vortex dynamics as for example discussed in Newton (2001).

3.1. Matrix representation of the algebra vh(2)

Matrix representations of Lie algebras allow a transition to Lie groups, as we will show in subsection 3.2. The main idea for the algebraic approach is to find matrices that represent the conserved quantities for the given system. For two-dimensional, inviscid incompressible vorticity dynamics these quantities are the linear momenta $\mathcal{P}_x$ and $\mathcal{P}_y$ and the circulation $\mathcal{Z}$. The matrix commutator of two $n \times n$-matrices $\mathbf{X}$ and $\mathbf{Y}$ is defined by

$$[\mathbf{X}, \mathbf{Y}] = \mathbf{XY} - \mathbf{YX}. \tag{30}$$

Regarding the notation, bold capital letters denote matrices in this paper. The different notations used in this work are summarized in table 2.

In the first step, we will introduce the matrices that represent the conserved quantities. Analogously to the above defined Nambu bracket we will consider the matrix commutator for a fixed enstrophy $\mathcal{E}$:

$$\begin{aligned} \text{Nambu bracket} &\longrightarrow \text{Matrix commutator} \\ \{\mathcal{F}, \mathcal{E}, \mathcal{G}\} &\longrightarrow [\mathbf{F}, \mathbf{G}]. \end{aligned} \tag{31}$$

Physically, applying the matrix commutator, we consider the dynamics on an enstrophy hyperplane. We will show that the matrix commutator of the matrices representing the conserved quantities provide the same physical bracket relations as the Nambu bracket for the conserved quantities given by functionals! The novel matrix representation of the two linear momenta $\boldsymbol{P_x}$ and $\boldsymbol{P_y}$ and the circulation $\boldsymbol{Z}$ are given by:

$$\mathbf{P_x} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{P_y} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{32}$$

Indeed, determining the bracket in terms of the matrix commutator leads to the same relations as the Nambu brackets for 2D vortex flows (28):

$$[\mathbf{P_x}, \mathbf{P_y}] = \mathbf{Z}, \qquad [\mathbf{P_y}, \mathbf{P_x}] = -\mathbf{Z}, \qquad [\mathbf{Z}, \mathbf{P_x}] = [\mathbf{Z}, \mathbf{P_y}] = 0. \tag{33}$$

Thus, we found a matrix representation for the second-step nilpotent Lie algebra $vh(2)$. Furthermore, the linear combination of the basis matrices reads as:

$$\mathbf{A} = \lambda_x \mathbf{P_x} + \lambda_y \mathbf{P_y} + \mu \mathbf{Z} = \begin{pmatrix} 0 & \lambda_x & \mu \\ 0 & 0 & \lambda_y \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_x, \lambda_y, \mu \in \mathbb{R}. \tag{34}$$

This representation has the classical nilpotent Heisenberg structure, mathematically discussed for example in Hall (2003).

3.2. Matrix representation of the group VH(2)

The Lie algebra of a matrix Lie group G is just the tangent space of G at the identity. Furthermore, the exponential map as it is defined in the matrix case coincides with the exponential map for general Lie groups (see e.g. Hall 2003). The transition of elements in the Lie algebra to elements of the corresponding Lie group can be achieved via the exponential map, which maps the matrix Lie algebra (34) to a corresponding matrix Lie group. Let $\boldsymbol{X}$ be a complex or real $n \times n$ matrix. Then the exponential map is defined by:

$$\exp \mathbf{X} = \sum_{k=0}^{\infty} \frac{\mathbf{X}^k}{k!} = \mathbb{E} + \mathbf{X} + \frac{1}{2}\mathbf{X}^2 + \frac{1}{6}\mathbf{X}^3 + \cdots \tag{35}$$

with the $n \times n$ unit matrix $\mathbb{E}$. For detailed proofs see e.g. the books of Hall (2003) or Procesi (2006). The exponential map reduces to the first k terms if the matrix $\boldsymbol{A}$ is nilpotent of order k. The matrix $\mathbf{A}$ as defined in (34) has the property $\boldsymbol{A}^{\mathbf{2}} \neq \mathbf{0}, \boldsymbol{A}^{\mathbf{3}} = \mathbf{0}$. Applying the exponential map (35) to the linear combination of the matrix representatives of the conserved quantities $\boldsymbol{A}$ (34) we obtain a matrix representation of a Lie group for two-dimensional, incompressible vortex dynamics:

$$\exp \mathbf{A} = \mathbb{E} + \mathbf{A} + \frac{1}{2}\mathbf{A}^2 = \begin{pmatrix} 1 & \lambda_x & \mu' \\ 0 & 1 & \lambda_y \\ 0 & 0 & 1 \end{pmatrix}, \tag{36}$$

with $\mu' = (\mu + \frac{1}{2}\lambda_x\lambda_y)$. From physical perspective, group elements have a different meaning than the elements of the Lie algebra that 'live' on the tangent space of the neutral element of the group. We summarize and interpret the notation for the algebra and the group elements analogously to the notation for classical mechanics discussed in Sudarshan and Mukunda (1974):

$$\begin{array}{lllllll} \text{algebra} & & & & & & \text{group} \\ \mathcal{P}: \text{surface integral} & \longrightarrow & \text{density of the momentum} & \longrightarrow & \text{velocity} & \longrightarrow & \mathbf{s} \\ \mathcal{Z}: \text{surface integral} & \longrightarrow & \text{density of the circulation} & \longrightarrow & \text{vorticity} & \longrightarrow & a \end{array} \tag{37}$$

with $\boldsymbol{s} = (s_x, s_y)$. See subsection 3.4 for a more detailed physical explanation. Giving the matrix entries a physical meaning, the matrices of form $\exp(\boldsymbol{A})$ (see (36)) together with the classical matrix product form a group derived from the vorticity equation. Mathematically, it is a subgroup of the so-called Heisenberg group that is defined as the set of upper triangular matrices.

After the transition from the algebra to the group representation, the basis matrix representation of a vortex Lie group for two-dimensional fluid dynamics is given by:

$$\mathbf{A} = \begin{pmatrix} 1 & s_x & a \\ 0 & 1 & s_y \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

where the group operation is given by the usual matrix product. We will call this group *vortex-Heisenberg group* and denote it $VH(2)$.

3.3. The group composition of the group $VH(2)$

Further, we aim for the group operation for group elements represented as vectors and the corresponding group composition. To achieve this goal, we start again from the matrix representation of the Lie algebra (34). Then, the group operation can be derived by applying Baker–Campbell–Hausdorff-formula to the matrices $\boldsymbol{A}$ and $\boldsymbol{A}'$ of form (34):

$$\mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{A}'} = \mathrm{e}^{\mathbf{A}+\mathbf{A}'+\frac{1}{2}[\mathbf{A},\mathbf{A}']+\frac{1}{12}[\mathbf{A},[\mathbf{A},\mathbf{A}']]-\frac{1}{12}[\mathbf{A}',[\mathbf{A},\mathbf{A}']]-\frac{1}{24}[\mathbf{A}',[\mathbf{A},[\mathbf{A},\mathbf{A}']]]-\cdots}. \tag{39}$$

For second step nilpotent Lie algebras, the Baker–Campbell–Hausdorff-formula (39) reduces to:

$$\mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{A}'} = \mathrm{e}^{\mathbf{A}+\mathbf{A}'+\frac{1}{2}[\mathbf{A},\mathbf{A}']}. \tag{40}$$

Let $\boldsymbol{A}$ and $\boldsymbol{A}'$ be two elements of the vortex group given by:

$$\mathbf{A} = \begin{pmatrix} 0 & s_x & a \\ 0 & 0 & s_y \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \mathbf{A}' = \begin{pmatrix} 0 & s'_x & a' \\ 0 & 0 & s'_y \\ 0 & 0 & 0 \end{pmatrix}. \tag{41}$$

Applying the Baker–Campbell–Hausdorff-formula to $\boldsymbol{A}$ and $\boldsymbol{A}'$ we obtain for the exponent on the right-hand side in (40):

$$\mathbf{A}+\mathbf{A}'+\frac{1}{2}[\mathbf{A},\mathbf{A}'] = \begin{pmatrix} 0 & (s_x+s'_x) & a+a'+\dfrac{1}{2}(s_xs'_y-s_ys'_x) \\ 0 & 0 & (s_y+s'_y) \\ 0 & 0 & 0 \end{pmatrix}. \tag{42}$$

Now, we compare the entries of the matrix (42) with the entries in $\mathbf{A}$. We see that the λ-coordinates add $(s_x+s'_x)$, $(s_y+s'_y)$ and the μ-entry becomes $a+a'+\frac{1}{2}(s_xs'_y-s_ys'_x)$. We can formulate the group operation $(\circ)$ for 2D incompressible, inviscid vortex dynamics componentwise:

$$((s_x,s_y),a)\circ((s'_x,s'_y),a') = ((s_x+s'_x,s_y+s'_y),a+a'+\frac{1}{2}(s_xs'_y-s_ys'_x)). \tag{43}$$

As an important property of vortex dynamics, the circulation $\mathcal{Z}$ can have positive and negative sign depending on the direction of the vortex rotation. It follows that the 2D Nambu bracket (28) and also the associated matrix commutator can have both sign. Taking the different signs into account leads to the following group operation:

$$((s_x,s_y),a)\circ((s'_x,s'_y),a') = ((s_x+s'_x,s_y+s'_y),a+a'\pm\frac{1}{2}(s_xs'_y-s_ys'_x)). \tag{44}$$

We will call the here derived group *vortex-Heisenberg group* and denote it $VH(2)$. See appendix A for the proof of the group properties. In the next section 4 we will show the analogies and differences of this vortex group to the group for mass point dynamics, which is called standard Heisenberg group.

Remark. We shortly discuss the sign in front of the cross product in the group operation (44). For three-dimensional fluids, the Nambu bracket is calculated with respect to the helicity, which can have positive or negative sign. This leads to a Nambu bracket, where different signs are possible, see (14). In two dimensions we consider the Nambu bracket with respect to the positive definite enstrophy. But it is important to remark that the circulation can have different signs, depending on the direction of rotation. Alternatively to the positive definite enstrophy discussed here, the 2D bracket could be defined with respect to a generalized enstrophy, considering an arbitrary exponent of the vorticity. For odd exponents this generalized quantity could be positive or negative, analogously to the helicity in three dimensions. This would lead to the same mathematical group operation as $VH(2)$ given in (44).

3.4. Physical interpretation

In the latter subsection we introduced vortex-Heisenberg group elements as pairs $(\boldsymbol{s}, a)$ with $\boldsymbol{s} = (\boldsymbol{s_x}, \boldsymbol{s_y})$ for two-dimensional fluid dynamics. We recall the zonal and meridional momentum with respect to the vorticity ζ:

$$\mathcal{P}_x[\zeta] = \int_F \mathrm{d}f\, \zeta y, \qquad \mathcal{P}_y[\zeta] = -\int_F \mathrm{d}f\, \zeta x. \tag{45}$$

Now, assume infinitesimal area elements of the momentum (45). Similar to the approach of Sudarshan and Mukunda (1974), we obtain:

$$\begin{array}{llllll} & \text{algebra} & & & \text{group} & \\ \mathcal{P}_x: & \text{surface integral} \longrightarrow & \text{density} & \longrightarrow & \exp((\zeta\ y \cdot s_x)^*) & \longrightarrow s_x = x - x' \\ \mathcal{P}_y: & \text{surface integral} \longrightarrow & \text{density} & \longrightarrow & \exp((-\zeta x \cdot s_y)^*) & \longrightarrow s_y = y - y' \\ \mathcal{Z}: & \text{surface integral} \longrightarrow & \text{density} & \longrightarrow & \exp((\zeta \cdot a)^*) & \longrightarrow a = F - F' \end{array} \tag{46}$$

where the asterisk denotes the scaling one over the action to obtain dimensionless exponents. Let a state of a vortex be characterized by its local position x. Furthermore, let F denote the area of the 2D vortex, for example the area of a high- or low pressure area. Then, the vector $\mathbf{s} = (s_x, s_y)$ has the unit m and represents the displacements of the vortex $s_x = \Delta x = x - x'$ and $s_y = \Delta y = y - y'$. The scalar quantity a denotes the change of the area element $a = \Delta F = F - F'$ of the vortex.

4. Comparison to the inertial motion of classical mass points

The Heisenberg group representation for 2D vortex dynamics (44) that we introduced in the last section holds a frequently discussed mathematical structure, which has been applied to different physical problems such as quantum mechanics. It is interesting to note that this group can also be found in classical mechanics, which only has been rarely discussed (see, e.g. Sudarshan and Mukunda 1974, Sorba 1976). Bolsinov *et al* (2005) considered four planar vortices and by reducing the system, the authors provide a relation of vortex dynamics to mass point dynamics. Regarding the methodological aspects of dynamical algebraic representations in physics, Zhang *et al* (1990) and Richaud and Penna (2017) show the applicability to quantum systems.

We will start with the algebraic representation of the inertial motion of mass points with respect to the Galilei transformation. Consider the six dimensional phase space given by the position vector $\mathbf{p} = (p_1, p_2, p_3) \in \mathbb{R}^3$ and the momentum $\mathbf{q} = (q_1, q_2, q_3) \in \mathbb{R}^3$ to describe the state of a point with mass m at a given time t. Then, the Hamiltonian representation of the set of equations is given by:

$$\frac{\mathrm{d}p_\alpha}{\mathrm{d}t} = -\frac{\partial H}{\partial q_\alpha}, \qquad \frac{\mathrm{d}q_\alpha}{\mathrm{d}t} = \frac{\partial H}{\partial p_\alpha}, \tag{47}$$

where the index $\alpha = 1, 2, 3$ denotes the direction in the three-dimensional configuration space; H is the Hamiltonian given by the kinetic energy. This set of equations leads to the classical canonical Poisson bracket:

$$\{F, G\}_\mathrm{P} = \sum_{\alpha=1}^{3} \left(\frac{\partial F}{\partial q_\alpha} \frac{\partial G}{\partial p_\alpha} - \frac{\partial F}{\partial p_\alpha} \frac{\partial G}{\partial q_\alpha} \right). \tag{48}$$

Adding the mass to the six-dimensional phase space of the momentum and position vector the dimension of the phase space becomes seven. The mass commutes with the components of $\mathbf{p}$ and $\mathbf{g}$. Additionally, we regard the generalization of the special Galilei transformation $\mathbf{g} = (g_1, g_2, g_3)$ and obtain the following relations of the components of the momentum, position vector and the mass m that define the standard Heisenberg algebra:

$$\{g_\alpha, p_\beta\}_\mathrm{P} = \delta_{ij} m, \quad \{g_\alpha, g_\beta\}_\mathrm{P} = 0, \quad \{p_\alpha, p_\beta\}_\mathrm{P} = 0, \quad \{m, g_\alpha\}_\mathrm{P} = 0, \quad \{m, p_\alpha\}_\mathrm{P} = 0, \tag{49}$$

for $\alpha, \beta = 1, 2, 3$. Comparing the nilpotent vortex-Heisenberg algebra $vh(2)$ with the nilpotent standard Heisenberg algebra for one mass point in a three-dimensional space $sh(3)$:

$$\begin{array}{ccc} \text{Vortex-Heisenberg algebra } vh(2) & \longleftrightarrow & \text{Standard Heisenberg algebra } sh(3) \\ \{\mathcal{P}_\alpha, \mathcal{E}, \mathcal{P}_\beta\} = \epsilon_{\alpha\beta}\mathcal{Z} \quad (\alpha, \beta = 1, 2) & & \{g_\alpha, p_\beta\}_P = \delta_{\alpha\beta} m \quad (\alpha, \beta = 1, 2, 3) \end{array} \tag{50}$$

we recognize fundamental differences comprising the nilpotent structure: an antisymmetric structure constant $\epsilon_{\alpha\beta}$ of $vh(2)$ and a symmetric structure constant $\delta_{\alpha\beta}$ of $sh(3)$. Furthermore, regarding $vh(2)$ the bracket is determined by one variable, the momentum (calculated w.r.t. the enstrophy), the indices α and β are space directions. Regarding $sh(3)$ two different variables (the Galilean transformation and the location) determine the bracket, the indices α and β indicate the space dimension.

4.1. Matrix representations of sh(1) and SH(1)

We can find a matrix representation for a state in the phase space characterized by the momentum q, the position p and the total mass m, such that they satisfy the bracket relations (49) with respect to the matrix commutator. For simplicity, consider one mass point in one spatial dimension:

$$\{g, p\} = m \tag{51}$$

and denote the corresponding algebra $sh(1)$.

$$\begin{array}{rcl} \text{Poisson-bracket} & \longrightarrow & \text{Matrix commutator} \\ \{x, y\}_\mathrm{P} & \longrightarrow & [\mathbf{X}, \mathbf{Y}] = \mathbf{X} \cdot \mathbf{Y} - \mathbf{Y} \cdot \mathbf{X}, \end{array} \tag{52}$$

where $\mathbf{X}$ and $\mathbf{Y}$ denote two $n \times n$ matrices and '$\cdot$' is the usual matrix multiplication.

Denoting the matrix basis representations of the Galilei transformation g, the momentum p and the mass m with $\boldsymbol{G}, \boldsymbol{P}$ and $\boldsymbol{M}$, the matrices read:

$$\mathbf{G} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{P} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{M} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (53)$$

Applying the matrix commutator (30), we obtain the same bracket relations as in (49). Then, the linear combination $\boldsymbol{C}$ of the matrix representations (53) reads as:

$$\mathbf{C} := s\mathbf{P} + v\mathbf{G} + \theta\mathbf{M} = \begin{pmatrix} 0 & v & \theta \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix}, \quad (54)$$

where s is a constant displacement in direction of the momentum, v a constant velocity generated by g and the variable θ is conjugated to the mass with respect to the action.

4.2. The matrix Lie group

Sudarshan and Mukunda (1974) consider the following componentwise transition from the algebra to the group:

$$\begin{array}{ccccc} \text{algebra} & & & & \text{group} \\ p & \longrightarrow & \exp((s \cdot p)^*) & \longrightarrow & s \\ g & \longrightarrow & \exp((v \cdot g)^*) & \longrightarrow & v \\ m & \longrightarrow & \exp((\theta \cdot m)^*) & \longrightarrow & \theta, \end{array} \quad (55)$$

where the asterisk denotes the scaling by the action to obtain dimensionless exponents. In the next step, we will derive the vector representation of the standard Heisenberg group. We consider two states (s, v, θ) and (s', v', θ') represented by the matrices $\boldsymbol{X}$ and $\boldsymbol{X}'$:

$$(s, v, \theta) \mapsto \begin{pmatrix} 0 & v & \theta \\ 0 & 0 & s \\ 0 & 0 & 0 \end{pmatrix} := \mathbf{X} \quad \text{and} \quad (s', v', \theta') \mapsto \begin{pmatrix} 0 & v' & \theta' \\ 0 & 0 & s' \\ 0 & 0 & 0 \end{pmatrix} := \mathbf{X}'. \quad (56)$$

Calculating the matrix commutator of the matrix commutator of X and X' and an additional state (s'', v'', θ''), represented by the matrix X'', the outcome vanishes, i.e.

$$\left[\left[\mathbf{X}, \mathbf{X}'\right], \mathbf{X}''\right] = 0. \quad (57)$$

Thus, this algebraic structure is nilpotent. This property allows for a smooth transition from the bracket, i.e. the algebra, to the group. We apply the Baker–Campbell–Hausdorff-formula (40) to obtain the group operation directly from the Lie algebra matrix representation:

$$\mathrm{e}^{\mathbf{X}}\,\mathrm{e}^{\mathbf{X}'} = \mathrm{e}^{\mathbf{X}+\mathbf{X}'+\frac{1}{2}[\mathbf{X},\mathbf{X}']}, \quad (58)$$

with the usual matrix addition and the matrix commutator (30). The exponent of (58) reads as:

$$\mathbf{X}+\mathbf{X}'+\frac{1}{2}[\mathbf{X},\mathbf{X}'] = \begin{pmatrix} 0 & v+v' & \theta+\theta'+\frac{1}{2}(v\cdot s'-s\cdot v') \\ 0 & 0 & s+s' \\ 0 & 0 & 0 \end{pmatrix}. \tag{59}$$

Then, ln (exp($\mathbf{X}$) exp($\mathbf{X}'$)) provides the associated group operation of the Heisenberg Lie group for mass point dynamics:

$$(v,s,\theta)\circ(v',s',\theta') = \left(v+v',s+s',\theta+\theta'+\frac{1}{2}(v\cdot s'-s\cdot v')\right). \tag{60}$$

Comparing (60) with (44), both group representations have an isomorphic structure, even though their physical meaning differs. From mathematical perspective, the mass is always positive and the circulation can have both signs. It follows that the vortex-Heisenberg group is a covering of the standard Heisenberg group of one mass point in one spatial direction denoted by $SH(1)$:

$$VH(2)/Z_2 = SH(1), \tag{61}$$

with the cyclic group Z_2; the multiplicative group comprising 1 and -1.

5. Algebra and group representation of 3D incompressible vortex dynamics

In order to generalize the group representation for incompressible 3D vortex dynamics we shortly recall the concept of quaternions. Quaternions can be seen as a generalization of the complex numbers. The algebra of quaternions is denoted $\mathbb{H}$ to honour William R Hamilton who introduced the quaternions in 1843. A quaternion q can be written as $\boldsymbol{q} = \boldsymbol{ai}+\boldsymbol{bj}+\boldsymbol{ck}+\boldsymbol{d}$ and its conjugate is given by $\overline{\mathbf{q}} = -ai-bj-ck+d$. We can write $\mathbb{H}$ as set

$$\mathbb{H} = \{(a,b,c,d) \mid a,b,c,d\in\mathbb{R}\} \tag{62}$$

with basis elements

$$i=(1,0,0,0),\qquad j=(0,1,0,0),\qquad k=(0,0,1,0),\qquad 1=(0,0,0,1). \tag{63}$$

Thus, the set $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$ and the imaginary part of quaternions can be identified with $\mathbb{R}^3$. But, in contrast to the multiplication of real numbers the multiplication of quaternions is not commutative. The multiplication of the basic elements are summarized in table 1.

5.1. Matrix representation of the algebra vh(3)

In general, a Lie algebra can be regarded as a tangent space at the identity element of a group, see section 3. The algebraic view on vortex dynamics offers several advantages. On the one hand, a Lie algebra forms a linear space. On the other hand, the algebra provides the possibility to map the dynamics of vortices represented by the algebra to a Lie group for vortex dynamics. This step can be seen as a 'structural integration method'.

The six-dimensional Lie algebra of the space of the elements $\{\mathcal{P}_x,\mathcal{P}_y,\mathcal{P}_z,\mathcal{Z}_x,\mathcal{Z}_y,\mathcal{Z}_z\}$ together with the Nambu bracket satisfies the necessary properties of being multi-linear and antisymmetric w.r.t. all arguments. Moreover, recalling

$$\{\mathcal{P}_\alpha,h,\mathcal{P}_\beta\} = \varepsilon_{\alpha\beta\gamma}\mathcal{Z}_\gamma,\qquad \{\mathcal{P}_\alpha,h,\mathcal{Z}_\beta\} = \{\mathcal{Z}_\alpha,h,\mathcal{Z}_\beta\} = 0 \tag{64}$$

Table 1. Quaternion multiplication table.

Quaternion				
$\cdot$	i	j	k	1
i	-1	k	$-j$	i
j	$-k$	-1	i	j
k	j	$-i$	-1	k
1	i	j	k	1

for $\alpha, \beta, \gamma \in \{1, 2, 3\}$ and keeping the helicity fixed, the Jacobi-identity is satisfied

$$\{\mathcal{Z}_\alpha, h, \{\mathcal{P}_\alpha, h, \mathcal{P}_\beta\}\} + \{\mathcal{P}_\beta, h, \{\mathcal{Z}_\alpha, h, \mathcal{P}_\alpha\}\} + \{\mathcal{P}_\alpha, h, \{\mathcal{P}_\beta, h, \mathcal{Z}_\alpha\}\} = 0, \tag{65}$$

because each summand vanishes. These properties hold for h being a fixed positive and negative real number, see the definition of the Nambu bracket (14). Therefore, all properties of the Lie algebra are here also satisfied in the continuous case for functionals; see definition 1. We call this Lie algebra *vortex-Heisenberg Lie algebra* and denote it $vh(3)$.

We note that the general Nambu bracket for discrete systems and for a non-fixed conserved quantity has to satisfy the Takhtajan identity that can be seen as a generalization of the Jacobi-identity obeying the Liouville theorem (Takhtajan 1994).

The vortex-Heisenberg algebra has important algebraic properties. The result of the bracket relation of two momenta components (64) are the components of the total flux of vorticity that commute with any other element of the space. Therefore, the vortex-Heisenberg algebra is nilpotent. Moreover, the Casimir functional $\mathcal{C}$ that commutes with all other functionals is given by

$$\mathcal{C} = \pm\lambda\ (\mathcal{P}_x\mathcal{Z}_x + \mathcal{P}_y\mathcal{Z}_y + \mathcal{P}_z\mathcal{Z}_z). \tag{66}$$

The Casimir functional is given by the scalar product of the linear momentum and the total flux of vorticity given by volume integrals. Therefore, it can be seen as a helicity-like quantity with respect to integral conserved quantities. The helicity is a natural property for three-dimensional fluid motion. We remark that this quantity vanishes for even spatial dimensions. The proof of $\mathcal{C}$ being a Casimir functional is straight forward using (64):

$$\begin{aligned}
\{\mathcal{C}, h, \mathcal{P}_x\} &= \pm\lambda\ (\{\mathcal{P}_x\mathcal{Z}_x + \mathcal{P}_y\mathcal{Z}_y + \mathcal{P}_z\mathcal{Z}_z, h, \mathcal{P}_x\}) \\
&= \pm\lambda\ (\{\mathcal{P}_x\mathcal{Z}_x, h, \mathcal{P}_x\} + \{\mathcal{P}_y\mathcal{Z}_y, h, \mathcal{P}_x\} + \{\mathcal{P}_z\mathcal{Z}_z, h, \mathcal{P}_x\}) \\
&= \pm\lambda\ \left(\mathcal{P}_y\{\mathcal{Z}_y, h, \mathcal{P}_x\} + \mathcal{Z}_y\{\mathcal{P}_y, h, \mathcal{P}_x\} + \mathcal{P}_z\{\mathcal{Z}_z, h, \mathcal{P}_x\} + \mathcal{Z}_z\{\mathcal{P}_z, h, \mathcal{P}_x\}\right) \\
&= \pm\lambda\ \left(\mathcal{Z}_y(-\mathcal{Z}_z) + \mathcal{Z}_z\mathcal{Z}_y\right) = 0
\end{aligned} \tag{67}$$

Analogously we obtain:

$$\{\mathcal{C}, h, \mathcal{P}_y\} = \{\mathcal{C}, h, \mathcal{P}_z\} = 0 \tag{68}$$

and

$$\{\mathcal{C}, h, \mathcal{Z}_x\} = \{\mathcal{C}, h, \mathcal{Z}_y\} = \{\mathcal{C}, h, \mathcal{Z}_z\} = 0. \tag{69}$$

We remark that the Casimir functional can also be written and proven analogously in terms of the matrix representation.

In order to derive a group for 3D hydrodynamical systems based on the vorticity equation, we will introduce a matrix representation of the basis elements of the vortex-Heisenberg algebra with respect to the basis of quaternions $\mathbb{H}$. Analogously to our approach in two dimensions we will introduce matrices representing the conserved quantities $\mathcal{P}_\alpha$ and $\mathcal{Z}_\alpha$ $(\alpha = 1, 2, 3)$ such that the corresponding matrix commutator satisfies the same relations as the Nambu bracket in (64).

We start by proposing matrix representations for the components of the momentum $\mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z$ that we will denote as $\boldsymbol{P_x}, \boldsymbol{P_y}, \boldsymbol{P_z}$ and matrix representations for the components of the total flux of vorticity $\mathcal{Z}_x$, $\mathcal{Z}_y$, $\mathcal{Z}_z$ that we will denote as $\mathbf{Z_x}, \mathbf{Z_y}, \mathbf{Z_z}$ with respect to quaternionian basis i, j, k. Then we will see that the matrix commutator of these matrices result in the same physical relations as the Nambu bracket for the corresponding functionals (64).

$$\mathbf{P_x} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -i & 0 & 0 & 0 & 0 \\ 0 & i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z_x} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (70)$$

$$\mathbf{P_y} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & j & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -j & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z_y} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & j \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (71)$$

$$\mathbf{P_z} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k \\ 0 & -k & 0 & 0 & 0 & 0 & 0 \\ k & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \qquad \mathbf{Z_z} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \qquad (72)$$

Applying the matrix commutator (52) with respect to a fixed positive helicity h to the above basis representation, we obtain:

$$[\mathbf{P_\alpha}, \mathbf{P_\beta}] = \epsilon_{\alpha\beta\gamma}\mathbf{Z_\gamma}, \qquad [\mathbf{P_\alpha}, \mathbf{Z_\beta}] = 0, \quad \text{and} \quad [\mathbf{Z_\alpha}, \mathbf{Z_\beta}] = 0 \qquad (73)$$

for α, β, $\gamma \in \{\mathrm{x, y, z}\}$, where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol. These relations coincide with the Nambu bracket (64) and therefore, these matrices represent $vh(3)$. The linear combination of the basis elements reads:

$$\mathbf{A} = \lambda_x\mathbf{P_x} + \lambda_y\mathbf{P_y} + \lambda_z\mathbf{P_z} + \mu_x\mathbf{Z_x} + \mu_y\mathbf{Z_y} + \mu_z\mathbf{Z_z} \qquad (74)$$

with the linear coefficients $\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z \in \mathbb{R}$. Inserting (70)–(72) in (74) results in:

$$\mathbf{A} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \lambda_x i \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_y j \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_z k \\ 0 & -\lambda_z k & \lambda_y j & 0 & 0 & 0 & \mu_x i \\ \lambda_z k & 0 & -\lambda_x i & 0 & 0 & 0 & \mu_y j \\ -\lambda_y j & \lambda_x i & 0 & 0 & 0 & 0 & \mu_z k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{75}$$

5.2. *Matrix representation of the group VH*(3)

In this section we will introduce a matrix representation as well as a vector representation of the vortex-Heisenberg group.

We recall that a matrix $\boldsymbol{A}$ of form (75) is second-step nilpotent, i.e. $\boldsymbol{A}^2 = \mathbf{0}$. Especially for nilpotent algebraic representations the exponential map provides a transition from the Lie algebra $\mathfrak{g}$ to the group G (Hall 2003). The exponential map (35) reduces to

$$\exp(\mathbf{A}) = \sum_{k=0}^{\infty} \frac{\mathbf{A}^k}{k!} = \mathbb{E} + \mathbf{A}, \tag{76}$$

where $\mathbb{E}$ denotes the identity matrix. Thus, applying the exponential map (76) to the matrix representation of the vortex-Heisenberg algebra (75) we obtain the following matrix representation of the vortex-Heisenberg group:

$$\exp\,\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \hat{\lambda}_x i \\ 0 & 1 & 0 & 0 & 0 & 0 & \hat{\lambda}_y j \\ 0 & 0 & 1 & 0 & 0 & 0 & \hat{\lambda}_z k \\ 0 & -\hat{\lambda}_z k & \hat{\lambda}_y j & 1 & 0 & 0 & \hat{\mu}_x i \\ \hat{\lambda}_z k & 0 & -\hat{\lambda}_x i & 0 & 1 & 0 & \hat{\mu}_y j \\ -\hat{\lambda}_y j & \hat{\lambda}_x i & 0 & 0 & 0 & 1 & \hat{\mu}_z k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{77}$$

with $\hat{\lambda}_\alpha = \frac{\lambda_\alpha}{\sqrt{2}}$, $\hat{\mu}_\alpha = \frac{\mu_\alpha}{\sqrt{2}}, \in \mathbb{R}$. Now, each group element can be represented by a matrix of form (77) and the group operation is given by the usual matrix product, see appendix B for the proof of the existence of the identity and inverse element and that the associativity property holds.

5.3. *The group composition of the group VH*(3)

Let us now consider two matrices $\mathbf{A}$ and $\mathbf{A}'$ of form (77). Such matrices commute with their commutator, i.e.

$$\left[\mathbf{A}, \left[\mathbf{A}, \mathbf{A}'\right]\right] = \left[\mathbf{A}', \left[\mathbf{A}, \mathbf{A}'\right]\right] = 0. \tag{78}$$

Then, the Baker–Campbell–Hausdorff-formula, which yields a mapping from an algebra to a group, reduces to:

$$\mathrm{e}^{\mathbf{A}}\,\mathrm{e}^{\mathbf{A}'} = \mathrm{e}^{\mathbf{A}+\mathbf{A}'+\frac{1}{2}[\mathbf{A},\mathbf{A}']}. \tag{79}$$

(See e.g. Hall 2003). To derive the vector representation of the group composition, we apply the Baker–Campbell–Hausdorff-formula (79) to two matrices A' and A of form (75). Then, the exponent on the right-hand side of (79) reads as

$$\mathbf{A}+\mathbf{A}' + \frac{1}{2}[\mathbf{A},\mathbf{A}']$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & (\hat{\lambda}_x + \hat{\lambda}'_x)i \\ 0 & 0 & 0 & 0 & 0 & 0 & (\hat{\lambda}_y + \hat{\lambda}'_y)j \\ 0 & 0 & 0 & 0 & 0 & 0 & (\hat{\lambda}_z + \hat{\lambda}'_z)k \\ 0 & (-\hat{\lambda}_z - \hat{\lambda}'_z)k & (\hat{\lambda}_y + \hat{\lambda}'_y)j & 0 & 0 & 0 & (\hat{\mu}_x + \hat{\mu}'_x + \frac{1}{2}\left(\hat{\lambda}_y \cdot \hat{\lambda}'_z - \hat{\lambda}'_y \cdot \hat{\lambda}_y\right) i \\ \left(\hat{\lambda}_z + \hat{\lambda}'_z\right) k & 0 & (-\hat{\lambda}_x - \hat{\lambda}'_x)i & 0 & 0 & 0 & (\hat{\mu}_y + \hat{\mu}'_y + \frac{1}{2}\left(\hat{\lambda}_z \cdot \hat{\lambda}'_x - \hat{\lambda}'_z \cdot \hat{\lambda}_x\right) j \\ (-\hat{\lambda}_y - \hat{\lambda}'_y)j & (\hat{\lambda}_x + \hat{\lambda}'_x)i & 0 & 0 & 0 & 0 & (\hat{\mu}_z + \hat{\mu}'_z + \frac{1}{2}(\hat{\lambda}_x \cdot \hat{\lambda}'_y - \hat{\lambda}'_x \cdot \hat{\lambda}_y)k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{80}$$

with scaled $\hat{\lambda}_\alpha = \frac{1}{\sqrt{2}}\lambda_\alpha$ and $\hat{\mu}_\alpha = \frac{1}{\sqrt{2}}\mu_\alpha,\ \alpha = x, y, z$. We recall that the linear combination **A** (see (74)) is composed of the representations of the linear momenta $\mathscr{P}$ and the total flux of circulation $\mathscr{Z}$ (Müller 2018). Similar to the 2D notations for the group and algebra generators we suggest the following notations:

$$\begin{array}{llll} \text{algebra} & & & \text{group} \\ \mathscr{P}: \ \text{volume integral} \longrightarrow \text{density of the momentum} & \longrightarrow \text{velocity} & \longrightarrow & \mathbf{s} = \mathbf{r} - \mathbf{r}' \\ \mathcal{Z}: \ \text{volume integral} \longrightarrow \text{density of the total flux of vorticity} & \longrightarrow \text{vorticity} & \longrightarrow & \mathbf{a} = \mathbf{a} - \mathbf{a}', \end{array} \tag{81}$$

as we will explain more in detail in the following subsection.

Comparing (75) with (80) leads to the following vortex-Heisenberg group operation $\circ$ with respect to the group elements $(\mathbf{s}, \mathbf{a})$ and $(\mathbf{s}', \mathbf{a}')$:

$$(\mathbf{s}, \mathbf{a}) \circ (\mathbf{s}', \mathbf{a}') = \left(\mathbf{s} + \mathbf{s}', \mathbf{a} + \mathbf{a}' + \frac{1}{2}\, \mathbf{s} \times \mathbf{s}'\right). \tag{82}$$

We recall that as a first challenge the matrix representations of the corresponding conserved quantities had to be found. In the second step, we obtained the algebra representation via the matrix commutator (83), which is isomorphic to the corresponding Nambu bracket.

In section 2.1, we have pronounced the importance of the sign of the helicity (14). For a fixed negative helicity, the matrix commutator changes its sign:

$$\begin{array}{lll} \text{Nambu bracket} & \longrightarrow & \text{Matrix commutator} \\ \{\mathcal{F}, h^+, \mathcal{G}\} & \longrightarrow & [\mathbf{F}, \mathbf{G}] = \mathbf{F} \cdot \mathbf{G} - \mathbf{G} \cdot \mathbf{F} \quad . \\ \{\mathcal{F}, h^-, \mathcal{G}\} & \longrightarrow & [\mathbf{F}, \mathbf{G}]_- = \mathbf{G} \cdot \mathbf{F} - \mathbf{F} \cdot \mathbf{G} \end{array} \tag{83}$$

Taking both signs of the helicity into account we obtain the following vortex-Heisenberg group operation $\circ$ for 3D inviscid fluid dynamics:

$$(\mathbf{s}'', \mathbf{a}'') = (\mathbf{s}, \mathbf{a}) \circ (\mathbf{s}', \mathbf{a}') = \left(\mathbf{s} + \mathbf{s}', \mathbf{a} + \mathbf{a}' \pm \frac{1}{2}\, \mathbf{s} \times \mathbf{s}'\right), \tag{84}$$

where the last term on the right-hand side has negative sign in case of a negative helicity. $VH(3)$ can be seen as central extension of the abelian group of the 3D translations by the total flux of vorticity.

5.4. Physical interpretation

So far, we considered the vortex-Heisenberg group elements as pairs $(\boldsymbol{s}, a)$ with $\boldsymbol{s} = (s_x, s_y)$ and $a \in \mathbb{R}$ for 2D inviscid, incompressible vortex dynamics, respectively $(\boldsymbol{s}, \boldsymbol{a})$ with $\boldsymbol{s} = (s_x, s_y, s_z)$ and $\boldsymbol{a} = (\boldsymbol{a_x}, \boldsymbol{a_y}, \boldsymbol{a_z})$ for 3D inviscid, incompressible vortex dynamics. We interpret the physical meaning of the group elements analogously to classical mechanics suggested by Sudarshan and Mukunda (1974). For incompressible 3D vortex dynamics, the momentum is given by:

$$\mathscr{P} = \int_V \mathrm{d}\tau\, \mathbf{v} = \frac{1}{2}\int_V \mathrm{d}\tau(\mathbf{r} \times \boldsymbol{\xi}) \tag{85}$$

and the flux of vorticity reads as:

$$\mathscr{Z} = \int_V \mathrm{d}\tau\, \boldsymbol{\xi}. \tag{86}$$

Now, we assume small volume elements of the momentum and the flux of vorticity leading to the following physical interpretations for the vortex-Heisenberg algebra and for the vortex-Heisenberg group:

$$\begin{array}{llllll} \text{algebra} & & & & & \text{group} \\ \mathscr{P} \longrightarrow & \text{density of the momentum} & \longrightarrow & \exp\left(\left(\frac{1}{2}(\mathbf{r}\times\boldsymbol{\xi})\cdot\mathbf{s}\right)^*\right) & \longrightarrow & \mathbf{s} = \mathbf{r} - \mathbf{r}' \\ \mathscr{Z} \longrightarrow & \text{density of the total flux of vorticity} & \longrightarrow & \exp((\boldsymbol{\xi}\cdot\mathbf{a})^*) & \longrightarrow & \mathbf{a} = \mathbf{a} - \mathbf{a}', \end{array} \tag{87}$$

where the asterisk in the exponent denotes the scaling by one over the action to obtain dimensionless exponents.

6. Including the angular momentum in the algebra and the group representation

Taking additionally to the above introduced algebra the components $\mathcal{L}_\alpha,\ \alpha = 1, 2, 3$, of the angular momentum $\mathcal{L}$ into account and calculating the Nambu bracket of the angular momentum component and the and linear momentum $\mathcal{P}$ with respect to the helicity h (see (88)) leads to a further algebra that we denote $v(3)$.

6.1. The algebra v(3)

The corresponding space has the elements $\{\mathcal{P}_x, \mathcal{P}_y, \mathcal{P}_z, \mathcal{Z}_x, \mathcal{Z}_y, \mathcal{Z}_z, \mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z\}$ such that the following Nambu bracket relations result in:

$$\begin{aligned} &\{\mathcal{P}_\alpha, h, \mathcal{P}_\beta\} = \varepsilon_{\alpha\beta\gamma}\mathcal{Z}_\gamma, && \{\mathcal{L}_\alpha, h, \mathcal{L}_\beta\} = \varepsilon_{\alpha\beta\gamma}\mathcal{L}_\gamma \\ &\{\mathcal{P}_\alpha, h, \mathcal{Z}_\beta\} = 0, && \{\mathcal{L}_\alpha, h, \mathcal{P}_\beta\} = \varepsilon_{\alpha\beta\gamma}\mathcal{P}_\gamma \\ &\{\mathcal{Z}_\alpha, h, \mathcal{Z}_\beta\} = 0, && \{\mathcal{L}_\alpha, h, \mathcal{Z}_\beta\} = 0 \end{aligned} \tag{88}$$

for $\alpha, \beta, \gamma \in \{\mathrm{x, y, z}\}$. We remark that the upper right Nambu bracket is the algebra $so(3)$ of rigid body rotations. Analogously to the vortex-Heisenberg algebra, the algebra $v(3)$ is antisymmetric in all arguments and it is also multilinear. Furthermore, for fixed h the Jacobi identity is satisfied, because for fixed h each summand of the Jacobi identity cancels.

6.2. The group V(3)

The inclusion of the angular momentum into the algebra leads to a further group. We suggest to call the extension of the vortex-Heisenberg Lie group Helmholtz vortex group, because it is based on Helmholtz' vorticity equation. We can formulate this group that we will denote $V(3)$ as semidirect product of the special orthogonal group $SO(3)$ and the vortex-Heisenberg group $VH(3)$:

$$V(3) = SO(3) \ltimes VH(3). \tag{89}$$

Considering the definition of a semidirect product, the corresponding homomorphism $\varphi : VH(3) \rightarrow V(3)$ is given by $x \mapsto \boldsymbol{R}x$, where $\mathbf{R}$ denotes a rotation matrix and is an element in $SO(3)$. But, when we include the angular momentum, i.e. the Euclidean rotation represented by $SO(3)$, we lose the nilpotency for the whole group. Therefore, this vortex group does not hold the typical Heisenberg group structure. We will first introduce the novel group and then show that it satisfies the group properties. In the following, we will call this group *Helmholtz vortex group*.

The Helmholtz vortex group is given by the set

$$V(3) = \{(\mathbf{s}, \mathbf{a}, \mathbf{R}) \mid \mathbf{s}, \mathbf{a} \in \mathbb{R}^3,\ \ \mathbf{R} \in SO(3)\} \tag{90}$$

together with the following group operation:

$$V(3): \quad (\mathbf{s}, \mathbf{a}, \mathbf{R}) \circ \left(\mathbf{s}', \mathbf{a}', \mathbf{R}'\right) = \left(\mathbf{s} + \mathbf{R}\mathbf{s}', \mathbf{a} + \mathbf{R}\mathbf{a}' \pm \frac{1}{2}\mathbf{s} \times \mathbf{R}\mathbf{s}', \mathbf{R}\mathbf{R}'\right) \tag{91}$$

for $\mathbf{s}, \mathbf{s}', \mathbf{a}, \mathbf{a}' \in \mathbb{R}^3$ and $\boldsymbol{R}, \boldsymbol{R}' \in SO(3)$. The sign before the cross-product term depends on the sign of the helicity that is taken as middle argument in the Nambu bracket. By extending the vortex-Heisenberg group by the angular momentum, i.e. the rigid body rotation, that is expressed by the rotation matrices $\boldsymbol{R} \in SO(3)$, the Helmholtz vortex group is a nine-dimensional group. This nine-dimensional group can be represented by the following semidirect product:

$$V(3) = SO(3) \ltimes VH(3). \tag{92}$$

The terms of the rigid body rotations in the Helmholtz vortex group operation (91) and (92) are similar embedded as in the group operation of the three-dimensional Euclidean group $E(3)$ that is given by:

$$E(3) = SO(3) \ltimes T(3). \tag{93}$$

Thus, $VH(3)$ can be seen as central extension of the abelian group of translations $T(3)$. Moreover, comparing the group operations of $V(3)$ given in (92) with $E(3)$:

$$E(3): \quad (\mathbf{s}, \mathbf{R}) \circ \left(\mathbf{s}', \mathbf{R}'\right) = \left(\mathbf{s} + \mathbf{R}\mathbf{s}', \mathbf{R}\mathbf{R}'\right) \tag{94}$$

we recognize that $V(3)$ extents $E(3)$ by the additional term $\mathbf{a} + \mathbf{R}\mathbf{a}' \pm \frac{1}{2}\mathbf{s} \times \mathbf{R}\mathbf{s}'$. In appendix C we prove that the Helmholtz vortex group satisfies all group properties, i.e. we show the existence of the inverse and identity elements and that the group operation does satisfy the associative property.

Moreover, the vortex-Heisenberg group can be embedded in Helmholtz vortex group, which can be seen by considering the neutral element of $SO(3)$, i.e. the identity $\mathbb{E}$:

$$(\mathbf{s}, \mathbf{a}) \cong (\mathbf{s}, \mathbf{a}, \mathbb{E}) \ \subset (\mathbf{s}, \mathbf{a}, \mathbf{R}). \tag{95}$$

Table 2. Notation.

$\mathscr{H}$	Kinetic energy
$\mathscr{P}$	Linear momentum
$\mathscr{L}$	Angular momentum
$\mathscr{Z}$	Total flux of vorticity
$\mathcal{Z}$	Circulation
$\mathcal{E}$	Enstrophy
h	Helicity
$\mathbf{v}$	Velocity
$\boldsymbol{\xi}$	3D vorticity vector
ζ	Vorticity
$\mathcal{F}, \mathcal{G}$	Functional that depend on $\boldsymbol{\xi}$
$\mathbf{X}, \mathbf{Y}$	Matrices
$\mathbf{P}_\alpha$	Matrix representations of $\mathcal{P}_\alpha$
$\mathbf{Z}_\alpha$	Matrix representations of $\mathcal{Z}_\alpha$
$\mathbf{R}$	$\in SO(3)$
$(\mathbf{s}, a)$	$= (s_x, s_y, a)$ group element of $VH(2)$
$(\mathbf{s}, \mathbf{a})$	$= (s_x, s_y, s_z, a_x, a_y, a_z)$ group element of $VH(3)$

Thus, the Helmholtz vortex group can be regarded as generalization of the previously discussed vortex-Heisenberg group.

7. Conclusion

We have shown how continuous Nambu mechanics allows for an algebraic approach of vortex dynamics. Névir and Blender (1993) introduced the continuous Nambu bracket for incompressible fluids. Moreover Névir (1998), calculated the brackets for all conserved quantities. These works provide the basis to derive a group and algebra representation for vortex dynamics. We started from the Nambu bracket for two- and three-dimensional incompressible vortical flows and introduced novel matrix representations of the algebra for two- and for three-dimensional vortex flows. We introduced a matrix representation for the components of the linear momentum and the circulation for 2D fluids, respectively the total flux of vorticity for inviscid 3D flows. The matrix commutator of these physical interpretable matrices result in the same relations as the Nambu bracket of the corresponding conserved quantities.

In the second step, we used these representations to derive the matrix and vector representations for the vortex-Heisenberg groups for two and three dimensions that we denote by $VH(2)$ and $VH(3)$. The name vortex-Heisenberg group is chosen, because the vortex-Heisenberg group for two-dimensional vortex dynamics is a covering of the standard-Heisenberg group for the inertial motion of mass points, even though it is based on different sets of equations. As an interesting property, $VH(2)$ and $VH(3)$ are nilpotent groups and therefore, they are solvable. Moreover, we included the angular momentum leading to a vortex group for 3D inviscid flows that is based on three conserved quantities, where each quantity has three components: the linear momentum, the total flux of vorticity, and the angular momentum. In this way, we introduced a nine dimensional group that captures spin-like vortical rotations as well as rigid body rotations about a given angle. This extension of the vortex-Heisenberg group looses its nilpotent structure and can be expressed as semi-direct product of the vortex-Heisenberg group and the classical rotational group $SO(3)$.

There are some advantages of the group theoretical approach: a group is a set together with a group operation. The elements of the set satisfy some properties such that the group is closed under the group operation. From physical and mathematical perspective, this seemingly harmless property has major consequences. At first, the Nambu formulation represents the vorticity equation, and the Nambu bracket generates the vortex-Heisenberg Lie algebra. This Lie algebra is the tangent space at the group identity and thus, the vortex-Heisenberg Lie group can be seen as novel integration of the Helmholtz vorticity equation. Calculating the vortex-Heisenberg group operation of two elements leads to a third element, where all three elements can be seen as a structural solution of the Helmholtz vorticity equation, because of the closure property. The existence of inverse group elements has a further important physical meaning, because we can recover the initial states from the subsequent states.

We think that this work will inspire a new way of analyzing 2D- and 3D vortex dynamics and might help deepening the understanding of vortex interactions.

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Data availability statement

No new data were created or analysed in this study.

Appendix A. Proof of the *VH*(2) group properties

The vortex-Heisenberg group $VH(2)$ is given by the set of elements $\{(\mathbf{s}, a)\ |a \in \mathbb{R},\ \mathbf{s} \in \mathbb{R}^2\}$ with the group operation:

$$((s_x, s_y), a) \circ ((s'_x, s'_y), a') = ((s_x + s'_x, s_y + s'_y), a + a' \pm \frac{1}{2}(s_x s'_y - s_y s'_x)), \quad (\mathrm{A.1})$$

where $e = (\mathbf{0}, 0)$ is the identity element and the inverse element is given by $(-\mathbf{s}, -a) \in VH(2)$.

Proof. Identity element.

$$(\mathbf{s}, a) \circ (\mathbf{0}, 0) = (\mathbf{s} + 0, a + 0 \pm \frac{1}{2}\mathbf{k} \cdot (\mathbf{s} \times \mathbf{0})) = (\mathbf{s}, a). \quad (\mathrm{A.2})$$

Inverse element.

$$\begin{aligned} &(\mathbf{s}, a) \circ (-\mathbf{s}, -a) = (\mathbf{s} - \mathbf{s}, a - a \pm \frac{1}{2}\mathbf{k} \cdot (\mathbf{s} \times (-\mathbf{s}))) = (\mathbf{0}, 0) \\ &(-\mathbf{s}, -a) \circ (\mathbf{s}, a) = (-\mathbf{s} + \mathbf{s}, -a + a \pm \frac{1}{2}\mathbf{k} \cdot (-\mathbf{s} \times \mathbf{s})) = (\mathbf{0}, 0). \end{aligned} \quad (\mathrm{A.3})$$

Moreover, the associate law is satisfied:

$$\begin{aligned}
&(\mathbf{s}'', a'') \circ \big((\mathbf{s}', a') \circ (\mathbf{s}, a)\big) \\
&\quad = \left(\mathbf{s}'' + \mathbf{s}' + \mathbf{s}, a'' + (a' \pm \frac{1}{2}\mathbf{k} \cdot (\mathbf{s}' \times \mathbf{s})) + a \pm \frac{1}{2}\mathbf{k} \cdot (\mathbf{s}'' \times (\mathbf{s}' + \mathbf{s}))\right) \\
&\quad = \left(\mathbf{s}'' + \mathbf{s}' + \mathbf{s}, a'' + a' + a \pm \frac{1}{2}\mathbf{k} \cdot (\mathbf{s}' \times \mathbf{s} + \mathbf{s}'' \times \mathbf{s}' + \mathbf{s}'' \times \mathbf{s})\right) \\
&\quad = \big((\mathbf{s}'', a'') \circ (\mathbf{s}', a')\big) \circ (\mathbf{s}, a)
\end{aligned} \quad . \tag{A.4}$$

Therefore, $VH(2)$ with group operation (A.1) forms a group. □

We denote the vortex-Heisenberg Lie group with capital letters $VH(2)$ that is derived from the vortex-Heisenberg Lie algebra $vh(2)$. Because the matrix group representation is a subgroup of the well known Heisenberg group, the matrix representation of $VH(2)$ introduced in section 3 satisfies all matrix group properties.

Appendix B. Proof of the *VH*(3) group properties

The set $M = \{(\mathbf{s}, \mathbf{a}) \mid \mathbf{s}, \mathbf{a} \in \mathbb{R}^3\}$ together with the following group operation

$$(\mathbf{s}, \mathbf{a}) \circ (\mathbf{s}', \mathbf{a}') = \left(\mathbf{s} + \mathbf{s}', \mathbf{a} + \mathbf{a}' \pm \frac{1}{2}\ \mathbf{s} \times \mathbf{s}'\right) \tag{B.1}$$

forms a group, the vortex-Heisenberg group $VH(3)$. We have to show the existence of the identity and the inverse element and that the associativity holds. The identity element of the $VH(3)$ is given by $e = (\mathbf{0}, \mathbf{0}) \in \mathbb{R}^6$ which corresponds to the unit matrix considering the matrix Lie group representation. The inverse element $g^{-1} \in VH(3)$ is given by:

$$g^{-1} = (-\mathbf{s}, -\mathbf{a}). \tag{B.2}$$

Furthermore, it is a left and a right inverse element. Moreover, the associative property holds.

Proof. *Identity element.*

Let $g = (\mathbf{s}, \mathbf{a})$ an arbitrary element in $VH(3)$. Applying (B.1) it is:

$$\begin{aligned}
e \circ g = (\mathbf{0}, \mathbf{0}) \circ (\mathbf{s}, \mathbf{a}) &= (\mathbf{0} + \mathbf{s}, \mathbf{0} + \mathbf{a} \pm \mathbf{s} \times \mathbf{0}) = (\mathbf{s}, \mathbf{a}) = g \\
&= (\mathbf{s}, \mathbf{a}) \circ (\mathbf{0}, \mathbf{0}) = g \circ e,
\end{aligned} \tag{B.3}$$

respectively for the matrix Lie group: multiplying the matrix representation $\boldsymbol{A}$ of the group

$$\mathbf{A} = \mathbf{A}(\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \lambda_x i \\ 0 & 1 & 0 & 0 & 0 & 0 & \lambda_y j \\ 0 & 0 & 1 & 0 & 0 & 0 & \lambda_z k \\ 0 & -\lambda_z k & \lambda_y j & 1 & 0 & 0 & \mu_x i \\ \lambda_z k & 0 & -\lambda_x i & 0 & 1 & 0 & \mu_y j \\ -\lambda_y j & \lambda_x i & 0 & 0 & 0 & 1 & \mu_z k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.4}$$

with the unity matrix

$$\mathbf{A}(e) = \mathbf{A}(\mathbf{0}, \mathbf{0}) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix} \tag{B.5}$$

proofs that the unit matrix is the identity element. We note that the group operation is here given by the matrix multiplication, see section 3.

Inverse element. Inserting the stated inverse element (B.2) into the group operation, we obtain:

$$g \circ g^{-1} = (\mathbf{s} - \mathbf{s}, \mathbf{a} - \mathbf{a} - \mathbf{s} \times \mathbf{s}) = e = g^{-1} \circ g. \tag{B.6}$$

Considering the matrix Lie group with the matrix multiplication we obtain:

$$\begin{aligned} &\mathbf{A}(\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z)\ \mathbf{A}(-\lambda_x, -\lambda_y, -\lambda_z, -\mu_x, -\mu_y, -\mu_z) \\ &= \mathbf{A}(-\lambda_x, -\lambda_y, -\lambda_z, -\mu_x, -\mu_y, -\mu_z)\ \mathbf{A}(\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z). \\ &= \mathbf{A}(e) \end{aligned} \tag{B.7}$$

Associate law. Moreover, the associate law is satisfied:

$$\begin{aligned} &(\mathbf{s}'', \mathbf{a}'')\circ \big((\mathbf{s}', \mathbf{a}') \circ (\mathbf{s}, \mathbf{a})\big) \\ &\quad = \left(\mathbf{s}'' + \mathbf{s}' + \mathbf{s}, \mathbf{a}'' + \left(\mathbf{a}' \pm \frac{1}{2}\mathbf{s}' \times \mathbf{s}\right) + \mathbf{a} \pm \frac{1}{2}\mathbf{s}'' \times (\mathbf{s}' + \mathbf{s})\right) \\ &\quad = \left(\mathbf{s}'' + \mathbf{s}' + \mathbf{s}, \mathbf{a}'' + \mathbf{a}' + \mathbf{a} \pm \frac{1}{2} \cdot (\mathbf{s}' \times \mathbf{s} + \mathbf{s}'' \times \mathbf{s}' + \mathbf{s}'' \times \mathbf{s})\right) \\ &\quad = \big((\mathbf{s}'', \mathbf{a}'') \circ (\mathbf{s}', \mathbf{a}')\big) \circ (\mathbf{s}, \mathbf{a}) \end{aligned}. \tag{B.8}$$

Moreover, we now use the fact that we can assign each state to a matrix. The group operation is associative, because the matrix multiplication of a squared matrix is always associative. Therefore, we have shown that $(VH(3), \circ)$ is a group.

Group homomorphism. In the last step, we show that the map

$$\mathbf{A} : VH(3) \longrightarrow GL \tag{B.9}$$

is a group homomorphism, i.e.

$$\mathbf{A}(g' \circ g) = \mathbf{A}(g') \cdot \mathbf{A}(g) \quad \forall\ g', g \in VH(3). \tag{B.10}$$

Let $g', g \in VH(3)$ with $g' = (\lambda_x{}', \lambda_y{}', \lambda_z{}', \mu_x{}', \mu_y{}', \mu_z{}')$ and $g := (\lambda_x, \lambda_y, \lambda_z, \mu_x, \mu_y, \mu_z)$. We obtain:

$$\mathbf{A}(g' \circ g) =$$
$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & (\lambda'_x + \lambda_x)i \\ 0 & 1 & 0 & 0 & 0 & 0 & (\lambda'_y + \lambda_y)j \\ 0 & 0 & 1 & 0 & 0 & 0 & (\lambda'_z + \lambda_z)k \\ 0 & -(\lambda'_z + \lambda_z)k & (\lambda'_y + \lambda_y)j & 1 & 0 & 0 & (\mu'_x + \mu_x + \frac{1}{2}(\lambda'_y\lambda_z - \lambda_y\lambda'_z))i \\ (\lambda'_z + \lambda_z)k & 0 & -(\lambda'_x + \lambda_x)i & 0 & 1 & 0 & (\mu'_y + \mu_y + \frac{1}{2}(\lambda'_z\lambda_x - \lambda_z\lambda'_x))j \\ -(\lambda'_y + \lambda_y)j & (\lambda'_x + \lambda_x)i & 0 & 0 & 0 & 1 & (\mu'_z + \mu_z + \frac{1}{2}(\lambda'_x\lambda_y - \lambda_x\lambda'_y))k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \tag{B.11}$$

$$= \mathbf{A}(g') \cdot \mathbf{A}(g). \tag{B.12}$$

Furthermore it is

$$\mathbf{A}(e) = \begin{pmatrix} 1 & & \\ & \ddots & \\ & & 1 \end{pmatrix}. \tag{B.13}$$

Because of (B.12) and (B.13) the map A is a group homomorphism and therefore it is a representation of the vortex-Heisenberg-group $VH(3)$. □

Proposition 1. *Let M be a $(n \times n)$ matrix with $\lim_{k\to\infty}(\mathbb{I}_n - M)^k = 0$. Then, M is regular and the inverse matrix of M is given by:*

$$M^{-1} = \sum_{k=0}^{\infty} (\mathbb{I}_n - M)^k. \tag{B.14}$$

Claim 1. Let A the above matrix representation of the $VH(3)$. A satisfies:

$$\mathbf{A}(g^{-1}) = \mathbf{A}^{-1}(g). \tag{B.15}$$

Proof. (of the claim)

Applying proposition 1, since

$$(\mathbb{I}_7 - \mathbf{A})^2 = 0. \tag{B.16}$$

The matrix $\mathbf{A}$ is regular and its determinant is one. Let now $g \in VH(3)$. It is

$$\mathbf{A}^{-1}(g) = \sum_{k=0}^{\infty} (\mathbb{I}_7 - \mathbf{A})^k = (\mathbb{I}_7 - \mathbf{A})^0 + (\mathbb{I}_7 - \mathbf{A})^1 + 0 + 0 + \cdots$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \lambda_x i \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_y j \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda_z k \\ 0 & -\lambda_z k & \lambda_y j & 0 & 0 & 0 & \mu_x i \\ \lambda_z k & 0 & -\lambda_x i & 0 & 0 & 0 & \mu_y j \\ -\lambda_y j & \lambda_x i & 0 & 0 & 0 & 0 & \mu_z k \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{B.17}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & -\lambda_x i \\ 0 & 1 & 0 & 0 & 0 & 0 & -\lambda_y j \\ 0 & 0 & 1 & 0 & 0 & 0 & -\lambda_z k \\ 0 & +\lambda_z k & -\lambda_y j & 1 & 0 & 0 & -\mu_x i \\ -\lambda_z k & 0 & +\lambda_x i & 0 & 1 & 0 & -\mu_y j \\ +\lambda_y j & +\lambda_x i & 0 & 0 & 0 & 1 & -\mu_z k \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{A}(g^{-1}). \tag{B.18}$$

□

Thus, we have derived a vector and a matrix representation of the vortex-Heisenberg group $VH(3)$ for three-dimensional incompressible, inviscid vortex dynamics.

Appendix C. Proof of the Helmholtz vortex group properties

The Helmholtz vortex group is given by the set

$$V(3) = \{(\mathbf{s}, \mathbf{a}, \mathbf{R}) \mid \mathbf{s}, \mathbf{a} \in \mathbb{R}^3,\ \mathbf{R} \in SO(3)\} \tag{C.1}$$

together with the following group operation:

$$V(3): \quad (\mathbf{s}, \mathbf{a}, \mathbf{R}) \circ (\mathbf{s}', \mathbf{a}', \mathbf{R}') = \left(\mathbf{s} + \mathbf{R}\mathbf{s}', \mathbf{a} + \mathbf{R}\mathbf{a}' \pm \frac{1}{2}\mathbf{s} \times \mathbf{R}\mathbf{s}', \mathbf{R}\mathbf{R}'\right) \tag{C.2}$$

for $\mathbf{s}, \mathbf{s}', \mathbf{a}, \mathbf{a}' \in \mathbb{R}^3$ and $\mathbf{R}, \mathbf{R}' \in SO(3)$. The sign before the cross-product term depends on the sign of the helicity that is taken as middle argument in the Nambu bracket.

Claim 2. The identity and inverse elements are given by

$$\begin{aligned} \text{Identity element:} \quad & (\mathbf{0}, \mathbf{0}, \mathbb{E}) \\ \text{Right and left inverse elements:} \quad & \left(-R^{-1}\mathbf{s}, -\mathbf{R}^{-1}\mathbf{a}, \mathbf{R}^{-1}\right), \end{aligned} \tag{C.3}$$

where $\mathbb{E}$ denotes the unit matrix. Moreover, the associate property holds.

Proof. We show that there is an identity element $(0, 0, \mathbb{E})$:

$$\begin{aligned}&(\mathbf{s}, \mathbf{a}, \mathbf{R}) \circ (0, 0, \mathbb{E}) = (\mathbf{s}, \mathbf{a}, \mathbf{R})\\&(0, 0, \mathbb{E}) \circ (\mathbf{s}, \mathbf{a}, \mathbf{R}) = (\mathbb{E}\ \mathbf{s}, \mathbb{E}\ \mathbf{a}, \mathbf{R}) = (\mathbf{s}, \mathbf{a}, \mathbf{R})\end{aligned} \tag{C.4}$$

and the existence of the inverse elements $\left(-R^{-1}\mathbf{s}, -R^{-1}\mathbf{a}, R^{-1}\right)$:

$$\begin{aligned}&(\mathbf{s}, \mathbf{a}, \mathbf{R}) \circ \left(-R^{-1}\mathbf{s}, -R^{-1}\mathbf{a}, R^{-1}\right)\\&\quad= (\mathbf{s} - \mathbf{R}\mathbf{R}^{-1}\mathbf{s}, \pm\frac{1}{2}\mathbf{s} \times (-\mathbf{R}\mathbf{R}^{-1}\mathbf{s}) + \mathbf{a} - \mathbf{R}\mathbf{R}^{-1}\mathbf{a}, \mathbf{R}\mathbf{R}^{-1})\\&\quad= (0, 0, \mathbb{E})\end{aligned} \tag{C.5}$$

and

$$\begin{aligned}&\left(-\mathbf{R}^{-1}\mathbf{s}, -\mathbf{R}\mathbf{R}^{-1}\mathbf{a}, \mathbf{R}^{-1}\right) \circ (\mathbf{s}, \mathbf{a}, \mathbf{R})\\&= (-\mathbf{R}^{-1}\mathbf{s} + \mathbf{R}^{-1}\mathbf{s}, \pm\frac{1}{2}(-\mathbf{R}^{-1}\mathbf{s}) \times \mathbf{R}^{-1}\mathbf{s} - \mathbf{R}^{-1}\mathbf{a} + \mathbf{R}^{-1}F, R^{-1}R).\\&= (0, 0, \mathbb{E})\end{aligned} \tag{C.6}$$

Further, we need to prove the associative property of the group operation. Let $(\boldsymbol{s}, \boldsymbol{a}, \boldsymbol{R})$, $(\boldsymbol{s}', \boldsymbol{a}', \boldsymbol{R}')$ and $(\boldsymbol{s}'', \boldsymbol{a}'', \boldsymbol{R}'')$ elements in Helmholtz vortex group.

$$\begin{aligned}&(\mathbf{s}'', \mathbf{a}'', \mathbf{R}'') \circ [\left(\mathbf{s}', \mathbf{a}', \mathbf{R}'\right) \circ (\mathbf{s}, \mathbf{a}, \mathbf{R})]\\&\quad= (\mathbf{s}'', \mathbf{a}'', \mathbf{R}'') \circ \left(\mathbf{s} + \mathbf{R}\mathbf{s}', \pm\frac{1}{2}\mathbf{s} \times \mathbf{R}'\mathbf{s}' + \mathbf{a} + \mathbf{R}\mathbf{a}', \mathbf{R}\mathbf{R}'\right)\\&\quad= (\mathbf{R}\mathbf{R}'\mathbf{s}'' + \mathbf{R}\mathbf{s}' + \mathbf{s}, \pm\frac{1}{2}(\mathbf{s} + \mathbf{R}\mathbf{s}') \times \mathbf{s}'') \pm \frac{1}{2}\mathbf{s} \times \mathbf{R}\mathbf{s}' + \mathbf{a} + \mathbf{R}\mathbf{a}'\\&\qquad+ \mathbf{R}\mathbf{R}'\mathbf{a}'', \mathbf{R}''\mathbf{R}'\mathbf{R})\\&\quad= \Bigg(\mathbf{R}\mathbf{R}'\mathbf{s}'' + \mathbf{R}\mathbf{s}' + \mathbf{s}, \pm\frac{1}{2}(\mathbf{s} \times \mathbf{R}\mathbf{R}'\mathbf{s}'' + \mathbf{R}\mathbf{s}' \times \mathbf{R}'\mathbf{R}\mathbf{s}'' + \mathbf{s} \times \mathbf{R}\mathbf{s}')\\&\qquad+ \mathbf{a} + \mathbf{R}\mathbf{a}' + \mathbf{R}\mathbf{R}'\mathbf{a}'', \quad \mathbf{R}''\mathbf{R}'\mathbf{R}\Bigg)\end{aligned}. \tag{C.7}$$

In the last step we applied $\boldsymbol{Rs} \times \boldsymbol{Rb} = \boldsymbol{R}(\boldsymbol{s} \times \boldsymbol{b})$ which holds for $\mathbf{s}, \mathbf{b} \in \mathbb{R}^3$, $\mathbf{R} \in SO(3)$. To satisfy the associate property last equation has to be equal to the following expression:

$$\begin{aligned}&[(\mathbf{s}'', \mathbf{a}'', \mathbf{R}'') \circ \left(\mathbf{s}', \mathbf{a}', \mathbf{R}'\right)] \circ (\mathbf{s}, \mathbf{a}, \mathbf{R})\\&\quad= \Bigg(\mathbf{R}\mathbf{R}'\mathbf{s}'' + \mathbf{R}\mathbf{s}' + \mathbf{s}, \pm\frac{1}{2}(\mathbf{s} \times \mathbf{R}\mathbf{R}'\mathbf{s}'' + \mathbf{R}\mathbf{s}' \times \mathbf{R}'\mathbf{R}\mathbf{s}'' + \mathbf{s} \times \mathbf{R}\mathbf{s}')\\&\qquad+ \mathbf{a} + \mathbf{R}\mathbf{a}' + \mathbf{R}\mathbf{R}'\mathbf{a}'', \quad \mathbf{R}''\mathbf{R}'\mathbf{R}\Bigg)\end{aligned}. \tag{C.8}$$

The associative law is satisfied, because (C.8) = (C.7). Therefore, $V(3)$ with the operation defined in (C.2) satisfies all group properties. □

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