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Abstract

In this work we study the global solvability of the primitive equations for the atmosphere coupled to moisture dynamics with phase changes for warm clouds, where water is present in the form of water vapor and in the liquid state as cloud water and rain water. This moisture model contains closures for the phase changes condensation and evaporation, as well as the processes of auto conversion of cloud water into rainwater and the collection of cloud water by the falling rain droplets. It has been used by Klein and Majda in [17] and corresponds to a basic form of the bulk microphysics closure in the spirit of Kessler [16] and Grabowski and Smolarkiewicz [12]. The moisture balances are strongly

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coupled to the thermodynamic equation via the latent heat associated to the phase changes. In [14] we assumed the velocity field to be given and proved rigorously the global existence and uniqueness of uniformly bounded solutions of the moisture balances coupled to the thermodynamic equation. In this paper we present the solvability of a full moist atmospheric flow model, where the moisture model is coupled to the primitive equations of atmospherical dynamics governing the velocity field. For the derivation of *a priori* estimates for the velocity field we thereby use the ideas of Cao and Titi [6], who succeeded in proving the global solvability of the primitive equations.

1. Introduction

Moisture and precipitation still cause major uncertainties in numerical weather prediction models and it is our aim here to develop further the rigorous analysis of atmospheric flow models. In a preceding paper [14] we assumed the velocity field to be given and studied the moisture model for water vapor, cloud water and rain water coupled to the thermodynamic equation through the latent heat in the setting of Klein and Majda [17] corresponding to a basic form of a bulk microphysics model in the spirit of Kessler [16] and Grabowski and Smolarkiewicz [12]. In this work we couple the moisture dynamics to the primitive equations of the atmospheric dynamics by taking over the ideas of Cao and Titi [6] for their recent breakthrough on the global solvability of the latter system. Moreover, cases of partial viscosities and diffusions, arising from the asymptotical analysis in [17], will be analyzed in a future work capitalizing on the results by Cao *et al* [2–4].

A study of a moisture model coupled to the primitive equations has already been carried out by Coti Zelati *et al* in [9]. The moisture model there consists of one moisture quantity coupled to temperature and contains only the process of condensation during upward motion, see e.g. [15]. Since the source term there is modeled via a heavy side function as a switching term between saturated and under saturated regions, the analysis, however, requires elaborate techniques. Coti Zelati *et al* in [1, 8, 10] therefore used an approach based on differential inclusions and variational techniques, which have then been coupled to the primitive equations in [9].

The moisture model we are analyzing here is physically more refined and consists of three moisture quantities for water vapor, cloud water and rain water. It contains besides the phase changes condensation and evaporation also the auto conversion of cloud water to rain water after a certain threshold is reached, as well as the collection of cloud water by the falling rain droplets.

In the remainder of the introduction we first state the moisture model in pressure coordinates, which have the advantage that under the assumption of hydrostatic balance the continuity equation takes the form of the incompressibility condition.

1.1. Governing equations

Solvability of the full geophysical governing equations (without moisture) is a long standing problem. Assuming hydrostatic balance

$$\frac{\partial p}{\partial z} = -g\rho, \tag{1}$$

where g denotes the gravitational acceleration, the equations reduce to the well-known primitive equations and only recently the global well-posedness of strong solutions could be proven by Cao and Titi [6] for the incompressible ocean dynamics. The density of air ρ in the atmosphere in comparison to the ocean however varies strongly with height, and the incompressibility assumption is only justified when describing shallow phenomena. Thus for the atmosphere in general the full compressible governing equations need to be considered. However, under the assumption of hydrostatic balance (1), which in particular guarantees the pressure to decrease monotonically with height, the pressure p can be used as the vertical coordinate. Switching to the pressure coordinates (x, y, p) has the main advantage that the continuity equation takes the form of the incompressibility condition

$$\partial_x u + \partial_y v + \partial_p \omega = 0 \quad \text{where} \quad \omega = \frac{\mathrm{d}p}{\mathrm{d}t}, \tag{2}$$

see e.g., Lions *et al* [19] and Petcu *et al* [20]. Thus the ideas of Cao and Titi [6] can be taken over for the atmospheric primitive equations in pressure coordinates, as we shall also see below and we therefore work in the following with the governing equations in the pressure coordinates and use hereafter the notation

$$\mathbf{u} = (u, v), \quad \nabla_h = (\partial_x, \partial_y), \quad \Delta_h = \partial_x^2 + \partial_y^2.$$

We note that the vertical velocity ω in pressure coordinates takes the opposite sign as the vertical velocity in the Cartesian coordinates, i.e., $\omega < 0$ for upward motion and $\omega > 0$ for downward motion. Moreover the derivatives as well as the velocity components have different units in vertical and horizontal directions. Nevertheless the total derivative in pressure coordinates reads

$$\frac{\mathrm{d}}{\mathrm{d}t} = \partial_t + \mathbf{u} \cdot \nabla_h + \omega \partial_p. \tag{3}$$

For the eddy viscosity closure of turbulence and molecular transport we use

$$\mathcal{D}^* = \mu_* \Delta_h + \nu_* \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p \right), \tag{4}$$

where $\bar{T} = \bar{T}(p)$ corresponds to some background distribution of the temperature, being uniformly bounded from above and below, and R is the individual gas constant. The operator $\mathcal{D}^*$ thereby provides a close approximation to the full Laplacian in Cartesian coordinates, see also [19, 20]. The governing equations in pressure coordinates (x, y, p) with corresponding velocities $(\mathbf{u}, \omega)$ become

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_h)\mathbf{u} + \omega \partial_p \mathbf{u} + \nabla_h \Phi + f(\mathbf{k} \times \mathbf{v})_h = \mathcal{D}^{\mathbf{u}} \mathbf{u}, \tag{5}$$

$$\partial_p \Phi + \frac{RT}{p} = 0, \tag{6}$$

$$\nabla_h \cdot \mathbf{u} + \partial_p \omega = 0, \tag{7}$$

where Φ denotes the geo potential $\partial_z \Phi = g$, and the second equation combines the ideal gas law (see (8) below) and the hydrostatic balance equation (1). Here $\mathbf{v} := (\mathbf{u}, \omega)$, $\mathbf{k} = (0, 0, 1)$, and $(\mathbf{k} \times \mathbf{v})_h$ are the first two components of $\mathbf{k} \times \mathbf{v}$. Now the density ρ does not appear in the system anymore, which is connected to the other thermodynamic quantities via the ideal gas law

$$p = R\rho T. \tag{8}$$

A typical measure for quantification of moisture are mixing ratios, which compare the density of the moisture component to the density of dry air (denoted as ρ_d). We assume to be in the warm cloud regime, where no ice and snow phases occur and water is therefore present in the form of water vapor (with density ρ_v) and in the liquid state as cloud water and rain water (with corresponding densities ρ_c and ρ_r), such that we have the mixing ratios

$$q_v = \frac{\rho_v}{\rho_d}, \quad q_c = \frac{\rho_c}{\rho_d}, \quad q_r = \frac{\rho_r}{\rho_d}. \tag{9}$$

For these mixing ratios we then have the moisture balances

$$\frac{\mathrm{d}q_v}{\mathrm{d}t} = S_{ev} - S_{cd} + \mathcal{D}^{q_v}q_v, \tag{10}$$

$$\frac{\mathrm{d}q_c}{\mathrm{d}t} = S_{cd} - S_{ac} - S_{cr} + \mathcal{D}^{q_c}q_c, \tag{11}$$

$$\frac{\mathrm{d}q_r}{\mathrm{d}t} + V\partial_p\left(\frac{p}{R_d\bar{T}}q_r\right) = S_{ac} + S_{cr} - S_{ev} + \mathcal{D}^{q_r}q_r, \tag{12}$$

where the total derivative is given according to (3) and the diffusion terms are as in (4). The source terms $S_{ev}, S_{cd}, S_{ac}, S_{cr}$ are, respectively, the rates of evaporation of rain water, the condensation of water vapor to cloud water and the inverse evaporation process, the auto-conversion of cloud water into rainwater by accumulation of microscopic droplets, and the collection of cloud water by falling rain. Moreover V denotes the terminal velocity of falling rain and is assumed to be constant.

The thermodynamic equation accounts for the diabatic source and sink terms, such as latent heating, radiation effects, but we will in the following only focus on the effect of latent heat in association with phase changes (see e.g., [8, 9, 14, 17]). The temperature equation in pressure coordinates then reads, see e.g., [11, 13],

$$\frac{\mathrm{d}T}{\mathrm{d}t} - \frac{R}{c_p}\frac{T}{p}\omega = \frac{L}{c_p}(S_{cd} - S_{ev}) + \mathcal{D}^{\mathrm{T}}T, \tag{13}$$

where the heat capacity c_p and the latent heat L are assumed to be constant.

To describe the state of the atmosphere a common thermodynamic quantity used instead of the temperature is the potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa}, \quad \text{where} \quad \kappa = \frac{R}{c_p}. \tag{14}$$

The potential temperature has the main advantage that the left-hand side of (13) simply reduces to $\frac{T}{\theta}\frac{\mathrm{d}}{\mathrm{d}t}\theta$. This property was essential in the preceding works [10, 14] to derive *a priori* non negativity and bounded ness of the temperature and moisture components.

Remark 1. In the present model the difference of the gas constants for dry air and water vapor as well as the dependence of the internal energy on the moisture components via different heat capacities is neglected. These additional terms that would arise in a more precise thermodynamical setting are small in principle and therefore often not taken into account. It has been, however, revealed in [13] that, e.g., in the presence of deep convective clouds the refined thermodynamical setting can be essential. The according moisture model has a much stronger coupling of the thermodynamic equation to the moisture components and will be investigated in a forthcoming paper.

1.2. Explicit expressions for the source terms

The threshold for phase changes is saturation, which is defined via the saturation mixing ratio q_{vs}. Saturation thereby is reached when $q_v = q_{vs}$, whereas the air is under saturated if $q_v < q_{vs}$ and oversaturated if $q_v > q_{vs}$, respectively. For the given function q_{vs} we pose the natural assumption to depend continuously on p and T and to vanish below and above some critical temperatures (given in Kelvin), i.e.

$$q_{vs}(p,T) = 0 \quad \text{for} \quad T \leqslant T_A \quad \text{and} \quad T \geqslant T_B, \tag{15}$$

for some $0 \leqslant T_A \leqslant T_B$, which is helpful for proving non negativity of the solution. Moreover we assume q_{vs} to be nonnegative, uniformly bounded and to be Lipschitz continuous with respect to T, i.e., we assume

$$|q_{vs}(p,T_1) - q_{vs}(p,T_2)| \leqslant C|T_1 - T_2|, \tag{16}$$

for a positive constant C. This constant actually depends on the pressure as it grows approximately inversely proportional to p. We are however only interested in the lower part of the atmosphere, where weather related phenomena are taking place. There the pressure is uniformly bounded from below (e.g. by 100 hPa), such that the constant C in (16) can be assumed to be positive and uniformly bounded. For more details we refer also to [14].

For the source terms of the mixing ratios we take over the setting of Klein and Majda [17] corresponding to a basic form of the bulk microphysics closure in the spirit of Kessler [18] and Grabowski and Smolarkiewicz [12], which has also been used in the preceding work [14]:

$$S_{ev} = C_{ev}T(q_r^+)^\beta(q_{vs} - q_v)^+, \quad \beta \in (0,1], \tag{17}$$

$$S_{cr} = C_{cr}q_c q_r, \tag{18}$$

$$S_{ac} = C_{ac}(q_c - q_{ac}^*)^+, \tag{19}$$

$$S_{cd} = C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+, \tag{20}$$

where $C_{ev}, C_{cr}, C_{ac}, C_{cd}, C_{cn}$ are dimensionless rate constants. Moreover, $(g)^+ = \max\{0, g\}$ and q_{ac}^* denotes the threshold for cloud water mixing ratio beyond which auto conversion of cloud water into precipitation becomes active.

Exponents $\beta \in (0,1)$ cause difficulties in the analysis, in particular for the uniqueness of the solutions. This problem however was overcome in [14] by introducing new unknowns, which allow for certain cancellation properties of the source terms and reveal advantageous monotonicity properties. This procedure in particular relies on the fact, that the evaporation constitutes a sink in the equation for temperature, i.e. for positive temperatures T the term S_{ev} is nonnegative and arises with a negative sign in the thermodynamic equation.

The rest of this paper is arranged as follows. In section 2, we formulate the full problem with boundary conditions and state the main results on the global existence and uniqueness of solutions. In section 3, we prove the existence and uniqueness, and the uniform *a priori* estimates of solutions to an approximate system, which is nothing but the original one by replacing S_{ev} (which may be only Hölder continuous in q_r) by $S_{ev,\varepsilon}$ (see (35) below) which is Lipschitz in q_r. In section 4, based on the results in section 3, we give the proof of the global existence result, and the uniqueness is also established by using the idea in our previous work [14].

Throughout this paper, unless explicitly specified, we use C to denote a generic positive constant depending only on the given functions in the boundary conditions, the initial data, and

the physical parameters appearing in the original system (but not on the parameter ε arising in the approximate system introduced in the next section).

2. Formulation of the problem and main results

Recall the momentum equation

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_h)\mathbf{u} + \omega \partial_p \mathbf{u} + f(\mathbf{k} \times \mathbf{v})_h + \nabla_h \Phi = \mathcal{D}^{\mathbf{u}} \mathbf{u}, \tag{21}$$

where $\mathbf{v} = (\mathbf{u}, \omega)$, $\mathbf{k} = (0, 0, 1)$, and $(\mathbf{k} \times \mathbf{v})_h$ are the first two components of $\mathbf{k} \times \mathbf{v}$. According to the incompressibility condition (6) and the boundary condition (24), we have a diagnostic equation for ω

$$\omega(t, x, y, p) = \int_p^{p_0} \nabla_h \cdot \mathbf{u}(t, x, y, s) \, \mathrm{d}s. \tag{22}$$

By the hydrostatic balance (6), we have

$$\Phi(t, x, y, p) = \Phi_s(x, y, t) + \int_p^{p_0} \frac{R}{\sigma} T(t, x, y, \sigma) \mathrm{d}\sigma. \tag{23}$$

For analyzing cloudy air phenomena either numerically or analytically a bounded domain of cylindrical form is a natural choice, see also [8, 14], and we thus consider a domain $\mathcal{M}$ defined as

$$\mathcal{M} = \{(x, y, p) \,|\, (x, y) \in \mathcal{M}', \quad p \in (p_1, p_0)\},$$

where $\mathcal{M}'$ is a smooth bounded domain in $\mathbb{R}^2$, and $0 < p_1 < p_0$. The boundary is given by

$$\Gamma_0 = \mathcal{M}' \times \{p_0\}, \quad \Gamma_1 = \mathcal{M}' \times \{p_1\}, \quad \Gamma_\ell = \partial\mathcal{M}' \times (p_1, p_0).$$

The boundary conditions are:

$$\Gamma_0 : \partial_p \mathbf{u} = -\alpha_{\mathbf{u}} \mathbf{u}, \quad \omega = 0, \quad \partial_p T = \alpha_{0T}(T_{b0} - T),$$

$$\partial_p q_j = \alpha_{0j}(q_{b0j} - q_j), \quad \text{for } j \in \{v, c, r\}, \tag{24}$$

$$\Gamma_1 : \partial_p \mathbf{u} = 0, \quad \omega = \partial_p T = \partial_p q_j = 0, \quad j \in \{v, c, r\}, \tag{25}$$

$$\Gamma_\ell : \mathbf{u} \cdot \mathbf{n} = 0, \quad \partial_n \mathbf{u} \times \mathbf{n} = 0, \quad \partial_{\mathbf{n}} T = \alpha_{\ell T}(T_{b\ell} - T),$$

$$\partial_n q_j = \alpha_{\ell j}(q_{b\ell j} - q_j), \quad \text{for } j \in \{v, c, r\}, \tag{26}$$

where $\alpha_{0j}, \alpha_{\ell j}, \alpha_{0T}, \alpha_{\ell T}$ are given nonnegative constants, and $T_{b0}, T_{b\ell}, q_{b0j}, q_{b\ell j}$, which can depend on time, are given nonnegative and sufficiently smooth functions. Here $\mathbf{n}$ denotes the outward normal direction on $\partial\mathcal{M}'$. Note that the boundary conditions (24)–(26) reduce to those in [9] if $\alpha_{\ell T}, \alpha_{\ell v}, \alpha_{\ell c}$, and $\alpha_{\ell r}$ are set to zero. The initial condition is

$$(\mathbf{u}, T, q_v, q_c, q_r)|_{t=0} = (\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0}). \tag{27}$$

Throughout this paper, we use the abbreviation

$$\|f\| = \|f\|_{L^2(\mathcal{M})}, \quad \|f\|_{L^p} = \|f\|_{L^p(\mathcal{M})}.$$

According to the weight in the vertical diffusion terms, we introduce the weighted norm

$$\|f\|_w = \left\| \left(\frac{gp}{R_d \bar{T}} \right) f \right\|,$$

which, since the weight $\frac{gp}{R_d \bar{T}}$ is uniformly bounded from above and below by positive constants, is equivalent to the L^2-norm. Moreover, we shall often use for convenience the notation

$$\|(f_1, \ldots, f_n)\|^2 = \sum_{j=1}^n \|f_j\|^2$$

We state our main result on the global existence and uniqueness of solutions to the fully coupled system in the following:

Theorem 1. *Assume that* $\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0} \in H^1(\mathcal{M})$ *and* $T_0, q_{v0}, q_{c0}, q_{r0} \in L^\infty(\mathcal{M})$, *with* $T_0, q_{v0}, q_{c0}, q_{r0} \geqslant 0$ *in* $\mathcal{M}$ *and* $\int_{p_0}^{p_1} \nabla_h \cdot \mathbf{u}_0 \mathrm{d}p = 0$ *on* $\mathcal{M}'$. *Then, system* (10)–(13), (21)–(23), *subject to* (24)–(27), *has a unique global in time solution* $(\mathbf{u}, T, q_v, q_c, q_r)$, *satisfying*

$$T, q_v, q_c, q_r \geqslant 0 \quad and \quad T, q_v, q_c, q_r \in L^\infty(0, \mathcal{T}; L^\infty(\mathcal{M})),$$

$$\mathbf{u}, T, q_v, q_c, q_r \in C([0, \mathcal{T}]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M})),$$

$$\partial_t \mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r \in L^2(0, \mathcal{T}; L^2(\mathcal{M})),$$

for any $\mathcal{T} \in (0, \infty)$.

   Some comments concerning the proof of theorem 1 are given in order. For the global existence of solutions, the key is to get appropriate *a priori* estimates for the solutions, which are mainly obtained by adopting the ideas from [14] for the moisture model with given velocity field and the work of Cao and Titi [6], who proved global well-posedness for the primitive equations for the ocean. In particular, we use as in [6] the ideas of decomposing the velocity into the barotropic (vertically averaged) and the baroclinic (the according deviation) components and using the Ladyzhenskaya type inequality (see lemma 2 in the appendix) to derive the $L^\infty(0, \mathcal{T}; L^6(\mathcal{M}))$ of the horizontal velocity, see the proof of proposition 6 in the below. Besides, similar to [6], due to the anisotropic property of the system (21)–(23), we also use anisotropic treatments to the horizontal derivatives and vertical derivatives, that is the *a priori* estimates for the vertical derivatives are carried out separately before working on the horizontal ones. Furthermore, as already mentioned before, we use the idea of introducing new unknowns and making use of cancellations as in [14] to overcome the difficulty cased by the exponent $\beta \in (0, 1)$ in the source term S_{ev} to prove the uniqueness. Yet there are some technical differences of arguments in the current paper comparing with those in [6, 14], due to the presence of the anti dissipative term $\frac{R}{c_p} \frac{T}{p} \omega$ and the latent heating term $\frac{L}{c_p}(S_{cd} - S_{ev,\varepsilon})$ in the thermodynamic equation (13), as well as the coupling of the momentum equations to the moisture system through the transport terms. The anti dissipative term makes the L^1 type estimate for T, i.e., proposition 4 in the below, become a necessary step to get the further estimates, while the latent heat provides a strong coupling to the moisture dynamics, which are in turn transported with the air velocity. These additional challenges due to the strong coupling of all solution components are overcome by careful derivations of *a priori* estimates, which need to be elaborated in the right order, as explained more detailed below. Moreover, since the velocity field is no longer assumed to be a given function as in [14], the required $L^\infty(0, \mathcal{T}; H^1(\mathcal{M}))$ of the moisture components cannot be derived by applying the parabolic estimates to the moisture system (10)–(12).

3. An approximated system: existence and *a priori* estimates

This section is devoted to proving the global existence, uniqueness, and uniform *a priori* estimates of the following ε-approximate system to (10)–(13), and (21)–(23),

$$\partial_t \mathbf{u} + (\mathbf{u} \cdot \nabla_h)\mathbf{u} + \omega \partial_p \mathbf{u} + \nabla_h \Phi + f(\mathbf{k} \times \mathbf{v})_h = \mathcal{D}^{\mathbf{u}}\mathbf{u}, \tag{28}$$

$$\partial_p \Phi + \frac{RT}{p} = 0 \tag{29}$$

$$\nabla_h \cdot \mathbf{u} + \partial_p \omega = 0, \tag{30}$$

$$\partial_t T + (\mathbf{u} \cdot \nabla_h)T + \omega \partial_p T - \frac{R}{c_p}\frac{T}{p}\omega = \frac{L}{c_p}(S_{cd}^+ - S_{ev,\varepsilon}^+) + \mathcal{D}^T T, \tag{31}$$

$$\partial_t q_v + \mathbf{u} \cdot \nabla_h q_v + \omega \partial_p q_v = S_{ev,\varepsilon}^+ - S_{cd}^+ + \mathcal{D}^{q_v} q_v, \tag{32}$$

$$\partial_t q_c + \mathbf{u} \cdot \nabla_h q_c + \omega \partial_p q_c = S_{cd}^+ - S_{ac}^+ - S_{cr}^+ + \mathcal{D}^{q_c} q_c, \tag{33}$$

$$\partial_t q_r + \mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r + V \partial_p \left(\frac{p}{R_d \bar{T}} q_r \right) = S_{ac}^+ + S_{cr}^+ - S_{ev,\varepsilon}^+ + \mathcal{D}^{q_r} q_r, \tag{34}$$

where $\varepsilon \in (0, 1)$ is fixed and

$$S_{ev,\varepsilon}^+ = C_{ev} R T^+ q_r^+ (q_r^+ + \varepsilon)^{\beta-1}(q_{vs}(p, T) - q_v)^+, \quad \varepsilon \in (0, 1), \tag{35}$$

$$S_{cr}^+ = C_{cr} q_c^+ q_r^+, \quad S_{ac}^+ = S_{ac} = C_{ac}(q_c - q_{ac}^*)^+, \tag{36}$$

$$S_{cd}^+ = C_{cd}(q_v^+ - q_{vs}(p, T))q_c^+ + C_{cn}(q_v - q_{vs}(p, T))^+. \tag{37}$$

Unlike the original S_{ev}, the corresponding approximation $S_{ev,\varepsilon}^+$ is Lipschitz with respect to q_r, and it approximates S_{ev} as ε tends to zero.

Since all the nonlinear terms $S_{ev,\varepsilon}^+, S_{cr}^+, S_{ac}^+$, and S_{cd}^+ are Lipschitz with respect to q_v, q_c, q_r, and T, the local, in time, existence and uniqueness of strong solutions to the initial boundary value problem of the ε-approximate system (28)–(34) follows the standard contraction mapping fixed point principle and we obtain the following proposition on the local, in time, existence and uniqueness result.

Proposition 1. *Assume that* $\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0} \in H^1(\mathcal{M})$ *and* $q_{v0}, q_{c0}, q_{r0} \in L^\infty(\mathcal{M})$, *with* $T_0, q_{v0}, q_{c0}, q_{r0} \geqslant 0$ *on* $\mathcal{M}$, *and* $\int_{p_0}^{p_1} \nabla_h \cdot \mathbf{u}_0 dp = 0$ *on* $\mathcal{M}'$. *Then, there is a positive time* $\mathcal{T}_0$ *depending only on the upper bound of* $\|(\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0})\|_{H^1(\mathcal{M})}$, *which is independent of* ε, *such that system* (28)–(34)*, subject to* (24)–(27)*, has a unique strong solution* $\mathbf{u}, T, q_v, q_c, q_r$ *on* $\mathcal{M} \times (0, \mathcal{T}_0)$, *satisfying*

$$\mathbf{u}, T, q_v, q_c, q_r \in C([0, \mathcal{T}_0]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}_0; H^2(\mathcal{M})),$$

$$\partial_t \mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r \in L^2(0, \mathcal{T}_0; L^2(\mathcal{M})).$$

In the following we let $(\mathbf{u}, T, q_v, q_c, q_r)$ be the solution obtained in proposition 1, and extend it to the maximal interval of existence $(0, \mathcal{T}_{\max})$, where $\mathcal{T}_{\max}$ is characterized as

$$\limsup_{\mathcal{T} \to \mathcal{T}_{\max}^-} \|(\mathbf{u}, T, q_v, q_c, q_r)\|_{H^1(\mathcal{M})} = \infty, \quad \text{if } \mathcal{T}_{\max} < \infty. \tag{38}$$

The following assumption will be made in the subsequent propositions throughout this section.

Assumption 1. Let all the assumptions in proposition 1 hold, and let the unique solution $(\mathbf{u}, T, q_v, q_c, q_r)$ obtained in proposition 1 be extended in the above way to the maximal time of existence $\mathcal{T}_{\max}$.

The aim of the rest of this section is to show that $\mathcal{T}_{\max} = \infty$, and to establish the *a priori* estimates that are independent of $\varepsilon \in (0, 1)$. Here at first non negativity and uniform bounded ness of (T, q_v, q_c, q_r) are proven. This in particular allows to derive estimates for the moisture mixing ratios in $L^\infty(0, \mathcal{T}; L^2(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^1(\mathcal{M}))$, which grow continuously in $\mathcal{T}$. The energy of the horizontal velocity is combined with the integral of the temperature, since this allows for the cancellation of the geopotential term involving the vertical velocity. This bound on the horizontal velocity then allows to obtain an $L^\infty(0, \mathcal{T}; L^2(\mathcal{M})) \cap L^2\left((0, \mathcal{T}; H^1(\mathcal{M}))\right)$ control of T. Due to the strong nonlinearity of the system the direct derivation of *a priori* estimates for the gradients of the solution components is not possible and we employ the main idea of Cao and Titi [6] by first bounding $\mathbf{u}$ in $L^\infty(0, \mathcal{T}; L^6(\mathcal{M}))$. The key idea here in [6] is to split the solution into the barotropic (vertically averaged) and the baroclinic component (the according deviation). Since the geopotential due to the hydrostatic assumption enters the equations in fact only as a two-dimensional surface, it is absent in the dynamics of the baroclinic mode, allowing to close the estimate for $\mathbf{u}$ in $L^\infty(0, \mathcal{T}; L^6(\mathcal{M}))$. Based upon this estimate then further an *a priori* estimate for $\partial_p \mathbf{u}$ and thereafter $\partial_p q_j$ can be derived. These and previous bounds allow further to control the horizontal gradients $\nabla_h \mathbf{u}$ and subsequently also $\nabla_h q_j$. Finally also a control of the gradient of T is derived, which completes the set of estimates allowing to bound all solution components in $L^\infty(0, \mathcal{T}; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M}))$.

As a first step we derive the non negativity and uniform bounded ness of (T, q_v, q_c, q_r) in the next proposition in a similar fashion to [14]:

Proposition 2. *Let assumption 3 hold, then the solution* (T, q_v, q_c, q_r) *satisfies*

$$0 \leqslant q_v \leqslant q_v^*, \quad 0 \leqslant q_c \leqslant q_c^*, \quad 0 \leqslant q_r \leqslant q_r^*, \quad 0 \leqslant T \leqslant T^*, \quad on \ \mathcal{M} \times (0, \mathcal{T}),$$

(39)

for any $\mathcal{T} \in (0, \mathcal{T}_{\max})$*, where*

$$q_v^* = \max \left\{ \|q_{v0}\|_{L^\infty(\mathcal{M})}, \|q_{b0v}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell v}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_{vs}^* \right\}$$

(40)

with $q_{vs}^* = \max q_{vs}$*. Moreover* q_c^*, q_r^*, T^* *are continuous in* $\mathcal{T}$ *and depend on the following quantities:*

$$q_c^* = C_{q_c}\left(\mathcal{T}, \|q_{c0}\|_{L^\infty(\mathcal{M})}, \|q_{b0c}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell c}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_v^*, q_{vs}^*\right), \quad (41)$$

$$q_r^* = C_{q_r}\left(\mathcal{T}, \|q_{r0}\|_{L^\infty(\mathcal{M})}, \|q_{b0r}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell r}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_c^*\right), \quad (42)$$

$$T^* = C_T\left(\mathcal{T}, \|T_0\|_{L^\infty(\mathcal{M})}, \|T_{b0}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|T_{b\ell}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_v^*, q_c^*, q_{vs}^*\right). \quad (43)$$

Proof. We only state the key steps of the proof and refer to the proof of proposition 3.2 in [14] for more details. Due to the vanishing of the anti dissipative term in the potential temperature θ, the latter is used instead of the temperature for the derivation of the maximum principle. As a first step then the non negativity of q_c, q_v, q_r and θ are derived in exactly this order by employing the Stampacchia method. Therefore the according equations are multiplied with the corresponding negative parts of the solution components and after integration and integration by parts the Gronwall inequality is applied to show that if the solution was nonnegative initially

it has to remain nonnegative for all times. The same method can then be used to show uniform bounded ness of q_v by deriving an L^2-estimate for $(q_v - q_v^*)^+$. The boundedness of q_c, q_r and θ again in this order follow by employing iterative estimations for cutoff functions in L^m and then passing to the limit as $m \to \infty$ (see proposition 3.2 in [14]). □

Due to the non negativity of q_v, q_c, q_r, and T we obtain also for the source terms

$$S_{cr}^+ = S_{cr}, \quad S_{cd}^+ = S_{cd} \quad \text{and} \quad S_{ev,\varepsilon}^+ = S_{ev,\varepsilon} = C_{ev} T q_r (q_r + \varepsilon)^{\beta-1} (q_{vs} - q_v)^+,$$

such that in the following we can drop the positive signs in the according notations. Proposition 2 in particular also implies the bounded ness of the source terms. To see the uniform bounded ness of $S_{ev,\varepsilon}$ we notice that since $\beta \in (0, 1)$, we have $q_r(q_r + \varepsilon)^{\beta-1} \leqslant q_r q_r^{\beta-1} = q_r^\beta$, and recall that $q_{vs}(p, T) = 0$, for $T \geqslant T_B$ (see (15)). Therefore, one has

$$0 \leqslant S_{ev,\epsilon}, S_{cr}, S_{ac}, |S_{cd}| \leqslant C, \tag{44}$$

where the constant C, as the upper bounds in proposition 2, depends on $\mathcal{T}$, the initial data and the given functions in the boundary conditions (24)–(26).

Proposition 3. *Let assumption 1 hold, then there exists a function $K_0(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that the estimate*

$$\int_0^{\mathcal{T}} \|\nabla(q_v, q_c, q_r)\|^2 \mathrm{d}t \leqslant K_0(\mathcal{T}),$$

holds for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. The conclusion follows from multiplying (32)–(34), respectively, with q_v, q_c, and q_r, performing integration, integrating by parts and using the boundary conditions as well as Young's inequality, where all integrals involving the source terms $S_{ev,\varepsilon}, S_{cr}, S_{ac}, S_{cd}$ are uniformly bounded, due to the *a priori* bounds obtained in proposition 2 concluding the proof. □

Furthermore we have the basic energy inequality contained in the following proposition.

Proposition 4 (Basic energy estimate). *Let assumption 1 hold, then there exists a function $K_1(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that*

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \int_{\mathcal{M}} (|\mathbf{u}|^2 + T) \mathrm{d}\mathcal{M} + \int_0^{\mathcal{T}} (\|\nabla_h \mathbf{u}\|^2 + \|\partial_p \mathbf{u}\|_w^2) \mathrm{d}t \leqslant K_1(\mathcal{T}),$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. Multiplying equation (28) by $\mathbf{u}$, integrating the resultant over $\mathcal{M}$, then it follows from integration by parts that

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\mathbf{u}\|^2 - \int_{\mathcal{M}} \mathcal{D}^{\mathbf{u}} \mathbf{u} \cdot \mathbf{u} \mathrm{d}\mathcal{M} = -\int_{\mathcal{M}} \nabla_h \Phi \cdot \mathbf{u} \mathrm{d}\mathcal{M}, \tag{45}$$

here we have used the facts that $(\mathbf{k} \times \mathbf{v})_h \cdot \mathbf{u} = \mathbf{u}^\perp \cdot \mathbf{u} = 0$, and that the integral involving the convection terms vanishes after integration by parts, due to the incompressibility condition

(30) and the boundary conditions (24)–(26). For the integral of the dissipation term, it follows from integration by parts and the boundary conditions (24)–(26) for **u** that

$$
-\int_{\mathcal{M}} \mathcal{D}^{\mathbf{u}}\mathbf{u} \cdot \mathbf{u}\mathrm{d}\mathcal{M} = -\int_{\mathcal{M}} \left[\mu_{\mathbf{u}}\Delta_h\mathbf{u} + \nu_{\mathbf{u}}\partial_p\left(\left(\frac{gp}{R_d\bar{T}}\right)^2\partial_p\mathbf{u} \right) \right] \cdot \mathbf{u}\mathrm{d}\mathcal{M}
$$

$$
= -\mu_{\mathbf{u}}\int_{\partial\Gamma_\ell} \partial_{\mathbf{n}}\mathbf{u} \cdot \mathbf{u}\mathrm{d}\Gamma_\ell - \nu_{\mathbf{u}}\int_{\mathcal{M}'} \frac{gp}{R_d\bar{T}}\partial_p\mathbf{u} \cdot \mathbf{u}\mathrm{d}\mathcal{M}' \Big|_{p=p_1}^{p_0}
$$

$$
+ \int_{\mathcal{M}} \left(\mu_{\mathbf{u}}|\nabla_h\mathbf{u}|^2 + \nu_{\mathbf{u}}\left(\frac{gp}{R_d\bar{T}}\right)^2|\partial_p\mathbf{u}|^2 \right)\mathrm{d}\mathcal{M}
$$

$$
= \frac{\nu_{\mathbf{u}}gp_0\alpha_{\mathbf{u}}}{R_d\bar{T}(p_0)}\int_{\mathcal{M}'}|\mathbf{u}|^2\mathrm{d}\mathcal{M}'\Big|_{p_0} + \mu_{\mathbf{u}}\|\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p\mathbf{u}\|_w^2
$$

$$
\geqslant \mu_{\mathbf{u}}\|\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p\mathbf{u}\|_w^2. \tag{46}
$$

For the term $-\int_{\mathcal{M}}\nabla_h\Phi \cdot \mathbf{u}\mathrm{d}\mathcal{M}$, it follows from integrating by parts, using the boundary conditions (24)–(26) for **u**, and equations (29) and (30), that

$$
-\int_{\mathcal{M}}\nabla_h\Phi \cdot \mathbf{u}\mathrm{d}\mathcal{M} = -\int_{\Gamma_\ell}\Phi\mathbf{u} \cdot \mathbf{n}\mathrm{d}\Gamma_\ell + \int_{\mathcal{M}}\Phi\nabla_h \cdot \mathbf{u}\mathrm{d}\mathcal{M}
$$

$$
= -\int_{\mathcal{M}}\Phi\partial_p\omega\mathrm{d}\mathcal{M} = \int_{\mathcal{M}}\partial_p\Phi\omega\mathrm{d}\mathcal{M} = -\int_{\mathcal{M}}\frac{RT}{p}\omega\mathrm{d}\mathcal{M}. \tag{47}
$$

Substituting (46) and (47) into (45) yields

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \mu_{\mathbf{u}}\|\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p\mathbf{u}\|_w^2 \leqslant -\int_{\mathcal{M}}\frac{RT}{p}\omega\mathrm{d}\mathcal{M}. \tag{48}
$$

Multiplying equation (31) by c_p, integrating the resultant over $\mathcal{M}$, and recalling the nonnegativity of T, it follows from integration by parts that

$$
c_p\frac{\mathrm{d}}{\mathrm{d}t}\|T\|_{L^1} = \int_{\mathcal{M}}\left(\mathcal{D}^{\mathrm{T}}T + \frac{RT}{p}\omega + L(S_{cd} - S_{ev,\varepsilon}) \right)\mathrm{d}\mathcal{M}. \tag{49}
$$

By the boundary conditions (24)–(26) for T, it follows from integration by parts that

$$
\int_{\mathcal{M}}\mathcal{D}^{\mathrm{T}}T\mathrm{d}\mathcal{M} = \int_{\mathcal{M}}\left(\mu_T\Delta_hT + \nu_T\partial_p\left(\frac{gp}{R_d\bar{T}}\partial_pT\right) \right)\mathrm{d}\mathcal{M}
$$

$$
= \mu_T\int_{\Gamma_\ell}\partial_{\mathbf{n}}T\mathrm{d}\Gamma_\ell + \nu_T\int_{\mathcal{M}'}\frac{gp}{R_d\bar{T}}\partial_pT\mathrm{d}\mathcal{M}'\Big|_{p=p_1}^{p_0}
$$

$$
= \mu_T\int_{\Gamma_\ell}\alpha_{\ell T}(T_{b\ell} - T)\mathrm{d}\Gamma_\ell + \frac{\nu_Tgp_0\alpha_{0T}}{R_d\bar{T}(p_0)}\int_{\mathcal{M}'}(T_{b0} - T)\mathrm{d}\mathcal{M}'\Big|_{p_0}
$$

$$
\leqslant \mu_T\int_{\Gamma_\ell}\alpha_{\ell T}T_{b\ell}\mathrm{d}\Gamma_\ell + \frac{\nu_Tgp_0\alpha_{0T}}{R_d\bar{T}(p_0)}\int_{\mathcal{M}'}T_{b0}\mathrm{d}\mathcal{M}' \leqslant C, \tag{50}
$$

where we note that C depends on the boundary data. Recalling (44) we have moreover

$$
\int_{\mathcal{M}}L(S_{cd} - S_{ev,\varepsilon})\mathrm{d}\mathcal{M} \leqslant C \tag{51}
$$

Thanks to (50) and (51), it follows from (49) that

$$c_p \frac{\mathrm{d}}{\mathrm{d}t} \|T\|_{L^1} \leqslant \int_{\mathcal{M}} \frac{RT}{p} \omega \mathrm{d}\mathcal{M} + C, \tag{52}$$

with the constant C depending again on the boundary data. Adding (48) and (52) yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{1}{2} \|\mathbf{u}\|^2 + c_p \|T\|_{L^1} \right) + \mu_\mathbf{u} \|\nabla_h \mathbf{u}\|^2 + \nu_\mathbf{u} \|\partial_p \mathbf{u}\|_w^2 \leqslant C,$$

from which by integration the conclusion of the proposition follows. $\qquad\square$

Thanks to the basic energy estimate, we can now derive the $L^\infty(L^2) \cap L^2(H^1)$ estimate on T contained in the following proposition.

Proposition 5. *Let assumption 1 hold, then there exists a function $K_2(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that*

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|T\|^2 + \int_0^{\mathcal{T}} (\|\nabla_h T\|^2 + \|\partial_p T\|_w^2) \mathrm{d}t \leqslant K_2(\mathcal{T}),$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. Performing integration by parts and using the boundary conditions for T we obtain

$$-\int_{\mathcal{M}} T \mathcal{D}^T T \mathrm{d}\mathcal{M} = -\int_{\mathcal{M}} \left[\mu_T \Delta_h T + \nu_T \partial_p \left(\left(\frac{gp}{R_d \bar{T}} \right)^2 \partial_p T \right) \right] T \mathrm{d}\mathcal{M}$$

$$= -\mu_T \int_{\Gamma_\ell} \partial_\mathbf{n} T T \mathrm{d}\Gamma_\ell - \nu_T \int_{\mathcal{M}'} \left(\frac{gp}{R_d \bar{T}} \right)^2 \partial_p T T \mathrm{d}\mathcal{M}' \Bigg|_{p=p_1}^{p_0}$$

$$+ \int_{\mathcal{M}} \left[\mu_T \nabla_h T \cdot \nabla_h T + \nu_T \left(\frac{gp}{R_d \bar{T}} \right)^2 \partial_p T \partial_p T \right] \mathrm{d}\mathcal{M}$$

$$= \mu_T \|\nabla_h T\|^2 + \nu_T \|\partial_p T\|_w^2 - \mu_T \int_{\Gamma_\ell} \alpha_{\ell T} (T_{b\ell} - T) T \mathrm{d}\Gamma_\ell$$

$$- \nu_T \alpha_{0T} \left(\frac{gp_0}{R_d \bar{T}(p_0)} \right)^2 \int_{\mathcal{M}'} (T_{b0} - T) T \mathrm{d}\mathcal{M}' \Bigg|_{p_0}$$

$$\geqslant \mu_T \|\nabla_h T\|^2 + \nu_T \|\partial_p T\|_w^2 - \mu_T \int_{\Gamma_\ell} \alpha_{\ell T} T_{b\ell} T \mathrm{d}\Gamma_\ell$$

$$- \nu_T \alpha_{0T} \left(\frac{gp_0}{R_d \bar{T}(p_0)} \right)^2 \int_{\mathcal{M}'} T_{b0} T \mathrm{d}\mathcal{M}' \Bigg|_{p_0}.$$

By the trace inequality, the boundary integrals in the above inequality can be bounded as

$$\mu_T \int_{\Gamma_\ell} \alpha_{\ell T} T_{b\ell} T \mathrm{d}\Gamma_\ell + \nu_T \alpha_{0T} \left(\frac{gp_0}{R_d \bar{T}(p_0)} \right)^2 \int_{\mathcal{M}'} T_{b0} T \mathrm{d}\mathcal{M}' \Bigg|_{p_0}$$

$$\leqslant C\|T\|_{L^2(\partial\mathcal{M})} \leqslant C\|T\|_{H^1(\mathcal{M})} \leqslant \frac{\mu_T}{4}\|\nabla_h T\|^2 + \frac{\nu_T}{4}\|\partial_p T\|_w^2 + C(1+\|T\|^2).$$

Therefore, we have

$$-\int_{\mathcal{M}} T\mathcal{D}^T T\mathrm{d}\mathcal{M} \geqslant \frac{3\mu_T}{4}\|\nabla_h T\|^2 + \frac{3\nu_T}{4}\|\partial_p T\|_w^2 - C(1+\|T\|^2). \tag{53}$$

Multiplying (31) with T and using (53) we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|T\|^2 + \frac{3\mu_T}{4}\|\nabla_h T\|^2 + \frac{3\nu_T}{4}\|\partial_p T\|_w^2$$
$$\leqslant \int_{\mathcal{M}} \frac{R}{c_p}\frac{\omega}{p}T^2\mathrm{d}\mathcal{M} + \int_{\mathcal{M}} \frac{L}{c_p}(S_{cd} - S_{ev,\varepsilon})T\mathrm{d}\mathcal{M},$$

from which, recalling (44), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|T\|^2 + \frac{3\mu_T}{2}\|\nabla_h T\|^2 + \frac{3\nu_T}{2}\|\partial_p T\|_w^2 \leqslant C\int_{\mathcal{M}}(|\omega|T^2 + 1 + T^2)\mathrm{d}\mathcal{M}. \tag{54}$$

By lemma 2 from the appendix and Young's inequality, we deduce

$$C\int_{\mathcal{M}}|\omega|T^2\mathrm{d}\mathcal{M} = \int_{\mathcal{M}}\left|\int_p^{p_0}\nabla_h \cdot \mathbf{u}\mathrm{d}p'\right|T^2\mathrm{d}\mathcal{M}$$
$$\leqslant C\int_{\mathcal{M}'}\int_{p_1}^{p_0}|\nabla_h\mathbf{u}|\mathrm{d}p\int_{p_1}^{p_0}T^2\mathrm{d}p\mathrm{d}\mathcal{M}' \leqslant C\|\nabla_h\mathbf{u}\|\|T\|(\|T\| + \|\nabla_h T\|)$$
$$\leqslant \frac{\mu_T}{2}\|\nabla_h T\|^2 + C(1+\|\nabla_h\mathbf{u}\|^2)\|T\|^2,$$

which, substituted into (54), leads to

$$\frac{\mathrm{d}}{\mathrm{d}t}\|T\|^2 + \mu_T\|\nabla_h T\|^2 + \nu_T\|\partial_p T\|_w^2 \leqslant C(1+\|\nabla_h\mathbf{u}\|_2^2)(1+\|T\|^2),$$

from which, by the Gronwall inequality and proposition 4, the conclusion follows. $\quad\square$

We have the following $L^\infty(L^2)$ estimate on the vertical derivative of the velocity, which was first established by Cao and Titi in [6] for the primitive equations without coupling to the moisture equations.

Proposition 6. *Let assumption 3 hold, then there exists a function $K_3(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that*

$$\sup_{0\leqslant t\leqslant\mathcal{T}}(\|\mathbf{u}\|_{L^6}^6 + \|\partial_p\mathbf{u}\|^2) + \int_0^{\mathcal{T}}(\|\partial_p\nabla_h\mathbf{u}\|^2 + \|\partial_p^2\mathbf{u}\|_w^2)\mathrm{d}t \leqslant K_3(\mathcal{T}),$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. The proof is adapted from that in [6]. Decompose the velocity into the barotropic and baroclinic modes $\bar{\mathbf{u}}$ and $\widetilde{\mathbf{u}}$ as follows

$$\mathbf{u} = \bar{\mathbf{u}} + \widetilde{\mathbf{u}},$$

where $\bar{f} := \frac{1}{p_0 - p_1} \int_{p_1}^{p_0} f \, \mathrm{d}p$. For the L^6-norm of $\widetilde{\mathbf{u}}$, following the arguments in [6], with tiny modifications on dealing with the integral involving Φ, we have (see the inequality above (66) in page 257 of [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\widetilde{\mathbf{u}}\|_{L^6}^6 + \int_{\mathcal{M}} \left(\mu_{\mathbf{u}}|\nabla_h\widetilde{\mathbf{u}}|^2|\widetilde{\mathbf{u}}|^4 + \nu_{\mathbf{u}}|\partial_p\widetilde{\mathbf{u}}|^2|\widetilde{\mathbf{u}}|^4\right)\mathrm{d}\mathcal{M}$$
$$\leqslant C\left(\|\bar{\mathbf{u}}\|^2\|\nabla\bar{\mathbf{u}}\|^2 + \|\nabla\widetilde{\mathbf{u}}\|^2 + \|\widetilde{\mathbf{u}}\|^2\right)\|\widetilde{\mathbf{u}}\|_6^6 + C\|\bar{T}\|^2\|\nabla\bar{T}\|^2.$$

Noticing $\|\bar{f}\| + \|\tilde{f}\| \leqslant C\|f\|$, $\|\nabla\bar{f}\| + \|\nabla\tilde{f}\| \leqslant C\|\nabla f\|$, it follows from proposition 4, proposition 5 and the Gronwall inequality that

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\widetilde{\mathbf{u}}\|_{L^6}^6 + \int_0^{\mathcal{T}} \int_{\mathcal{M}} \left(|\nabla_h\widetilde{\mathbf{u}}|^2|\widetilde{\mathbf{u}}|^4 + |\partial_p\widetilde{\mathbf{u}}|^2|\widetilde{\mathbf{u}}|^4\right)\mathrm{d}\mathcal{M}\,\mathrm{d}t \leqslant K_3'(\mathcal{T}), \tag{55}$$

for a continuous function K_3' on $[0, \infty)$, which depends on the initial and boundary data. For the barotropic mode $\bar{\mathbf{u}}$ it holds that (see the first inequality in page 259 of [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\nabla_h\bar{\mathbf{u}}\|^2 + \mu_{\mathbf{u}}\|\Delta_h\bar{\mathbf{u}}\|^2 \leqslant C\|\bar{\mathbf{u}}\|^2\|\nabla_h\bar{\mathbf{u}}\|^4 + C\left(\|\nabla_h\bar{\mathbf{u}}\|^2 + \int_{\mathcal{M}}|\nabla_h\widetilde{\mathbf{u}}|^2|\widetilde{\mathbf{u}}|^4\mathrm{d}\mathcal{M} + \|\bar{\mathbf{u}}\|^2\right),$$

from which, by proposition 4 and (55), it follows from the Gronwall inequality that

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\nabla_h\bar{\mathbf{u}}\|^2 + \int_0^{\mathcal{T}} \|\Delta_h\bar{\mathbf{u}}\|^2\mathrm{d}t \leqslant K_3''(\mathcal{T}), \tag{56}$$

for a continuous function K_3'' on $[0, \infty)$. Note that by the Sobolev embedding inequality and proposition 4 one can easily obtain from (55) and (56) that

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\mathbf{u}\|_{L^6} \leqslant K_3'''(\mathcal{T}),$$

for a continuous function K_3''' on $[0, \infty)$ depending on the initial and boundary data. For the vertical gradient we have the following bound (see the inequality above (75) in page 260 of [6])

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\partial_p\mathbf{u}\|^2 + \mu_{\mathbf{u}}\|\partial_p\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p^2\mathbf{u}\|_w^2 \leqslant C(\|\nabla_h\bar{\mathbf{u}}\|^4 + \|\widetilde{\mathbf{u}}\|_{L^6}^4)\|\partial_p\mathbf{u}\|^2 + C\|T\|^2,$$

from which, by (55) and (56), proposition 5 and the Gronwall inequality it follows that

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\partial_p\mathbf{u}\|^2 + \int_0^{\mathcal{T}} \left(\|\partial_p\nabla_h\mathbf{u}\|^2 + \|\partial_p^2\mathbf{u}\|_w^2\right)\mathrm{d}t \leqslant K_3''''(\mathcal{T}),$$

for a continuous function K_3'''' on $[0, \infty)$ depending on the initial and boundary data, which completes the proof. $\qquad\square$

We are now ready to establish the estimates for the vertical derivative of the moisture components.

Proposition 7. *Let assumption 1 hold, then there exists a function* $K_4(t)$, *which depends on the initial and boundary data, and is continuous for all* $t \geqslant 0$, *such that*

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\partial_p q_j\|^2 + \int_0^{\mathcal{T}} \|\nabla \partial_p q_j\|^2 \mathrm{d}t \leqslant K_4(\mathcal{T}), \quad j \in \{v, c, r\},$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. We only prove the estimate for q_r, since the derivation for the other moisture quantities follows the same steps. Multiplying equation (34) by $-\partial_p^2 q_r$ and integrating over $\mathcal{M}$ yield

$$- \int_{\mathcal{M}} \partial_t q_r \partial_p^2 q_r \mathrm{d}\mathcal{M} + \int_{\mathcal{M}} \mathcal{D}^{q_r} q_r \partial_p^2 q_r \mathrm{d}\mathcal{M}$$

$$= \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \partial_p^2 q_r \mathrm{d}\mathcal{M} + V \int_{\mathcal{M}} \partial_p \left(\frac{p q_r}{R_d \bar{T}} \right) \partial_p^2 q_r \mathrm{d}\mathcal{M}$$

$$+ \int_{\mathcal{M}} (S_{ev,\varepsilon} - S_{ac} - S_{cr}) \partial_p^2 q_r \mathrm{d}\mathcal{M}, \tag{57}$$

Recalling (44), it follows from the Hölder and Young inequalities that

$$V \int_{\mathcal{M}} \partial_p \left(\frac{p q_r}{R_d \bar{T}} \right) \partial_p^2 q_r \mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (S_{ev,\varepsilon} - S_{ac} - S_{cr}) \partial_p^2 q_r \mathrm{d}\mathcal{M}$$

$$\leqslant \frac{\nu_{q_r}}{16} \|\partial_p^2 q_r\|^2 + C \left(\|\partial_p q_r\|^2 + 1 \right). \tag{58}$$

For the first term on the left-hand side of (57), using the boundary conditions (24) and (25), it follows from integration by parts that

$$- \int_{\mathcal{M}} \partial_t q_r \partial_p^2 q_r \mathrm{d}\mathcal{M}$$

$$= - \int_{\mathcal{M}'} \partial_t q_r \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} + \int_{\mathcal{M}} \partial_t \partial_p q_r \partial_p q_r \mathrm{d}\mathcal{M}$$

$$= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|\partial_p q_r\|^2 - \alpha_{0r} \int_{\mathcal{M}'} \partial_t q_r (q_{b0r} - q_r) \mathrm{d}\mathcal{M}' \bigg|_{p_0}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\partial_p q_r\|^2}{2} + \alpha_{0r} \int_{\mathcal{M}'} \left(\frac{q_r^2}{2} - q_r q_{b0r} \right) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right) + \alpha_{0r} \int_{\mathcal{M}'} q_r \partial_t q_{b0r} \mathrm{d}\mathcal{M}' \bigg|_{p_0}$$

$$\geqslant \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\partial_p q_r\|^2}{2} + \alpha_{0r} \int_{\mathcal{M}'} \left(\frac{q_r^2}{2} - q_r q_{b0r} \right) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right) - C(\|\partial_p q_r\| + 1), \tag{59}$$

where in the last step we have used lemma 1 (see appendix), proposition 2 and the Hölder inequality to estimate

$$\left| \int_{\mathcal{M}'} q_r \partial_t q_{b0r} \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right| \leqslant C(\|q_r\|_{L^1(\mathcal{M})} + \|\partial_p q_r\|_{L^1(\mathcal{M})}) \leqslant C(1 + \|\partial_p q_r\|).$$

We next bound the integrals involving the diffusion terms. It should be noticed that the following derivations are formal, since the regularity of q_r does not guarantee the validity of

the integrals on the boundary $\Gamma_0 = \mathcal{M}' \times \{p_0\}$ directly. However, due to the $C^2(\overline{\mathcal{M}} \times [0, T])$ functions, enjoying the boundary conditions (24)–(26) for q_r, being dense in the space $\{f \,|\, f \in L^2(0, \mathcal{T}; H^2(\mathcal{M})), \partial_t f \in L^2(0, \mathcal{T}; L^2(\mathcal{M}))\}$, which q_r belongs to, one can rigorously justify the desired estimates (65) below for q_r in the standard way (i.e., choosing a sequence $\{q_{rn}\}_{n=1}^{\infty}$ in $C^2(\overline{\mathcal{M}} \times [0, \mathcal{T}])$ satisfying the corresponding boundary conditions (24)–(26), proving that (65) holds for each q_{rn}, and passing the limit $n \to \infty$).

Using the boundary conditions (25), it follows from integration by parts that

$$
\begin{aligned}
\int_{\mathcal{M}} \mathcal{D}^{q_r} q_r \partial_p^2 q_r \mathrm{d}\mathcal{M} &= \int_{\mathcal{M}} \left[\mu_{q_r} \Delta_h q_r + \nu_{q_r} \partial_p \left(\left(\frac{gp}{R_d \overline{T}} \right)^2 \partial_p q_r \right) \right] \partial_p^2 q_r \mathrm{d}\mathcal{M} \\
&= \mu_{q_r} \int_{\mathcal{M}'} \Delta_h q_r \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} - \mu_{q_r} \int_{\mathcal{M}} \partial_p \Delta_h q_r \partial_p q_r \mathrm{d}\mathcal{M} + \nu_{q_r} \|\partial_p^2 q_r\|_w^2 \\
&\quad + \nu_{q_r} \int_{\mathcal{M}} \partial_p \left(\frac{gp}{R_d \overline{T}} \right)^2 \partial_p q_r \partial_p^2 q_r \mathrm{d}\mathcal{M} \\
&= \mu_{q_r} \left(\int_{\mathcal{M}'} \Delta_h q_r \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} - \int_{\Gamma_\ell} \partial_p \partial_{\mathbf{n}} q_r \partial_p q_r \mathrm{d}\Gamma_\ell + \|\nabla_h \partial_p q_r\|^2 \right) \\
&\quad + \nu_{q_r} \|\partial_p^2 q_r\|_w^2 + \nu_{q_r} \int_{\mathcal{M}} \partial_p \left(\frac{gp}{R_d \overline{T}} \right)^2 \partial_p q_r \partial_p^2 q_r \mathrm{d}\mathcal{M} \\
&\geqslant \mu_{q_r} \left(\int_{\mathcal{M}'} \Delta_h q_r \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} - \int_{\Gamma_\ell} \partial_p \partial_{\mathbf{n}} q_r \partial_p q_r \mathrm{d}\Gamma_\ell \right) + \mu_{q_r} \|\nabla_h \partial_p q_r\|^2 \\
&\quad + \frac{3\nu_{q_r}}{4} \|\partial_p^2 q_r\|_w^2 - C\|\partial_p q_r\|^2.
\end{aligned}
\tag{60}
$$

Integrating by parts and using the boundary conditions (24) and (26), we deduce

$$
\begin{aligned}
&\int_{\mathcal{M}'} \Delta_h q_r \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} \\
&= \int_{\partial\mathcal{M}'} \partial_{\mathbf{n}} q_r \partial_p q_r \mathrm{d}\partial\mathcal{M}' \bigg|_{p_0} - \int_{\mathcal{M}'} \nabla_h q_r \cdot \nabla_h \partial_p q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} \\
&= \alpha_{\ell r} \alpha_{0r} \int_{\partial\mathcal{M}'} (q_{b\ell r} - q_r)(q_{b0r} - q_r) \mathrm{d}\partial\mathcal{M}' \bigg|_{p_0} + \alpha_{0r} \int_{\mathcal{M}'} \nabla_h q_r \cdot \nabla_h (q_r - q_{b0r}) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \\
&\geqslant -\frac{\alpha_{\ell r} \alpha_{0r}}{4} \int_{\partial\mathcal{M}'} (q_{b\ell r} - q_{b0r})^2 \mathrm{d}\partial\mathcal{M}' \bigg|_{p_0} - \frac{\alpha_{0r}}{4} \int_{\mathcal{M}'} |\nabla_h q_{b0r}|^2 \mathrm{d}\mathcal{M}' \bigg|_{p_0} \geqslant -C, \tag{61}
\end{aligned}
$$

where we have used

$$
(q_{b\ell r} - q_r)(q_{b0r} - q_r) = \left(q_r - \frac{q_{b\ell r} + q_{b0r}}{2} \right)^2 - \frac{(q_{b\ell r} - q_{b0r})^2}{4} \geqslant -\frac{(q_{b\ell r} - q_{b0r})^2}{4},
\tag{62}
$$

$$
\nabla_h q_r \cdot \nabla_h (q_r - q_{b0r}) = \left| \nabla_h q_r - \frac{\nabla_h q_{b0r}}{2} \right|^2 - \frac{|\nabla_h q_{b0r}|^2}{4} \geqslant -\frac{|\nabla_h q_{b0r}|^2}{4}.
\tag{63}
$$

Using the boundary condition (26), one has

$$
-\int_{\Gamma_\ell} \partial_p \partial_{\mathbf{n}} q_r \partial_p q_r \mathrm{d}\Gamma_\ell = \alpha_{\ell r} \int_{\Gamma_\ell} \partial_p (q_r - q_{b\ell r}) \partial_p q_r \mathrm{d}\Gamma_\ell
$$

$$
= \alpha_{\ell r} \int_{\Gamma_\ell} \left[\left(\partial_p q_r - \frac{\partial_p q_{b\ell r}}{2} \right)^2 - \frac{|\partial_p q_{b\ell r}|^2}{4} \right] \mathrm{d}\Gamma_\ell
$$

$$
\geqslant -\frac{\alpha_{\ell r}}{4} \int_{\Gamma_\ell} |\partial_p q_{b\ell r}|^2 \mathrm{d}\Gamma_\ell \geqslant -C. \tag{64}
$$

Substituting (61) and (64) into (60) yields

$$
\int_{\mathcal{M}} \mathcal{D}^{q_r} q_r \partial_p^2 q_r \mathrm{d}\mathcal{M} \geqslant \frac{3}{4} \left(\mu_{q_r} \| \nabla_h \partial_p q_r \|^2 + \nu_{q_r} \| \partial_p^2 q_r \|_w^2 \right) - C \left(\| \partial_p q_r \|^2 + 1 \right). \tag{65}
$$

For the term involving the convection, it follows from integration by parts and the boundary condition (25) that

$$
\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \partial_p^2 q_r \mathrm{d}\mathcal{M}
$$

$$
= \int_{\mathcal{M}'} (\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \partial_p q_r \mathrm{d}\mathcal{M}' \Big|_{p_0} - \int_{\mathcal{M}} (\partial_p \mathbf{u} \cdot \nabla_h q_r - \nabla_h \cdot \mathbf{u} \partial_p q_r) \partial_p q_r \mathrm{d}\mathcal{M}
$$

$$
- \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h \partial_p q_r + \omega \partial_p^2 q_r) \partial_p q_r \mathrm{d}\mathcal{M}
$$

$$
= \alpha_{0r} \int_{\mathcal{M}'} \mathbf{u} \cdot \nabla_h q_r (q_{b0r} - q_r) \mathrm{d}\mathcal{M}' \Big|_{p_0} - \int_{\mathcal{M}} (\partial_p \mathbf{u} \cdot \nabla_h q_r - \nabla_h \cdot \mathbf{u} \partial_p q_r) \partial_p q_r \mathrm{d}\mathcal{M}. \tag{66}
$$

By proposition 2, lemma 1 (see appendix), and Young's inequality, we obtain

$$
\alpha_{0r} \left| \int_{\mathcal{M}'} \mathbf{u} \cdot \nabla_h q_r (q_{b0r} - q_r) \mathrm{d}\mathcal{M}' \Big|_{p_0} \right|
$$

$$
\leqslant C \int_{\mathcal{M}'} |\mathbf{u} \cdot \nabla_h q_r| \mathrm{d}\mathcal{M}' \Big|_{p_0} \leqslant C(\| \mathbf{u} \cdot \nabla_h q_r \|_{L^1(\mathcal{M})} + \| \partial_p (\mathbf{u} \cdot \nabla_h q_r) \|_{L^1(\mathcal{M})})
$$

$$
\leqslant C(\| \mathbf{u} \cdot \nabla_h q_r \|_{L^1(\mathcal{M})} + \| \partial_p \mathbf{u} \cdot \nabla_h q_r \|_{L^1(\mathcal{M})} + \| \mathbf{u} \cdot \nabla_h \partial_p q_r \|_{L^1(\mathcal{M})})
$$

$$
\leqslant \frac{\mu_{q_r}}{12} \| \nabla_h \partial_p q_r \|_w^2 + C(\| \mathbf{u} \|^2 + \| \nabla_h q_r \|^2 + \| \partial_p \mathbf{u} \|^2)
$$

$$
\leqslant \frac{\mu_{q_r}}{12} \| \nabla_h \partial_p q_r \|_w^2 + C(\| \nabla_h q_r \|^2 + 1), \tag{67}
$$

where in the last step we have used proposition 4 and proposition 6. Note that the boundary condition (26) for $\mathbf{u}$ implies $\partial_p \mathbf{u} \cdot \mathbf{n} = 0$ on Γ_ℓ. Thanks to this, by proposition 2 and proposition 6, it follows from integration by parts and the Young inequality that

$$
-\int_{\mathcal{M}} \partial_p \mathbf{u} \cdot \nabla_h q_r \partial_p q_r \mathrm{d}\mathcal{M} = \int_{\mathcal{M}} (\partial_p \nabla_h \cdot \mathbf{u} \partial_p q_r + \partial_p \mathbf{u} \cdot \nabla_h \partial_p q_r) q_r \mathrm{d}\mathcal{M}
$$
$$
\leqslant \frac{\mu_{q_r}}{12} \|\nabla_h \partial_p q_r\|^2 + C(\|\partial_p \nabla_h \mathbf{u}\|^2 + \|\partial_p q_r\|^2 + 1).
$$

(68)

Noticing $\int_{p_1}^{p_0} \nabla_h \cdot \mathbf{u} \mathrm{d}p = 0$, we have

$$
|\nabla_h \cdot \mathbf{u}(x, y, p)| = \left| \frac{1}{p_0 - p_1} \int_{p_0}^{p_1} \nabla_h \cdot \mathbf{u} \mathrm{d}q + \frac{1}{p_0 - p_1} \int_{p_1}^{p_0} \int_q^p \nabla_h \cdot \partial_p \mathbf{u} \mathrm{d}\xi \mathrm{d}q \right|
$$
$$
= \left| \frac{1}{p_0 - p_1} \int_{p_1}^{p_0} \int_q^p \nabla_h \cdot \partial_p \mathbf{u} \mathrm{d}\xi \mathrm{d}q \right| \leqslant \int_{p_1}^{p_0} |\nabla_h \partial_p \mathbf{u}| \mathrm{d}q, \quad (69)
$$

for any $p \in [p_1, p_0]$. Based upon this fact, it follows from lemma 2 (see appendix) that

$$
\left| \int_{\mathcal{M}} \nabla_h \cdot \mathbf{u}(\partial_p q_r)^2 \mathrm{d}\mathcal{M} \right| \leqslant \int_{\mathcal{M}'} \int_{p_1}^{p_0} |\nabla_h \partial_p \mathbf{u}| \mathrm{d}p \int_{p_1}^{p_0} |\partial_p q_r|^2 \mathrm{d}p \mathrm{d}\mathcal{M}'
$$
$$
\leqslant C \|\nabla_h \partial_p \mathbf{u}\| \|\partial_p q_r\| (\|\partial_p q_r\| + \|\nabla_h \partial_p q_r\|)
$$
$$
\leqslant \frac{\mu_{q_r}}{12} \|\nabla_h \partial_p q_r\|^2 + C(1 + \|\nabla_h \partial_p \mathbf{u}\|^2) \|\partial_p q_r\|^2. \quad (70)
$$

Substituting (67), (68), and (70) into (66), one obtains the estimate

$$
\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \partial_p^2 q_r \mathrm{d}\mathcal{M}
$$
$$
\leqslant \frac{\mu_{q_r}}{4} \|\nabla_h \partial_p q_r\|^2 + C(\|\nabla_h q_r\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2 + 1)(1 + \|\partial_p q_r\|^2). \quad (71)
$$

Substituting (58), (59), (65), and (71) into (57) leads to

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\partial_p q_r\|^2}{2} + \alpha_{0r} \int_{\mathcal{M}'} \left(\frac{q_r^2}{2} - q_r q_{b0r} \right) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right) + \frac{1}{2} (\mu_{q_r} \|\nabla_h \partial_p q_r\|^2 + \nu_{q_r} \|\partial_p^2 q_r\|_w^2)
$$
$$
\leqslant C(\|\nabla_h \partial_p \mathbf{u}\|^2 + \|\nabla_h q_r\|^2 + 1)(1 + \|\partial_p q_r\|^2).
$$

from which, noticing that

$$
\int_{\mathcal{M}'} (q_r^2 - 2q_r q_{b0r}) \mathrm{d}\mathcal{M}' \bigg|_{p_0} = \int_{\mathcal{M}'} \left[(q_r - q_{b0r})^2 - q_{b0r}^2 \right] \mathrm{d}\mathcal{M}' \bigg|_{p_0}
$$
$$
\geqslant -\int_{\mathcal{M}'} q_{b0r}^2 \mathrm{d}\mathcal{M}' \bigg|_{p_0} \geqslant -C,
$$

the conclusion follows, by proposition 3, proposition 6, and the Gronwall inequality. $\qquad \square$

Based on the estimates on the vertical derivatives of the velocity and moisture components, we can now bound the corresponding horizontal derivatives.

Proposition 8. *Let assumption 1 hold, then there exists a function $K_5(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that*

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\nabla_h \mathbf{u}\|^2 + \int_0^{\mathcal{T}} \left(\|\Delta_h \mathbf{u}\|^2 + \|\partial_p \nabla_h \mathbf{u}\|^2 \right) \mathrm{d}t \leqslant K_5(\mathcal{T}),$$

and

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\nabla_h q_j\|^2 + \int_0^{\mathcal{T}} \left(\|\Delta_h q_j\|^2 + \|\nabla_h \partial_p q_j\|_w^2 \right) \mathrm{d}t \leqslant K_5(\mathcal{T}), \quad j \in \{v, c, r\},$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. Following the arguments in section 3.3.3 of [6], we have for the estimate of the horizontal gradient,

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_h \mathbf{u}\|^2 + \mu_{\mathbf{u}} \|\Delta_h \mathbf{u}\|^2 + \nu_{\mathbf{u}} \|\partial_p \nabla_h \mathbf{u}\|^2$$

$$\leqslant C \left(\|\mathbf{u}\|_{L^6}^4 + \|\nabla_h \mathbf{u}\|^2 \|\partial_p \mathbf{u}\|^2 \right) \|\nabla_h \mathbf{u}\|^2 + C \int_{\mathcal{M}} \left(\int_{p_1}^p |\nabla_h T| \mathrm{d}\sigma \right) |\Delta_h \mathbf{u}| \mathrm{d}\mathcal{M},$$

from which, by proposition 6 and Young's inequality, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_h \mathbf{u}\|^2 + \mu_{\mathbf{u}} \|\Delta_h \mathbf{u}\|^2 + \nu_{\mathbf{u}} \|\partial_p \nabla_h \mathbf{u}\|^2$$

$$\leqslant C(1 + \|\nabla_h \mathbf{u}\|^2) \|\nabla_h \mathbf{u}\|^2 + C \|\nabla_h T\| \|\Delta_h \mathbf{u}\|$$

$$\leqslant C(1 + \|\nabla_h \mathbf{u}\|^2) \|\nabla_h \mathbf{u}\|^2 + C \|\nabla_h T\|^2 + \frac{\mu_{\mathbf{u}}}{4} \|\Delta_h \mathbf{u}\|^2, \tag{72}$$

leading to

$$\frac{\mathrm{d}}{\mathrm{d}t} \|\nabla_h \mathbf{u}\|^2 + \frac{3\mu_{\mathbf{u}}}{4} \|\Delta_h \mathbf{u}\|^2 + \nu_{\mathbf{u}} \|\partial_p \nabla_h \mathbf{u}\|^2 \leqslant C(1 + \|\nabla_h \mathbf{u}\|^2) \|\nabla_h \mathbf{u}\|^2 + C \|\nabla_h T\|^2. \tag{73}$$

Next, we estimate the horizontal gradient of q_r. Multiplying equation (34) by $-\Delta_h q_r$ and integrating over $\mathcal{M}$ yield

$$\int_{\mathcal{M}} (-\partial_t q_r + D^{q_r} q_r) \Delta_h q_r \mathrm{d}\mathcal{M} \tag{74}$$

$$= \int_{\mathcal{M}} \left[V \partial_p \left(\frac{p q_r}{R_d \bar{T}} \right) + S_{ac} + S_{cr} - S_{ev,\varepsilon} \right] \Delta_h q_r \mathrm{d}\mathcal{M} + \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \Delta_h q_r \mathrm{d}\mathcal{M}.$$

Recalling (44), it follows from the Young inequality, proposition 2 and proposition 7 that

$$\int_{\mathcal{M}} \left[V \partial_p \left(\frac{p q_r}{R_d \bar{T}} \right) + S_{ac} + S_{cr} - S_{ev,\varepsilon} \right] \Delta_h q_r \mathrm{d}\mathcal{M}$$

$$\leqslant \frac{\mu_{q_r}}{4} \| \Delta_h q_r \|^2 + C \left(\| q_r \|^2 + \| \partial_p q_r \|^2 \right) \leqslant \frac{\mu_{q_r}}{8} \| \Delta_h q_r \|^2 + C. \tag{75}$$

Integrating by parts and using the boundary condition (26), we obtain

$$- \int_{\mathcal{M}} \partial_t q_r \Delta_h q_r \mathrm{d}\mathcal{M}$$

$$= - \int_{\Gamma_\ell} \partial_t q_r \partial_{\mathbf{n}} q_r \mathrm{d}\Gamma_\ell + \int_{\mathcal{M}} \partial_t \nabla_h q_r \cdot \nabla_h q_r \mathrm{d}\mathcal{M}$$

$$= \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \| \nabla_h q_r \|^2 - \alpha_{\ell v} \int_{\Gamma_\ell} \partial_t q_r (q_{b\ell r} - q_r) \mathrm{d}\Gamma_\ell$$

$$= \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\| \nabla_h q_r \|^2}{2} + \alpha_{\ell r} \int_{\Gamma_\ell} \left(\frac{q_r^2}{2} - q_r q_{b\ell r} \right) \mathrm{d}\Gamma_\ell \right) + \alpha_{\ell r} \int_{\Gamma_\ell} q_r \partial_t q_{b\ell r} \mathrm{d}\Gamma_\ell$$

$$\geqslant \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\| \nabla_h q_r \|^2}{2} + \alpha_{\ell r} \int_{\Gamma_\ell} \left(\frac{q_r^2}{2} - q_r q_{b\ell r} \right) \mathrm{d}\Gamma_\ell \right) - C(1 + \| \nabla_h q_r \|), \tag{76}$$

where in the last step we have used

$$\left| \int_{\Gamma_\ell} q_r \partial_t q_{b\ell r} \mathrm{d}\Gamma_\ell \right| \leqslant C \| q_r \|_{L^1(\Gamma_\ell)} \leqslant C(\| q_r \| + \| \nabla_h q_r \|) \leqslant C(1 + \| \nabla_h q_r \|),$$

which is guaranteed by the trace inequality and proposition 2.

Recalling the expression of $\mathcal{D}^{q_r} q_r$, we have

$$\int_{\mathcal{M}} \mathcal{D}^{q_r} q_r \Delta_h q_v \mathrm{d}\mathcal{M} = \mu_{q_r} \| \Delta_h q_r \|^2 + \nu_{q_r} \int_{\mathcal{M}} \partial_p \left(\left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \right) \Delta_h q_v \mathrm{d}\mathcal{M}. \tag{77}$$

Moreover integration by parts yields

$$\int_{\mathcal{M}} \partial_p \left(\left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \right) \Delta_h q_v \mathrm{d}\mathcal{M}$$

$$= \int_{\mathcal{M}'} \left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \Delta_h q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} - \int_{\mathcal{M}} \left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \partial_p \Delta_h q_r \mathrm{d}\mathcal{M}$$

$$= \int_{\mathcal{M}'} \left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \Delta_h q_r \mathrm{d}\mathcal{M}' \bigg|_{p_0} + \| \nabla_h \partial_p q_r \|_w^2$$

$$- \int_{\Gamma_\ell} \left(\frac{g p}{R_d \bar{T}} \right)^2 \partial_p q_r \partial_p \partial_{\mathbf{n}} q_r \mathrm{d}\Gamma_\ell. \tag{78}$$

We denote hereafter $\delta_0 = \alpha_{0r}\left(\frac{gp_0}{R_d\bar{T}(p_0)}\right)^2$. Then, using the boundary condition (24) and (62) and (63), it follows from integration by parts that

$$
\begin{aligned}
\int_{\mathcal{M}'}\left(\frac{gp}{R_d\bar{T}}\right)^2\partial_p q_r\Delta_h q_r\mathrm{d}\mathcal{M}'\bigg|_{p_0} &= \delta_0\int_{\mathcal{M}'}(q_{b0r}-q_r)\Delta_h q_r\mathrm{d}\mathcal{M}'\bigg|_{p_0}\\
&= \delta_0\left(\int_{\partial\mathcal{M}'}(q_{b0r}-q_r)\partial_\mathbf{n}q_r\mathrm{d}\partial\mathcal{M}'+\int_{\mathcal{M}'}\nabla_h(q_r-q_{b0r})\cdot\nabla_h q_r\mathrm{d}\mathcal{M}'\right)\bigg|_{p_0}\\
&= \delta_0\left(\alpha_{\ell r}\int_{\partial\mathcal{M}'}(q_{b\ell r}-q_r)(q_{b0r}-q_r)\mathrm{d}\partial\mathcal{M}'+\int_{\mathcal{M}'}\nabla_h q_r\cdot\nabla_h(q_r-q_{b0r})\mathrm{d}\mathcal{M}\right)\bigg|_{p_0}\\
&\geqslant -\frac{\delta_0}{4}\left(\alpha_{\ell r}\int_{\partial\mathcal{M}'}(q_{b\ell r}-q_{b0r})^2\mathrm{d}\partial\mathcal{M}'+\int_{\mathcal{M}'}|\nabla_h q_{b0r}|^2\mathrm{d}\mathcal{M}'\right)\bigg|_{p_0}\geqslant -C, \qquad (79)
\end{aligned}
$$

where we note that here and in the subsequent estimates the constant C depends on the initial and boundary data. Using the boundary condition (26), we have moreover

$$
\begin{aligned}
-\int_{\Gamma_\ell}\left(\frac{gp}{R_d\bar{T}}\right)^2&\partial_p q_r\partial_p\partial_\mathbf{n}q_r\mathrm{d}\Gamma_\ell\\
&= \alpha_{\ell r}\int_{\Gamma_\ell}\left(\frac{gp}{R_d\bar{T}}\right)^2\partial_p q_r(\partial_p q_r-\partial_p q_{b\ell r})\mathrm{d}\Gamma_\ell\\
&= \alpha_{\ell r}\int_{\Gamma_\ell}\left(\frac{gp}{R_d\bar{T}}\right)^2\left[\left(\partial_p q_r-\frac{\partial_p q_{b\ell r}}{2}\right)^2-\frac{|\partial_p q_{b\ell r}|^2}{4}\right]\mathrm{d}\Gamma_\ell\\
&\geqslant -\frac{\alpha_{\ell r}}{4}\int_{\Gamma_\ell}\left(\frac{gp}{R_d\bar{T}}\right)^2(\partial_p q_{b\ell r})^2\mathrm{d}\Gamma_\ell\geqslant -C. \qquad (80)
\end{aligned}
$$

Substituting (79) and (80) into (78) yields

$$
\int_{\mathcal{M}}\partial_p\left(\left(\frac{gp}{R_d\bar{T}}\right)^2\partial_p q_r\right)\Delta_h q_v\mathrm{d}\mathcal{M}\geqslant \|\nabla_h\partial_p q_r\|_w^2-C,
$$

and, consequently, it follows from (77) that

$$
\int_{\mathcal{M}}\mathcal{D}^{q_r}q_r\Delta_h q_v\mathrm{d}\mathcal{M}\geqslant \mu_{q_r}\|\Delta_h q_r\|^2+\nu_{q_r}\|\nabla_h\partial_p q_r\|_w^2-C. \qquad (81)
$$

Note that

$$
\begin{aligned}
|u(x,y,p)| &= \left|\frac{1}{p_0-p_1}\int_{p_1}^{p_0}u\mathrm{d}\sigma+\frac{1}{p_0-p_1}\int_{p_1}^{p_0}\left(\int_\sigma^p\partial_p u\mathrm{d}\xi\right)\mathrm{d}\sigma\right|\\
&\leqslant \frac{1}{p_0-p_1}\int_{p_1}^{p_0}|u|\mathrm{d}\sigma+\int_{p_1}^{p_0}|\partial_p u|\mathrm{d}\sigma,
\end{aligned}
$$

for any $p \in [p_1, p_0]$. Thanks to this, it follows from lemma 2 and Young's inequality that

$$
\begin{aligned}
\int_{\mathcal{M}} &(\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \Delta_h q_r \mathrm{d}\mathcal{M} \\
&\leqslant C \int_{\mathcal{M}'} \int_{p_1}^{p_0} (|u| + |\partial_p u|) \mathrm{d}p \int_{p_1}^{p_0} |\nabla_h q_r| |\Delta_h q_r| \mathrm{d}p \mathrm{d}\mathcal{M}' \\
&\quad + \int_{\mathcal{M}'} \int_{p_1}^{p_0} |\nabla_h \mathbf{u}| \mathrm{d}p \int_{p_1}^{p_0} |\partial_p q_r| |\Delta_h q_r| \mathrm{d}p \mathrm{d}\mathcal{M}' \\
&\leqslant C \left[\|\mathbf{u}\|^{\frac{1}{2}} \left(\|\mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) + \|\partial_p \mathbf{u}\|^{\frac{1}{2}} \left(\|\partial_p \mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \partial_p \mathbf{u}\|^{\frac{1}{2}} \right) \right] \|\nabla_h q_r\|^{\frac{1}{2}} \\
&\quad \times \left(\|\nabla_h q_r\|^{\frac{1}{2}} + \|\nabla_h^2 q_r\|^{\frac{1}{2}} \right) \|\Delta_h q_r\| + C \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \left(\|\nabla_h^2 \mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) \\
&\quad \times \|\partial_p q_r\|^{\frac{1}{2}} \left(\|\partial_p q_r\|^{\frac{1}{2}} + \|\nabla_h \partial_p q_r\|^{\frac{1}{2}} \right) \|\Delta_h q_r\| \\
&\leqslant C \left[\left(1 + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) + \left(1 + \|\nabla_h \partial_p \mathbf{u}\|^{\frac{1}{2}} \right) \right] \|\nabla_h q_r\|^{\frac{1}{2}} \left(\|\nabla_h q_r\|^{\frac{1}{2}} + \|\nabla_h^2 q_r\|^{\frac{1}{2}} \right) \\
&\quad \times \|\Delta_h q_r\| + C \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \left(\|\nabla_h^2 \mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) \left(1 + \|\nabla_h \partial_p q_r\|^{\frac{1}{2}} \right) \|\Delta_h q_r\|,
\end{aligned}
\tag{82}
$$

where, we have used proposition 2, proposition 6, and proposition 7. By standard elliptic estimates and proposition 2, we have

$$
\|\nabla_h^2 \mathbf{u}\| \leqslant C \|\Delta_h \mathbf{u}\|, \quad \|\nabla_h^2 q_i\| \leqslant C(\|\Delta_h q_i\| + \|q_i\| + 1) \leqslant C(\|\Delta_h q_i\| + 1),
\tag{83}
$$

for $i \in \{v, c, r\}$. Then it follows from (82) and Young's inequality that

$$
\begin{aligned}
\int_{\mathcal{M}} &(\mathbf{u} \cdot \nabla_h q_r + \omega \partial_p q_r) \Delta_h q_r \mathrm{d}\mathcal{M} \\
&\leqslant C \left[\left(1 + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) + \left(1 + \|\nabla_h \partial_p \mathbf{u}\|^{\frac{1}{2}} \right) \right] \|\nabla_h q_r\|^{\frac{1}{2}} \left(1 + \|\nabla_h q_r\|^{\frac{1}{2}} + \|\Delta_h q_r\|^{\frac{1}{2}} \right) \\
&\quad \times \|\Delta_h q_r\| + C \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \left(1 + \|\Delta_h \mathbf{u}\|^{\frac{1}{2}} + \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \right) \left(1 + \|\nabla_h \partial_p q_r\|^{\frac{1}{2}} \right) \|\Delta_h q_r\| \\
&\leqslant \frac{\mu_{q_r}}{8} \|\Delta_h q_r\|^2 + \frac{\mu_{\mathbf{u}}}{12} \|\Delta_h \mathbf{u}\|^2 + C(1 + \|\nabla_h \mathbf{u}\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2)(1 + \|\nabla_h q_r\|^2) \\
&\quad + C(1 + \|\nabla_h \partial_p q_r\|^2)(1 + \|\nabla_h \mathbf{u}\|^2).
\end{aligned}
\tag{84}
$$

Thanks to (75), (76), (81), and (84), it follows from (74) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t} &\left(\frac{\|\nabla_h q_r\|^2}{2} + \alpha_{\ell r} \int_{\Gamma_\ell} \left(\frac{q_r^2}{2} - q_r q_{b\ell r} \right) \mathrm{d}\Gamma_\ell \right) + \frac{3\mu_{q_r}}{4} \|\Delta_h q_r\|^2 + \nu_{q_r} \|\nabla_h \partial_p q_r\|_w^2 \\
&\leqslant \frac{\mu_{\mathbf{u}}}{12} \|\Delta_h \mathbf{u}\|^2 + C(1 + \|\nabla_h \mathbf{u}\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2)(1 + \|\nabla_h q_r\|^2) \\
&\quad + C(1 + \|\nabla_h \partial_p q_r\|^2)(1 + \|\nabla_h \mathbf{u}\|^2).
\end{aligned}
$$

The same estimate as above also holds for q_v, q_c, such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\nabla_h q_j\|^2}{2} + \alpha_{\ell j}\int_{\Gamma_\ell}\left(\frac{q_j^2}{2} - q_j q_{b\ell j}\right)\mathrm{d}\Gamma_\ell\right) + \frac{3\mu_{q_j}}{4}\|\Delta_h q_j\|^2 + \nu_{q_j}\|\nabla_h\partial_p q_j\|_w^2$$

$$\leqslant \frac{\mu_{\mathbf{u}}}{12}\|\Delta_h\mathbf{u}\|^2 + C(1 + \|\nabla_h\mathbf{u}\|^2 + \|\nabla_h\partial_p\mathbf{u}\|^2)(1 + \|\nabla_h q_j\|^2)$$

$$+ C(1 + \|\nabla_h\partial_p q_j\|^2)(1 + \|\nabla_h\mathbf{u}\|^2), \tag{85}$$

for $j \in \{v, c, r\}$.

Summing (73) with (85) for $j \in \{v, c, r\}$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\|\nabla_h\mathbf{u}\|^2 + \sum_{j\in\{v,c,r\}}\left(\frac{\|\nabla_h q_j\|^2}{2} + \alpha_{\ell j}\int_{\Gamma_\ell}\left(\frac{q_j^2}{2} - q_j q_{b\ell j}\right)\mathrm{d}\Gamma_\ell\right)\right]$$

$$+ \sum_{j\in\{v,c,r\}}\left(\frac{\mu_{q_j}}{2}\|\Delta_h q_j\|^2 + \nu_{q_j}\|\nabla_h\partial_p q_j\|_w^2\right) + \frac{\mu_{\mathbf{u}}}{2}\|\Delta_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p\nabla_h\mathbf{u}\|^2$$

$$\leqslant C\left(1 + \|\nabla_h\mathbf{u}\|^2 + \|\nabla_h T\|^2 + \|\nabla_h\partial_p\mathbf{u}\|^2 + \sum_{j\in\{v,c,r\}}\|\nabla_h\partial_p q_j\|^2\right)$$

$$\times \left(1 + \|\nabla_h\mathbf{u}\|^2 + \sum_{j\in\{v,c,r\}}\|\nabla_h q_j\|^2\right),$$

from which, noticing that

$$\int_{\Gamma_\ell}\left(\frac{q_j^2}{2} - q_j q_{b\ell j}\right)\mathrm{d}\Gamma_\ell = \frac{1}{2}\int_{\Gamma_\ell}\left((q_j - q_{b\ell j})^2 - q_{b\ell j}^2\right)\mathrm{d}\Gamma_\ell \geqslant -\frac{1}{2}\int_{\Gamma_\ell}q_{b\ell j}^2\mathrm{d}\Gamma_\ell \geqslant -C,$$

the conclusion follows by the Gronwall inequality and propositions 4–7. $\qquad\square$

Finally, we have the control of the gradient of the temperature.

Proposition 9. *Let assumption 1 hold, then there exists a function $K_6(t)$, which depends on the initial and boundary data, and is continuous for all $t \geqslant 0$, such that*

$$\sup_{0\leqslant t\leqslant\mathcal{T}}\|\nabla_h T\|^2 + \int_0^{\mathcal{T}}\left(\|\nabla_h^2 T\|^2 + \|\nabla_h\partial_p T\|_w^2\right)\mathrm{d}t \leqslant K_6(\mathcal{T}),$$

and

$$\sup_{0\leqslant t\leqslant\mathcal{T}}\|\partial_p T\|^2 + \int_0^{\mathcal{T}}\left(\|\nabla_h\partial_p T\|^2 + \|\partial_p^2 T\|_w^2\right)\mathrm{d}t \leqslant K_6(\mathcal{T}),$$

for any $\mathcal{T} \in [0, \mathcal{T}_{\max})$.

Proof. We first estimate the vertical derivative $\partial_p T$. Multiplying the thermodynamic equation (31) by $-\partial_p^2 T$ and integrating the resultant over $\mathcal{M}$ yields

$$
\int_{\mathcal{M}} (-\partial_t T + \mathcal{D}^T T) \partial_p^2 T \mathrm{d}\mathcal{M}
$$

$$
= \int_{\mathcal{M}} \left[\mathbf{u} \cdot \nabla_h T + \omega \partial_p T - \frac{R}{c_p} \frac{T}{p} \omega + \frac{L}{c_p} (S_{cd} - S_{ev,\varepsilon}) \right] \partial_p^2 T \mathrm{d}\mathcal{M}. \tag{86}
$$

Following the derivation of (59), (65), and (71), we obtain

$$
-\int_{\mathcal{M}} \partial_t T \partial_p^2 T \mathrm{d}\mathcal{M} \geqslant \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\partial_p T\|^2}{2} + \alpha_{0T} \int_{\mathcal{M}'} \left(\frac{T^2}{2} - T T_{b0} \right) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right) - C(\|\partial_p T\| + 1), \tag{87}
$$

$$
\int_{\mathcal{M}} \mathcal{D}^T T \partial_p^2 T \mathrm{d}\mathcal{M} \geqslant \frac{3}{4} \left(\mu_T \|\nabla_h \partial_p T\|^2 + \nu_T \|\partial_p^2 T\|_w^2 \right) - C(\|\partial_p T\|^2 + 1), \tag{88}
$$

and

$$
\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h T + \omega \partial_p T) \partial_p^2 T \mathrm{d}\mathcal{M}
$$

$$
\leqslant \frac{\mu_T}{12} \|\nabla_h \partial_p T\|^2 + C(\|\nabla_h T\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2 + 1)(1 + \|\partial_p T\|^2). \tag{89}
$$

By lemma 2, proposition 4, proposition 5, proposition 8, Young's inequality, and recalling (83), we deduce

$$
-\int_{\mathcal{M}} \frac{RT}{c_p p} \omega \partial_p^2 T \mathrm{d}\mathcal{M} \leqslant C \int_{\mathcal{M}'} \int_{p_1}^{p_0} |\nabla_h \mathbf{u}| \mathrm{d}p \int_{p_1}^{p_0} |T| |\partial_p^2 T| \mathrm{d}p \mathrm{d}\mathcal{M}
$$

$$
\leqslant C \|\nabla_h \mathbf{u}\|^{\frac{1}{2}} \|\nabla_h^2 \mathbf{u}\|^{\frac{1}{2}} \|T\|^{\frac{1}{2}} \left(\|T\|^{\frac{1}{2}} + \|\nabla_h T\|^{\frac{1}{2}} \right) \|\partial_p^2 T\|
$$

$$
\leqslant \frac{\mu_T}{12} \|\partial_p^2 T\|^2 + C(\|\Delta_h \mathbf{u}\|^2 + \|\nabla_h T\|^2 + 1). \tag{90}
$$

Recalling (44), it follows from Young's inequality and proposition 5 that

$$
\int_{\mathcal{M}} \frac{L}{c_p} (S_{cd} - S_{ev,\varepsilon}) \partial_p^2 T \mathrm{d}\mathcal{M} \leqslant C \int_{\mathcal{M}} |\partial_p^2 T| \mathrm{d}\mathcal{M} \leqslant \frac{\mu_T}{12} \|\partial_p^2 T\|^2 + C. \tag{91}
$$

Substituting (87)–(91) into (86), and recalling propositions 4 and 5, one obtains

$$
\frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\partial_p T\|^2}{2} + \alpha_{0T} \int_{\mathcal{M}'} \left(\frac{T^2}{2} - T T_{b0} \right) \mathrm{d}\mathcal{M}' \bigg|_{p_0} \right) + \left(\frac{\mu_T}{2} \|\nabla_h \partial_p T\|^2 + \frac{\nu_T}{2} \|\partial_p^2 T\|_w^2 \right)
$$

$$
\leqslant C(\|\nabla_h T\|^2 + \|\Delta_h \mathbf{u}\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2 + 1)(1 + \|\partial_p T\|^2),
$$

from which, noticing that

$$
\int_{\mathcal{M}'} (T^2 - 2 T T_{b0}) \mathrm{d}\mathcal{M}' \bigg|_{p_0} = \int_{\mathcal{M}'} \left[(T - T_{b0})^2 - T_{b0}^2 \right] \mathrm{d}\mathcal{M}' \bigg|_{p_0} \geqslant -\int_{\mathcal{M}'} T_{b0}^2 \mathrm{d}\mathcal{M}' \bigg|_{p_0} \geqslant -C,
$$

by the Gronwall inequality, propositions 5, 6 and 8, we obtain

$$\sup_{0 \leqslant t \leqslant \mathcal{T}} \|\partial_p T\|^2 + \frac{1}{2} \int_0^{\mathcal{T}} (\mu_T \|\nabla_h \partial_p T\|^2 + \nu_T \|\partial_p^2 T\|_w^2) \mathrm{d}t \leqslant K_6'(\mathcal{T}). \tag{92}$$

Next, we estimate the horizontal gradient $\nabla_h T$. Multiplying equation (31) by $-\Delta_h T$ and integrating the resultant over $\mathcal{M}$ yield

$$\int_{\mathcal{M}} (-\partial_t T + \mathcal{D}^T T) \Delta_h T \mathrm{d}\mathcal{M}$$
$$= \int_{\mathcal{M}} \left[\mathbf{u} \cdot \nabla_H T + \omega \partial_p T - \frac{R}{c_p} \frac{T}{p} \omega + \frac{L}{c_p} (S_{cd} - S_{ev,\varepsilon}) \right] \Delta_h T \mathrm{d}\mathcal{M}. \tag{93}$$

Following the derivations in (76) and (81), we obtain

$$-\int_{\mathcal{M}} \partial_t T \Delta_h T \mathrm{d}\mathcal{M} \geqslant \frac{\mathrm{d}}{\mathrm{d}t} \left(\frac{\|\nabla_h T\|^2}{2} + \alpha_{\ell T} \int_{\Gamma_\ell} \left(\frac{T^2}{2} - TT_{b\ell} \right) \mathrm{d}\Gamma_\ell \right)$$
$$- C(1 + \|\nabla_h T\|), \tag{94}$$

and

$$\int_{\mathcal{M}} \mathcal{D}^T T \Delta_h T \mathrm{d}\mathcal{M} \geqslant \mu_T \|\Delta_h T\|^2 + \nu_T \|\nabla_h \partial_p T\|_w^2 - C. \tag{95}$$

Similar to (84), we get

$$\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h T + \omega \partial_p T) \Delta_h T \mathrm{d}\mathcal{M}$$
$$\leqslant \frac{\mu_T}{6} \|\Delta_h T\|^2 + \frac{\mu_{\mathbf{u}}}{12} \|\Delta_h \mathbf{u}\|^2 + C(1 + \|\nabla_h \mathbf{u}\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2)(1 + \|\nabla_h T\|^2)$$
$$+ C(1 + \|\nabla_h \partial_p T\|^2)(1 + \|\nabla_h \mathbf{u}\|^2),$$

from which, by proposition 8, one obtains

$$\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h T + \omega \partial_p T) \Delta_h T \mathrm{d}\mathcal{M} \leqslant C(1 + \|\nabla_h \partial_p T\|^2 + \|\nabla_h \partial_p \mathbf{u}\|^2)(1 + \|\nabla_h T\|^2)$$
$$+ \frac{\mu_T}{6} \|\Delta_h T\|^2 + \frac{\mu_{\mathbf{u}}}{12} \|\Delta_h \mathbf{u}\|^2. \tag{96}$$

Following the derivations in (90) and (91), we get

$$-\int_{\mathcal{M}} \frac{RT}{c_p p} \omega \Delta_h T \mathrm{d}\mathcal{M} \leqslant \frac{\mu_T}{6} \|\Delta_h T\|^2 + C(\|\Delta_h \mathbf{u}\|^2 + \|\nabla_h T\|^2 + 1). \tag{97}$$

and

$$\int_{\mathcal{M}} \frac{L}{c_p} (S_{cd} - S_{ev,\varepsilon}) \Delta_h T \mathrm{d}\mathcal{M} \leqslant \frac{\mu_T}{6} \|\Delta_h T\|^2 + C. \tag{98}$$

Substituting (94)–(98) into (93) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\|\nabla_h T\|^2}{2} + \alpha_{\ell T}\int_{\Gamma_\ell}\left(\frac{T^2}{2} - TT_{b\ell}\right)\mathrm{d}\Gamma_\ell\right) + \frac{1}{2}(\mu_T\|\Delta_h T\|^2 + \nu_T\|\nabla_h\partial_p T\|_w^2)$$
$$\leqslant C(1 + \|\nabla_h\partial_p T\|^2 + \|\Delta_h\mathbf{u}\|^2 + \|\nabla_h\partial_p\mathbf{u}\|^2)(1 + \|\nabla_h T\|^2),$$

from which, noticing that

$$\int_{\Gamma_\ell}\left(\frac{T^2}{2} - TT_{b\ell}\right)\mathrm{d}\Gamma_\ell = \frac{1}{2}\int_{\Gamma_\ell}\left((T - T_{b\ell})^2 - T_{b\ell}^2\right)\mathrm{d}\Gamma_\ell \geqslant -\frac{1}{2}\int_{\Gamma_\ell}T_{b\ell}^2\mathrm{d}\Gamma_\ell \geqslant -C,$$

it follows from the Gronwall inequality, propositions 6 and 8, (92), and (83) that

$$\sup_{0\leqslant t\leqslant\mathcal{T}}\|\nabla_h T\|^2 + \int_0^{\mathcal{T}}\left(\|\nabla_h^2 T\|^2 + \|\nabla_h\partial_p T\|_w^2\right)\mathrm{d}t \leqslant K_6''(\mathcal{T}),$$

where $K_6''(t)$ depends on the initial and boundary data, and is continuous for all $t \geqslant 0$. Combining this with (92) yields the conclusion. $\square$

As a corollary of propositions 1–9, we have the following global existence and *a priori* estimates for system (28)–(34), subject to (24)–(27).

Corollary 1. *Assume that* $\mathbf{u}_0, T_0, q_{v0}, q_{c0}, q_{r0} \in H^1(\mathcal{M})$ *and* $T_0, q_{v0}, q_{c0}, q_{r0} \in L^\infty(\mathcal{M})$*, with* $T_0, q_{v0}, q_{c0}, q_{r0} \geqslant 0$ *on* $\mathcal{M}$ *and* $\int_{p_0}^{p_1}\nabla_h\cdot\mathbf{u}_0\mathrm{d}p = 0$ *on* $\mathcal{M}'$*. Then, system* (28)–(34)*, subject to* (24)–(27) *has a unique global strong solution* $(\mathbf{u}, T, q_v, q_c, q_r)$*, satisfying*

$$T, q_v, q_c, q_r \geqslant 0, \quad and \quad T, q_v, q_c, q_r \in L^\infty(0, \mathcal{T}; L^\infty),$$

$$\mathbf{u}, T, q_v, q_c, q_r \in C([0, \mathcal{T}]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M})),$$

$$\partial_t\mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r \in L^2(0, \mathcal{T}; L^2(\mathcal{M})),$$

and the a priori *estimate*

$$\sup_{0\leqslant t\leqslant\mathcal{T}}(\|(T, q_v, q_c, q_r)\|_{L^\infty(\mathcal{M})} + \|(\mathbf{u}, T, q_v, q_c, q_r)\|_{H^1(\mathcal{M})}) \leqslant K(\mathcal{T}),$$

and

$$\int_0^{\mathcal{T}}(\|(\mathbf{u}, q_v, q_c, q_r)\|_{H^2(\mathcal{M})}^2 + \|(\partial_t\mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r)\|^2)\mathrm{d}t \leqslant K(\mathcal{T}),$$

for any $\mathcal{T} \in (0, \infty)$*, where K is a continuous function on* $[0, \infty)$*, depending only on the initial and boundary data, which is independent of* $\varepsilon \in (0, 1)$*.*

Proof. We need to prove $\mathcal{T}_{\max} = \infty$. Assume, by contradiction, that $\mathcal{T}_{\max} < \infty$. By propositions 2–9, we have the estimate

$$\sup_{0\leqslant t\leqslant\mathcal{T}}\|(\mathbf{u}, T, q_v, q_c, q_r)\|_{H^1(\mathcal{M})} \leqslant C_0,$$

for any $\mathcal{T} \in (0, \mathcal{T}_{\max})$, and C_0 is a positive constant depending on the initial and boundary data, but which is independent of $\mathcal{T} \in (0, \mathcal{T}_{\max})$. This contradicts (38) and, thus, $\mathcal{T}_{\max} = \infty$. The *a priori* estimates except those involving the time derivatives follow directly from propositions 2–9; while the desired estimates for the time derivative follow from those in propositions 2–9, using equations (28), (31)–(34). The proof is lengthy but standard, and therefore omitted. $\square$

4. Global existence and uniqueness

We are now ready to prove the global existence and uniqueness result, i.e., theorem 2.

Proof of theorem 1.   (i) Existence. By corollary 3 for any $\varepsilon \in (0,1)$, there is a unique global solution $(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon})$ satisfying

$$T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon} \geqslant 0, \tag{99}$$

and the *a priori* estimates

$$\sup_{0\leqslant t\leqslant \mathcal{T}} (\|(T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon})\|_{L^\infty(\mathcal{M})} + \|(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon})\|_{H^1(\mathcal{M})}) \leqslant K(\mathcal{T}),$$

$$\tag{100}$$

and

$$\int_0^{\mathcal{T}} (\|(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon})\|_{H^2(\mathcal{M})}^2 + \|(\partial_t \mathbf{u}_\varepsilon, \partial_t T_\varepsilon, \partial_t q_{v\varepsilon}, \partial_t q_{c\varepsilon}, \partial_t q_{r\varepsilon})\|^2)\mathrm{d}t \leqslant K(\mathcal{T}),$$

$$\tag{101}$$

for any $\mathcal{T} \in (0, \infty)$, for a continuous function K on $[0, \infty)$ independent of ε. Thanks to the *a priori* estimates (100) and (101), by the Banach–Alaoglu theorem, and using Cantor's diagonal argument to ε, there is a subsequence (still denoted by $(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon})$), and $(\mathbf{u}, T, q_v, q_c, q_r)$, such that

$$(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon}) \rightharpoonup^* (\mathbf{u}, T, q_v, q_c, q_r) \quad \text{in } L^\infty(0, \mathcal{T}; H^1(\mathcal{M})), \tag{102}$$

$$(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon}) \rightharpoonup (\mathbf{u}, T, q_v, q_c, q_r) \quad \text{in } L^2(0, \mathcal{T}; H^2(\mathcal{M})), \tag{103}$$

and

$$(\partial_t \mathbf{u}_\varepsilon, \partial_t T_\varepsilon, q_{v\varepsilon}, \partial_t q_{c\varepsilon}, \partial_t q_{r\varepsilon}) \rightharpoonup (\partial_t \mathbf{u}, \partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r) \quad \text{in } L^2(0, \mathcal{T}; L^2(\mathcal{M})),$$

$$\tag{104}$$

where $\rightharpoonup$ and $\rightharpoonup^*$, respectively, denote the weak and weak-$*$ convergences in the corresponding spaces. By the Aubin–Lions compactness lemma (see e.g. [21]), it follows from (102)–(104) that

$$(\mathbf{u}_\varepsilon, T_\varepsilon, q_{v\varepsilon}, q_{c\varepsilon}, q_{r\varepsilon}) \to (\mathbf{u}, T, q_v, q_c, q_r) \quad \text{in } C([0, \mathcal{T}]; L^2(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^1(\mathcal{M})),$$

$$\tag{105}$$

from which, recalling (99) and (100), one has

$$(T, q_v, q_c, q_r) \in L^\infty(0, \mathcal{T}; L^\infty(\mathcal{M})), \quad T, q_v, q_c, q_r \geqslant 0. \tag{106}$$

Thanks to the convergences (102)–(105), we can take the limit as $\varepsilon \to 0$ to show that $(\mathbf{u}, T, q_v, q_c, q_r)$ is a solution to system (10)–(13), (21)–(23), subject to (24)–(27).

(ii) Uniqueness. Let $(\mathbf{u}_i, T_i, q_{vi}, q_{ci}, q_{ri})$, $i = 1, 2$, be two solutions, and denote by $(\mathbf{u}, T, q_v, q_c, q_r)$ their difference. Then $\mathbf{u}$ satisfies

$$\partial_t \mathbf{u} + (\mathbf{u_1} \cdot \nabla_h)\mathbf{u} + \omega_1 \partial_p \mathbf{u} + f(k \times \mathbf{v})_h - \mathcal{D}^\mathbf{u}\mathbf{u} = -(\mathbf{u} \cdot \nabla_h)\mathbf{u}_2 - \omega\partial_p\mathbf{u}_2 - \nabla_h\Phi,$$

3232

where $\Phi = \Phi_1 - \Phi_2$. Multiplying the above equation by **u** and integrating over $\mathcal{M}$ yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 - \int_{\mathcal{M}} \mathcal{D}^{\mathbf{u}}\mathbf{u} \cdot \mathbf{u}\mathrm{d}\mathcal{M} = -\int_{\mathcal{M}}[(\mathbf{u}\cdot\nabla_h)\mathbf{u}_2 + \omega\partial_p\mathbf{u}_2 + \nabla_h\Phi]\cdot\mathbf{u}\mathrm{d}\mathcal{M}.$$
(107)

As in (46), we have

$$-\int_{\mathcal{M}}\mathcal{D}^{\mathbf{u}}\mathbf{u}\cdot\mathbf{u}\mathrm{d}\mathcal{M} \geqslant \mu_{\mathbf{u}}\|\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p u\|_w^2.$$
(108)

By (23), the boundary condition (26) for **u**, and noticing that $\int_{p_1}^{p_0}\nabla_h\cdot\nabla\mathbf{u}\mathrm{d}p = 0$, we obtain from integration by parts

$$-\int_{\mathcal{M}}\nabla_h\Phi\cdot\mathbf{u}\mathrm{d}\mathcal{M} = \int_{\mathcal{M}}\Phi\nabla_h\cdot\mathbf{u}\mathrm{d}\mathcal{M} = \int_p^{p_0}\frac{RT}{\sigma}\mathrm{d}\sigma\nabla_h\cdot\mathbf{u}\mathrm{d}\mathcal{M}$$

$$\leqslant C\|\nabla_h\mathbf{u}\|\|T\| \leqslant \frac{\mu_{\mathbf{u}}}{6}\|\nabla_h\mathbf{u}\|^2 + C\|T\|^2.$$
(109)

Noticing that $|f| \leqslant \frac{1}{p_0-p_1}\int_{p_1}^{p_0}|f|\mathrm{d}p + \int_{p_1}^{p_0}|\partial_p f|\mathrm{d}p$, it follows from lemma 2 (see appendix) that

$$\int_{\mathcal{M}}(\mathbf{u}\cdot\nabla_h)\mathbf{u}_2\cdot\mathbf{u}\mathrm{d}\mathcal{M} \leqslant C\int_{\mathcal{M}'}\int_{p_1}^{p_0}(|\nabla_h\mathbf{u}_2| + |\nabla_h\partial_p\mathbf{u}_2|)\mathrm{d}p\int_{p_1}^{p_0}|\mathbf{u}|^2\mathrm{d}\mathcal{M}'$$

$$\leqslant C\|(\nabla_h\mathbf{u}_2,\nabla_h\partial_p\mathbf{u}_2)\|\|\mathbf{u}\|\|(\mathbf{u},\nabla_h\mathbf{u})\|$$

$$\leqslant \frac{\mu_{\mathbf{u}}}{6}\|\nabla_h\mathbf{u}\|^2 + C(1 + \|(\nabla_h\mathbf{u}_2,\nabla_h\partial_p\mathbf{u}_2)\|^2)\|\mathbf{u}\|^2. \quad (110)$$

Again by lemma 2 and Young's inequality, we have moreover

$$-\int_{\mathcal{M}}\omega\partial_p\mathbf{u}_2\mathbf{u}\mathrm{d}\mathcal{M} \leqslant \int_{\mathcal{M}'}\int_{p_1}^{p_0}|\nabla_h\mathbf{u}|\mathrm{d}p\int_{p_1}^{p_0}|\partial_p\mathbf{u}_2|\|\mathbf{u}|\mathrm{d}p\mathrm{d}\mathcal{M}'$$

$$\leqslant C\|\nabla_h\mathbf{u}\|\|\partial_p\mathbf{u}_2\|^{\frac{1}{2}}\|(\partial_p\mathbf{u_2},\nabla_h\partial_p\mathbf{u}_2)\|^{\frac{1}{2}}\|\mathbf{u}\|^{\frac{1}{2}}\|(\mathbf{u},\nabla_h\mathbf{u})\|^{\frac{1}{2}}$$

$$\leqslant \frac{\mu_{\mathbf{u}}}{6}\|\nabla_h\mathbf{u}\|^2 + C\left(1 + \|\partial_p\mathbf{u}_2\|^2\|(\partial_p\mathbf{u}_2,\nabla_h\partial_p\mathbf{u}_2)\|^2\right)\|\mathbf{u}\|^2.$$
(111)

Substituting (108)–(111) into (107) and using the regularities of $(\mathbf{u}_i, T_i, q_{vi}, q_{ci}, q_{ri})$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\mathbf{u}\|^2 + \mu_{\mathbf{u}}\|\nabla_h\mathbf{u}\|^2 + \nu_{\mathbf{u}}\|\partial_p\mathbf{u}\|_w^2 \leqslant C(1 + \|\nabla_h\partial_p\mathbf{u}_2\|^2)\left(\|\mathbf{u}\|^2 + \|T\|^2\right).$$
(112)

For the estimates of the differences for the temperature and moisture components, we proceed in a similar fashion to [14] by introducing the new unknowns as

$$Q_i = q_{vi} + q_{ri}, \quad H_i = T_i - \frac{L}{c_p}(q_{ci} + q_{ri}),$$

3233

$i = 1, 2$, and let $Q = Q_1 - Q_2, H = H_1 - H_2$ and $q_j = q_{j1} - q_{j2}$ for $j \in \{v, c, r\}$ be the corresponding differences. The source terms for these quantities reveal helpful cancellation properties allowing to prove uniqueness of Q, q_c, q_r, H, which then further implies the uniqueness of the solution in terms of the original unknowns T, q_v, q_c, q_r. Following the argument of proposition 4.1 in [14], we have the estimates

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|Q\|^2 + \frac{\mu_{q_v}}{4}\|\nabla_h Q\|^2 + \frac{\nu_{q_v}}{4}\|\partial_p Q\|_w^2$$

$$\leqslant \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h Q_2 + \omega \partial_p Q_2) Q \mathrm{d}\mathcal{M} + C\|(Q, q_r, q_c, H)\|^2$$

$$+ C_Q \left(\mu_{q_r}\|\nabla_h q_r\|^2 + \nu_{q_r}\|\partial_p q_r\|_w^2 \right), \tag{113}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q_c\|^2 + \mu_{q_c}\|\nabla_h q_c\|^2 + \nu_{q_c}\|\partial_p q_c\|_w^2$$

$$\leqslant \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_{c2} + \omega \partial_p q_{c2}) q_c \mathrm{d}\mathcal{M} + C\|(Q, q_r, q_c, H)\|^2, \tag{114}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|q_r\|^2 + \frac{\mu_{q_r}}{4}\|\nabla_h q_r\|^2 + \frac{\nu_{q_r}}{4}\|\partial_p q_r\|_w^2$$

$$\leqslant \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h q_{r2} + \omega \partial_p q_{r2}) q_r \mathrm{d}\mathcal{M} + C\|(Q, q_r, q_c, H)\|^2, \tag{115}$$

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|H\|^2 + \frac{\mu_T}{2}\|\nabla_h H\|^2 + \frac{\nu_T}{2}\|\partial_p H\|_w^2$$

$$\leqslant \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h H_2 + \omega \partial_p H_2) H \mathrm{d}\mathcal{M} + C_H \left(\mu_{q_c}\|\nabla_h q_c\|^2 + \mu_{q_r}\|\nabla_h q_r\|^2 \right)$$

$$+ C_H(\nu_{q_c}\|\partial_p q_c\|^2 + \nu_{q_r}\|\partial_p q_r\|^2) + C\|(H, q_r, q_c)\|^2. \tag{116}$$

For the integrals involving the convection terms in the above inequalities, thanks to the boundary conditions (24)–(26) for $\mathbf{u}$ and ω, it follows from integration by parts that

$$\int_{\mathcal{M}} (\mathbf{u} \cdot \nabla \varphi_2 + \omega \partial_p \varphi_2) \varphi \mathrm{d}\mathcal{M} = - \int_{\mathcal{M}} (\mathbf{u} \cdot \nabla_h \varphi + \omega \varphi) \varphi_2 \mathrm{d}\mathcal{M}$$

$$\leqslant \eta(\|\nabla_h \mathbf{u}\|^2 + \|\nabla_h \varphi\|^2) + C_\eta \|\varphi_2\|_{L^\infty}^2 (\|\mathbf{u}\|^2 + \|\varphi\|^2)$$

$$\leqslant \eta(\|\nabla_h \mathbf{u}\|^2 + \|\nabla_h \varphi\|^2) + C_\eta(\|\mathbf{u}\|^2 + \|\varphi\|^2), \tag{117}$$

for $\varphi \in \{Q, q_c, q_r, H\}$, and for any positive η, where in the last step we have used the uniform boundedness of the moisture components. Multiplying (114) and (115) by a sufficient large positive number A, adding the resultants with (114) and (115), and using (117), with η sufficiently small, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \left(\|(Q, H)\|^2 + A\|(q_r, q_c)\|^2 \right) \leqslant 4\eta\|\nabla_h \mathbf{u}\|^2 + C_\eta \|(\mathbf{u}, Q, q_r, q_c, H)\|^2, \tag{118}$$

for a sufficiently small positive η.

Combining (112) with (118) and choosing η sufficiently small, one obtains

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|(Q,H)\|^2 + A\|(q_r, q_c)\|^2 + \|\mathbf{u}\|^2\right)$$

$$\leqslant C(1 + \|\nabla_h \partial_p \mathbf{u}_2\|^2)\|(\mathbf{u}, Q, q_r, q_c, H)\|^2.$$

Since due to the regularities of the solution we have $\|\nabla_h \partial_p \mathbf{u}_2\|^2 \in L^1((0, \mathcal{T}))$ for any $\mathcal{T} \in (0, \infty)$, the conclusion follows using Gronwall's inequality. $\qquad\square$

Acknowledgments

Appendix

Lemma 1. *For any measurable function f satisfying* $f, \partial_p f \in L^1(\mathcal{M})$ *the following estimate holds*

$$\sup_{p_1 \leqslant p \leqslant p_0} \|f\|_{L^1(\mathcal{M}')} \leqslant \frac{\|f\|_{L^1(\mathcal{M})}}{p_0 - p_1} + \|\partial_p f\|_{L^1(\mathcal{M})}.$$

Proof. For any $p, q \in [p_1, p_0]$

$$\int_{\mathcal{M}'} |f(x,y,p)|\mathrm{d}\mathcal{M}' = \int_{\mathcal{M}'} |f(x,y,q)|\mathrm{d}\mathcal{M}' + \int_q^p \int_{\mathcal{M}'} \partial_p |f(x,y,\xi)|\mathrm{d}\mathcal{M}'\mathrm{d}\xi.$$

Integrating the above equality with respect to q over (p_1, p_0) yields

$$\int_{\mathcal{M}'} |f(x,y,p)|\mathrm{d}\mathcal{M}' = \frac{1}{p_0 - p_1}\int_{p_1}^{p_0}\left(\int_{\mathcal{M}'}|f|\mathrm{d}\mathcal{M}' + \int_q^p \int_{\mathcal{M}'} \partial_p|f(x,y,\xi)|\mathrm{d}\mathcal{M}'\mathrm{d}\xi\right)\mathrm{d}q$$

$$\leqslant \frac{\|f\|_{L^1(\mathcal{M})}}{p_0 - p_1} + \|\partial_p f\|_{L^1(\mathcal{M})},$$

from which, by taking the superium in p, the conclusion follows. $\qquad\square$

We will also use the following lemma from [2], where we also refer to [5, 7] for some similar inequalities.

Lemma 2 (See lemma 2.1 in [2]). *The following inequalities hold*

$$\int_{\mathcal{M}'}\left(\int_{p_1}^{p_0}|\phi|\mathrm{d}p\right)\left(\int_{p_1}^{p_0}|\varphi\psi|\mathrm{d}p\right)\mathrm{d}\mathcal{M}'$$

$$\leqslant C\|\phi\|\|\varphi\|^{\frac{1}{2}}\left(\|\varphi\|^{\frac{1}{2}} + \|\nabla_h\varphi\|^{\frac{1}{2}}\right)\|\psi\|^{\frac{1}{2}}\left(\|\psi\|^{\frac{1}{2}} + \|\nabla_h\psi\|^{\frac{1}{2}}\right),$$

and

$$\int_{\mathcal{M}'} \left(\int_{p_1}^{p_0} |\phi| \mathrm{d}p \right) \left(\int_{p_1}^{p_0} |\varphi\psi| \mathrm{d}p \right) \mathrm{d}\mathcal{M}'$$

$$\leqslant C\|\phi\|^{\frac{1}{2}} \left(\|\phi\|^{\frac{1}{2}} + \|\nabla_h \phi\|^{\frac{1}{2}} \right) \|\varphi\|^{\frac{1}{2}} \left(\|\varphi\|^{\frac{1}{2}} + \|\nabla_h \varphi\|^{\frac{1}{2}} \right) \|\psi\|,$$

for any measurable functions ϕ, φ, and ψ such the quantities on the right-hand sides are finite, where C is a positive constant depending only on p_1, p_0, and $\mathcal{M}'$.

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