Atmospheric predictability: the origins of the finite-time behaviour

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ABSTRACT

First proposed by Edward N. Lorenz in 1969, the existence of a finite-time barrier in deterministically predicting atmospheric flows is now well-accepted in the community of dynamical meteorology. The present work argues via numerical simulation that Lorenz's model may be over-simplified. The apparent contradiction between a finite-time barrier to predictability and the proof of well-posedness of the incompressible two-dimensional Navier-Stokes equations, regardless of the slope of the kinetic energy spectrum, is reconciled through understanding of this slope's practical role in a particular error bound of the analytic proof.

1. Introduction

Now an accepted fact in dynamical meteorology, the existence of a finite-time barrier in deterministically predicting atmospheric flows was first conceptually shown by Lorenz (1969). In his seminal paper, he made the distinction between fluid systems whose error at any future time can be made arbitrarily small by suitably reducing the initial error, and those whose error at any future time cannot be reduced below a certain limit unless the initial error is zero. These systems were characterised in terms of *range of predictability* (or simply *predictability*; the reader is referred to Appendix A for a motivation of the concept): the former category has an infinite range and the latter has only a finite range. By modelling atmospheric flows by the two-dimensional (2D) barotropic vorticity equation and assuming a $-\frac{5}{3}$ exponent in the power-law relationship of the kinetic energy (KE) spectrum (the *spectral slope*, as a power-law in a log-log plot is a straight line with the slope being the exponent) for the unperturbed flow, he argued that such flows have a finite range of predictability.

Although the barotropic vorticity equation with large-scale forcing produces a steeper spectral slope of -3, and unbalanced dynamics are required to produce a spectral slope of $-\frac{5}{3}$ in more realistic models (Sun and Zhang 2016), it has been shown that predictability is determined much more by the spectral slope than by the nature of the dynamics (Rotunno and Snyder 2008). Thus, it is appropriate to use the barotropic vorticity system to study predictability with a range of spectral slopes.

Closely related to this system are the incompressible 2D Navier-Stokes (2D-NS) equations, whose well-posedness was first rigorously shown by Ladyzhenskaya also in the second half of the twentieth century (Robinson 2001). As we will see in Section 4, it is not difficult to show that well-posedness implies an infinite range of predictability in the sense of Lorenz.

The present paper attempts to bridge the gap between the finite predictability result of Lorenz and the infinite predictability corollary of Ladyzhenskaya's proof, in the context of incompressible 2D flows. Section 2 reviews Lorenz's argument of its finite-time behaviour. In Section 3 we reproduce Lorenz's numerical results and discuss the predictability in the directly simulated 2D barotropic vorticity model. An account of the well-posedness and infinite predictability of the incompressible 2D-NS equations is presented in Section 4, with which we reconcile Lorenz's result of finite predictability in Section 5. The major findings are summarised in Section 6.

2. Lorenz's argument of finite predictability

The model of Lorenz (1969) is based on the dimensionless 2D barotropic vorticity equation

$$\frac{\partial \theta}{\partial t} + J(\psi, \theta) = 0, \qquad \theta = \Delta\psi \tag{1}$$

where ψ is the velocity streamfunction (related to the velocity $\boldsymbol{u}$ by $\boldsymbol{u} = -\nabla \times (\psi\hat{\mathbf{k}})$), $\Delta = \nabla \cdot \nabla, \nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ and $J(A,B) = \frac{\partial A}{\partial x}\frac{\partial B}{\partial y} - \frac{\partial A}{\partial y}\frac{\partial B}{\partial x}$. Assuming a doubly periodic domain, Lorenz expanded the variables ψ and θ in Fourier series and re-wrote the linearised error equation of (1) in Fourier components. Then he made various assumptions to an ensemble of error fields for the linearised error equation of (1), most notably homogeneity and a slight generalisation of the quasi-normal closure. The resulting equation was then passed into the large-domain and continuous-spectrum limit.

The derivation would have been more straightforward if the domain were the whole $\mathbb{R}^2$ space and the variables were Fourier-transformed rather than expanded in Fourier series. We have checked that this method indeed returns the same equation as the limiting equation of Lorenz, up to a constant multiplicative factor.

A further assumption of isotropy simplifies the equation, which was then discretised and numerically approximated. Depending on the specification of a KE spectrum for the unperturbed flow, a matrix of constant coefficients C was constructed so that the vector Z of error KE at different scales (each scale K collectively represents wavenumbers $k = 2^{K-1}$ to $k = 2^K$) evolves according to the linear model

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} Z = CZ, \qquad \text{or equivalently} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} Z \\ W \end{pmatrix} = \begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}\begin{pmatrix} Z \\ W \end{pmatrix}. \tag{2}$$

As Rotunno and Snyder (2008) mentioned, the computation of C involves computing integrals of nearly singular functions. We have been cautious about these integrations and have made sure that our integrations for C are accurate, some details of which are provided in Appendix B.

To time-integrate equation (2), it is necessary that the initial conditions for Z and its first derivative W are specified. Lorenz did not explicitly give an initial condition for W, although it is safe to assume $W(t = 0) \equiv 0$ as was prescribed in the predictability experiments of Rotunno and Snyder (2008). The non-linear effects were accounted for by removing the corresponding components of Z and C when the error KE saturated at a particular scale. Time-integration with the resulting lower-dimensional system was carried on, until all scales became saturated.

As Lorenz noted down the saturation times t_K of scale K, he found that the successive differences $t_K - t_{K+1}$ behaved approximately proportional to $2^{-\beta K}$ with β depending on the spectral slope. He therefore concluded that, given an initial error at an infinitesimally small scale, the range of predictability is finite if and only if the telescoping series

$$t_K = \sum_{j=K}^{\infty} (t_j - t_{j+1}) = \sum_{j=K}^{\infty} 2^{-\beta j} \tag{3}$$

is summable, which is the case if and only if $\beta > 0$. By observing $\beta = \frac{2}{3}$ for the atmospherically relevant spectral slope of $-\frac{5}{3}$, he concluded finite predictability for the atmosphere. Additionally, he found that $\beta = \frac{1}{3}$ for a hypothetical spectral slope of $-\frac{7}{3}$. Lorenz thus hypothesised by linear extrapolation that the range of predictability would be infinite if the spectral slope were steepened to -3.

3. Numerical simulations

We performed a series of numerical simulations, first on the Lorenz model (2) followed by a forced-dissipative version of the full 2D barotropic vorticity system (1), to see whether infinite predictability is indeed achieved with a KE spectral slope of -3 as Lorenz hypothesised.

a. Lorenz's model

Rotunno and Snyder (2008) solved for the growth of the error KE spectrum for a background spectral slope of $-p$ where $p = 3$. In order to assess the range of predictability in Lorenz's framework, we extended their calculations to study the relationship between K and t_K.

Having computed the matrix C as in Rotunno and Snyder (2008), we solved the linear matrix system (2) explicitly, that is, by writing out the general solution in terms of the eigenvalues and eigenvectors of

$$\begin{pmatrix} 0 & I \\ C & 0 \end{pmatrix}$$

and projecting the initial condition onto such an eigenspace to determine the constants of the general solution. An advantage of this exact approach to solving the system is that it eliminates the unstable effects brought over by the positive eigenvalues in any numerical scheme, such as those used in Lorenz (1969), Rotunno and Snyder (2008) and its extension by Durran and Gingrich (2014).

Figure 1 shows the saturation times t_K as a function of the scale K for the -3 spectrum as in Rotunno and Snyder (2008), and Figure 2 shows the evolution of the error spectrum. Note that in Figure 1 t_K is plotted instead of $t_K - t_{K+1}$ against K, but the choice makes little difference when $\beta > 0$ since if $t_K - t_{K+1}$ is proportional to $2^{-\beta K}$ then so is t_K (cf. equation (3)). It is clear that the saturation times t_K scale as $2^{-\beta K}$ with a small but positive β (0.05) along the inertial range, so that the sum in equation (3) is still finite for $p = 3$, contrary to Lorenz's prediction. Indeed, arguing in the same way as Lorenz, our result indicates finite predictability for a -3 spectrum which is contrary to Lorenz's hypothesis, although we acknowledge that $\beta = 0.05$ is just marginally away from the critical value of zero.

Additionally, we also re-ran Lorenz's model with background KE spectra of $p = \frac{5}{3}$ and $p = \frac{7}{3}$. It turned out that the K-t_K relationship is not that straightforward: in some cases it is hard to even fit a β to the relationship because the graphs look so different to $2^{-\beta K}$. Yet, even with our 'nicest' results, the central assumption to Lorenz's hypothesis (that β varies linearly with p) does not hold: we found that $\beta = 0.67, 0.27$ for $p = \frac{5}{3}, \frac{7}{3}$ respectively based on our computations. Since we have no access to his original code, we are unable to explain our divergence with Lorenz who concluded $\beta = \frac{1}{3}$ for $p = \frac{7}{3}$.

b. Forced-dissipative 2D barotropic vorticity equation

Among the spectral slopes where our values of β differ from Lorenz's, the -3 spectrum deserves particular attention because the difference amounts to a qualitative contrast between finite and infinite ranges of predictability. To further investigate this, we performed direct numerical simulations (DNS) on this $p = 3$ spectrum in the form of identical-twin experiments (pairs of runs which only differ in the initial condition), and assessed the predictability following Lorenz's methodology with necessary adaptations.

First of all, equation (1) had to be restricted to a doubly periodic domain and be made forced-dissipative:

$$\frac{\partial \theta}{\partial t} + J(\psi, \theta) = f + d, \qquad \theta = \Delta \psi. \tag{4}$$

The forcing and dissipation, however small, are necessary for generating statistically stationary KE spectra in the DNS. To generate a -3 spectral slope, forcing was applied at the large scale: $f(t)$ was chosen to be an independent white-noise process for each 2D wavevector whose scalar wavenumber k falls in the narrow band ($\pm 10\%$) around $k = 20$. The dissipation d was a highly scale-selective hyperviscosity $d \sim -\Delta^6 \theta$.

It is worth noting that equation (4) would also be the vorticity form of the incompressible 2D-NS equations

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p + f(x,t) + \nu \Delta \boldsymbol{u}, \qquad \nabla \cdot \boldsymbol{u} = 0 \tag{5}$$

if d were chosen to be $d = \nu \Delta \theta, \nu > 0$. We would have liked to run these DNS on the 2D-NS equations which will be discussed in Section 4, but the build-up of KE at the smallest scales as a numerical artefact was so strong that we had to either increase ν – which would substantially shorten the inertial range and thus reduce the reliability of our conclusions – or choose a more scale-selective dissipation. We opted for the latter.

We performed five pairs of identical-twin runs on equation (4) by varying the random seed that generated the pre-perturbation (original) initial condition. Within each pair, notably, the realisations of the large-scale stochastic forcing $f(t)$ in the control and perturbed runs were identical. The model was pseudo-spectral with a truncation wavenumber of $k_t = 512$, in which the $J(\psi, \theta)$ term was computed in the physical domain via a pair of Fast Fourier Transforms with the spectral de-aliasing filter proposed by Hou and Li (2007). The original initial condition for each of the five cases was an already-developed homogeneous and (approximately) isotropic turbulence with a clean logarithmically corrected -3 spectrum in the inertial range (Figure 3), which has been shown to be a more accurate description of the large-scale-forced 2D turbulent spectrum for finite inertial ranges (Bowman 1996).

The perturbations were introduced spectrally at each of the 2D wavevectors $\boldsymbol{k}$ for a specified value of $k_p = |\boldsymbol{k}|$. A random phase shift independently drawn from a uniform distribution was applied on a pre-determined part $\gamma \in [0,1]$ of the spectral coefficients $\hat{\psi}(\boldsymbol{k})$ and thus $\hat{\theta}(\boldsymbol{k})$, where the hat indicates Fourier coefficients. It can be shown that $\gamma(k_p)$ and $\mathbb{E}(E_e(k_p))$, the expected value of the one-dimensional error KE spectral density at wavenumber k_p, are related by $\mathbb{E}(E_e(k_p)) = 2\gamma^2 E(k_p)$, where $E(\cdot)$ is the one-dimensional KE spectral density of the background flow. By specifying the relative error $\frac{\mathbb{E}(E_e(k_p))}{E(k_p)}$, we could work out γ and thus generate the perturbation fields, to which we added the original initial conditions to obtain the perturbed initial conditions.

Since our truncation wavenumber $k_t = 512$ corresponds to $K = 9$ of Lorenz's paper, there would only be 8 values of $t_K - t_{K+1}$, among which only 4 or 5 would be in the inertial range. It would be inaccurate to determine β from such a few data points, so we have transformed Lorenz's argument to incorporate information from all wavenumbers k, not only from the scale K as a collection of wavenumbers.

To transform the argument, recall that $t_K \sim 2^{-\beta K}$ when $\beta > 0$. Since $k \sim 2^K$ and both t_K and T represent saturation times, we may write $T \sim k^{-\beta}$ and conclude that the T should vary with k as a power-law if Lorenz's results hold. The argument will break down when β becomes zero, that is, when the threshold for infinite predictability is reached.

In this study, the perturbations were introduced at $k_p = 256$. The saturation threshold was chosen to be 1.315 times the KE spectral density of the control flow, or equivalently 0.6575 times the maximum permissible error energy, in accordance with Lorenz (1969) (we applied sensitivity tests and found that the results are largely insensitive to the saturation threshold). Figures 4 and 5 show respectively the evolution of the error KE spectrum, and the saturation times T across different wavenumbers k which fit the $T \sim k^{-\beta}$ relationship for $\beta = 0.24$, averaged over the five cases. (The five cases exhibited very similar qualitative behaviour, showing that our results are robust to initial conditions, hence justifying the use of averaging to obtain smoother results.) Based on the transformed version of Lorenz's argument, our result also suggests finite predictability for a (logarithmically corrected) -3 spectrum, this time with greater confidence as β is further away from zero.

4. Aspects from PDE theory: the incompressible 2D Navier-Stokes equations

Although one may argue that the β found in the DNS (Section 3(b)) confirmed our earlier result of finite predictability with Lorenz's model (Section 3(a)), the non-trivial difference in the values of β raises a concern that Lorenz's argument may be over-simplified, particularly in respect of the implied extrapolation of equation (3) to an infinitesimally small scale and the assumed linear p-β relationship.

A very different approach to the problem of finite or infinite predictability is via use of the more rigorous mathematical theory of partial differential equations (PDEs). The incompressible 2D-NS equations (5), where we shall drop the word 'incompressible' for the remainder of the paper, are always useful as a pedagogical first step towards understanding and modelling the motion of real fluid flows in the atmosphere. As such, the analytical properties of the 2D-NS problem have been extensively studied. We are however unaware of anyone taking this approach to address predictability until recently.

Well-posedness and implications on predictability

Unlike their three-dimensional counterpart whose regularity problem remains open, the initial-value problem for the 2D-NS equations on the torus (i.e. a doubly periodic domain) has been proven to be well-posed, by which we mean the existence of a unique solution that depends continuously on the initial conditions. Proofs of its well-posedness, for both strong and weak solutions respectively, can be found in the book by Robinson (2001). In the present paper we shall use his proof for weak solutions to demonstrate that the 2D-NS system is infinitely predictable. To set the context, a summary of the uniqueness proof is provided below. Interested readers may refer to Robinson's book for a full proof.

First of all, the 2D-NS equations (5) are cast in the form of an ordinary differential equation in an appropriate function space depending on an arbitrary, fixed positive time T. An equation for the error velocity field $w = u - v$ of two solutions u and v is formulated, and its inner product with w itself is taken to obtain an equation for the time-evolution of the error energy $\frac{1}{2}\|w\|^2$, where $\|\cdot\|$ is the L^2 norm on the torus. This equation contains a term which can be bounded above by Ladyzhenskaya's inequalities (Robinson 2001) specific to the 2D-NS equations. After some work one uses Grönwall's inequality to show that

$$\|w(t)\|^2 \leq \exp\left(\int_0^t \frac{M}{\nu}\|\nabla u(s)\|^2 \,\mathrm{d}s\right)\|w(0)\|^2, \qquad t \in [0,T],$$

where M is a positive constant provided by Ladyzhenskaya's inequalities. Uniqueness follows by setting $w(0) = 0$. One can also show the continuous dependence on initial conditions, since

$$\|w(t)\| \leq \sqrt{\exp\left(\int_0^T \frac{M}{\nu}\|\nabla u(s)\|^2 \,\mathrm{d}s\right)}\|w(0)\| =: L(T)\|w(0)\|, \qquad t \in [0,T], \tag{6}$$

i.e. errors are Lipschitz in time.

As an immediate corollary to inequality (6), the 2D-NS system is infinitely predictable (Palmer et al. 2014). Indeed, if a prediction is defined to lose its skill when $\|w(t)\| > \varepsilon$, then for any given time $T \in \mathbb{R}^+$, the prediction is skilful for at least up to T when the initial error $\|w(0)\|$ can be made sufficiently small, that is, smaller than $\frac{\varepsilon}{L(T)}$.

It is important to note that in the present Section the KE spectral slope plays no role in determining the finiteness or infiniteness of predictability of the 2D-NS equations. The above argument applies to 2D-NS systems of any spectral slope.

5. Reconciling the contradiction with Lorenz

At first glance, our result of infinite predictability derived in Section 4 seems to contradict Lorenz's result in Section 2 for any $p < 3$. However, we have not discussed the role of the KE spectral slope in $L(T)$ which, as we will see in the following, reconciles the conflict.

Central to our argument is the inequality (6) presented above. For simplicity, suppose the real system has only one inertial range of slope $-p$ (without any logarithmic correction) in its KE spectrum so that $|\hat{u}(k)|^2 \sim k^{-p}$ (note the change of notation: the hat now represents Fourier coefficients in the space of one-dimensional wavenumbers k) between its large-scale cutoff wavenumber k_1 and small-scale cutoff wavenumber k_2. Then

$$\|\nabla u_s\|^2 = \int_0^\infty k^2 |\hat{u_s}|^2 \,\mathrm{d}k = \int_0^{k_1} k^2 |\hat{u_s}|^2 \,\mathrm{d}k + A_0 \int_{k_1}^{k_2} k^{2-p} \,\mathrm{d}k + \int_{k_2}^\infty k^2 |\hat{u_s}|^2 \,\mathrm{d}k \qquad (A_0 \text{ constant}), \quad (7)$$

where the subscript s distinguishes the system itself from a model for the system which we will denote with subscript m. The three terms on the right-hand-side of equation (7) represent contributions from the large scale, the inertial range and the viscous range respectively. Compared to the first two terms, the term representing the viscous range is assumed to be small. In particular, the integrand is assumed to decay rapidly enough so that $\|\nabla u_s\|^2$ remains finite (this is in fact part of the definition of the function space to which u_s belongs).

Now, suppose the model truncates at wavenumber $k_t \ll k_2$ and numerical dissipation kicks in at wavenumber $k_0 \in (k_1, k_t)$. For the model,

$$\|\nabla u_m\|^2 = \int_0^{k_t} k^2 |\hat{u_m}|^2 \, \mathrm{d}k = \int_0^{k_1} k^2 |\hat{u_m}|^2 \, \mathrm{d}k + A_0 \int_{k_1}^{k_0} k^{2-p} \, \mathrm{d}k + \int_{k_0}^{k_t} k^2 |\hat{u_m}|^2 \, \mathrm{d}k. \tag{8}$$

Again, we may neglect the contribution from the viscous range, so that

$$\|\nabla u_m\|^2 \sim \int_0^{k_1} k^2 |\hat{u_m}|^2 \, \mathrm{d}k + A_0 \int_{k_1}^{k_0} k^{2-p} \, \mathrm{d}k. \tag{9}$$

Because $k_0, k_t \ll k_2$, the second integral in relation (9) with $p < 3$ *appears* to diverge as the resolution (k_0, k_t) increases. Combining this with inequality (6), $L(T)$ – until k_2 is reached – grows exponentially with k_0, leading to a breakdown of the Lipschitz-continuous dependence on initial conditions in inequality (6). To keep the error $\|w(t)\|$ under control, the initial error $\|w(0)\|$ would have to decrease exponentially, but decreasing the scale of the initial error without changing its magnitude relative to the background KE spectral density (Lorenz's thought experiment) would only give a polynomial decrease. The corollary of infinite predictability therefore fails to hold. Hence the range of predictability is finite in practice, even though the system is infinitely predictable, because infinite predictability cannot be achieved without making the model resolution so high that its effective resolution k_0 (and the scale of the initial error) falls within the viscous range of the real system, let alone the large-scale error has to be constrained to zero (Durran and Gingrich 2014).

This concept, known as 'asymptotic ill-posedness', was put forward by Palmer et al. (2014) as they argued that whether the three-dimensional Navier-Stokes system is well-posed is practically irrelevant to the well-established theory of finite predictability. We have now extended the discussion to the 2D-NS system and given a mathematical basis to the concept in our context.

When $p > 3$, the second integral in relation (9) does not appear to diverge as $k_0 \to k_2$. This means one may indeed approximate $\|\nabla u_s\|^2$ by the $\|\nabla u_m\|^2$ in relation (9) with a sufficiently large value of k_0. So would $L(T)$ of inequality (6) be approximated without regard to the model resolution, making it possible for $\|w(t)\| \leq \varepsilon$ by making $\|w(0)\|$ small enough in scale and thus achieving infinite predictability.

So far our argument for the cases $p < 3$ and $p > 3$ are in harmony with Lorenz's result in Section 2. For the borderline case $p = 3$, our argument suggests practically finite predictability, since $\|\nabla u_m\|^2 \sim \text{constant} + \int_{k_1}^{k_0} k^{-1}\, \mathrm{d}k = \text{constant} + \log \frac{k_0}{k_1}$ which appears to diverge as $k_0 \to k_2$. This disagrees with Lorenz. Even with the logarithmic correction

$$|\hat{u}(k)|^2 \sim k^{-3}\left[\log\left(\frac{k}{k_r}\right)\right]^{-\frac{1}{3}} \qquad (k_r > 0 \text{ constant}),$$

or more generally

$$|\hat{u}(k)|^2 \sim k^{-3}\left[A_1 \log\left(\frac{k}{k_r}\right) + A_2\right]^{-\frac{1}{3}} \qquad (A_1, A_2, k_r > 0 \text{ constants}),$$

to the -3 spectrum (Bowman 1996), an easy calculation along the previous lines still suggests that the range of predictability is practically finite. As such, we are unable to explain the disagreement and we leave the problem open.

For models and systems with multiple inertial ranges, only the range immediately before viscous effects become important pertains to our argument concerning the large-k_0 behaviour. This applies to the real atmosphere where $p = \frac{5}{3}$ (Nastrom and Gage 1985). Since k_t for atmospheric models is smaller than k_2 by 'at least seven or eight orders of magnitude' (Palmer et al. 2014), the crucial assumption to our discussion ($k_t \ll k_2$) is satisfied and we conclude that atmospheric predictability is indeed practically finite.

6. Conclusions

Half a century on since Lorenz's pioneering argument of finite atmospheric predictability, we revisited his original argument by (i) re-running his simplified model of the 2D barotropic vorticity equation, (ii) directly simulating the full model and (iii) comparing his conclusions with the well-posedness of the 2D-NS equations as proven by Ladyzhenskaya.

Although his main conclusion – that atmospheric predictability is inherently finite because the KE spectral slope is shallower than -3 – has now become an 'accepted part of the canon of dynamical meteorology' (Rotunno and Snyder 2008), the details behind the conclusion were re-assessed. The linearity assumption between the spectral slope $-p$ and β fails in his own simplified model (Sections 2 and 3(a)) although we were unable to provide a simple, non-technical explanation. We also saw a substantially different β in the DNS (Section 3(b)) than in Lorenz's model, which may be an indication that his model is inadequate in simulating the error growth. In both cases, nevertheless, the hypothesis of infinite predictability ($\beta = 0$) for $p = 3$ based on linear extrapolation was refuted.

The 2D-NS equations that closely relates to the 2D barotropic vorticity equation were used to address the predictability problem from a more rigorous perspective. The forced-dissipative system was shown to be infinitely predictable regardless of the spectral slope (Section 4). However, we found that $p = 3$ serves as a cutoff between *practically* finite and infinite predictability by noting how quickly the initial error has to be brought down with increasing resolution in order to maintain the bound for the error at future times (Section 5). This echoes Lorenz's original conclusions except the borderline case $p = 3$ itself, in which case our result of finite predictability agrees with our own computations of Lorenz's model and the DNS.

Until recently, KE spectra in global weather forecast models had only resolved the synoptic-scale -3 range. As model resolutions start to extend into the $-\frac{5}{3}$ range, the strong constraints on the range of predictability envisaged by Lorenz will become relevant. However, the limits on predictability arising from initial errors on the large scales will also limit predictability in practice

(Durran and Gingrich 2014), and the interplay between the two could be an interesting area to explore.

By providing another approach to attacking the problem of predictability (via the analytic theory of the 2D-NS equations), we look forward to similar results on more atmospherically relevant PDEs such as the surface quasi-geostrophic equations (Held et al. 1995), and a more active contribution from mathematicians on this topic.

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APPENDIX A

Motivating the concept *range of predictability*

Standard theory in dynamical systems dictates that the dynamics of the error ε can be completely described by the time t and the initial error ε_0, so that $\varepsilon = \varepsilon(t, \varepsilon_0)$. Suppose now that the skill S of a forecast can be quantified by a continuously decreasing function of some norm $\|\cdot\|$ (such as the kinetic energy) of the error. In such a case, we can write $S = S(\|\varepsilon\|) = S(\varepsilon) = S(t, \varepsilon_0)$. If we further assume that the error norm increases with t, we can infer that $S(t, \varepsilon_0)$ is monotonically decreasing in time. We acknowledge that the assumption cannot be rigorously defended in the context of atmospheric predictions, since it is typically observed in an average sense only.

Perhaps a first question to the understanding of predictability can be formulated as follows: how long does it take for an initial error ε_0 to grow so that a prediction loses its skill (defined by $S < \alpha$ where α is a threshold of skill, and typically achieved in fluid flows by saturation of the error kinetic energy spectrum at specified scales)? The answer $\tilde{T}$, known as the range of predictability, is the solution to $S(t, \varepsilon_0) = \alpha$ for the specified ε_0. The monotonicity assumption of S guarantees the uniqueness of the solution $\tilde{T}$.

By formulating this question for different initial error fields we can regard $\tilde{T}$ as a function of ε_0. It is clear from the very definition of deterministic systems that $\varepsilon_0 = 0$ implies $\tilde{T}(\varepsilon_0) = \infty$. However, it is not quite obvious as to whether $\tilde{T}$ could be made arbitrarily large by reducing $\|\varepsilon_0\|$ to anything positive below a threshold, or equivalently whether the equality $\liminf_{\|\varepsilon_0\| \to 0} \tilde{T}(\varepsilon_0) = \infty$ holds, because $\tilde{T}$ may behave irregularly at small $\|\varepsilon_0\|$ – or at least appear to.

To see the equivalence, we unwrap the statement $\liminf_{\|\varepsilon_0\| \to 0} \tilde{T}(\varepsilon_0) = \infty$ to get

$$
\begin{aligned}
& \liminf_{\|\varepsilon_0\| \to 0} \tilde{T}(\varepsilon_0) = \infty \\
\Leftrightarrow \quad & \forall R \in \mathbb{R},\ \liminf_{\|\varepsilon_0\| \to 0} \tilde{T}(\varepsilon_0) \geq R \\
\Leftrightarrow \quad & \forall R \in \mathbb{R},\ \sup_{\varepsilon' > 0} \inf_{\|\varepsilon_0\| \in (0, \varepsilon')} \tilde{T}(\varepsilon_0) \geq R \\
\Leftrightarrow \quad & \forall R \in \mathbb{R},\ \exists\, \varepsilon' > 0 \text{ such that } \inf_{\|\varepsilon_0\| \in (0, \varepsilon')} \tilde{T}(\varepsilon_0) \geq R \\
\Leftrightarrow \quad & \tilde{T} \text{ could be made arbitrarily large by reducing } \|\varepsilon_0\| \text{ to anything positive below a threshold.}
\end{aligned}
$$

With this in mind, a system is said to have an *infinite range of predictability*, or be *infinitely predictable*, if the range of predictability could be made arbitrarily large by reducing the initial error to a small yet positive value. Systems that fail to satisfy such a condition are referred to as *finitely predictable*.

APPENDIX B

Some details regarding the computation of the matrix C

The integrations were performed using `scipy.integrate.nquad` on Python which returned a warning message ‘`IntegrationWarning: Extremely bad integrand behavior occurs at some points of the integration interval`’ about the integrand’s singular behaviour, even if the integration domain were confined to the support of the integrand so that resources were not wasted in integrating zero regions. The warning disappeared by applying a change of coordinates (from logarithmic to Cartesian) in the integrand and accordingly the integration limits, which sped up the wall-clock time of the computation by a factor of about 9 as well. The entries of C computed by these two methods differ by no more than 0.0025%. Based on these observations, we are confident that our computations are accurate.

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LIST OF FIGURES

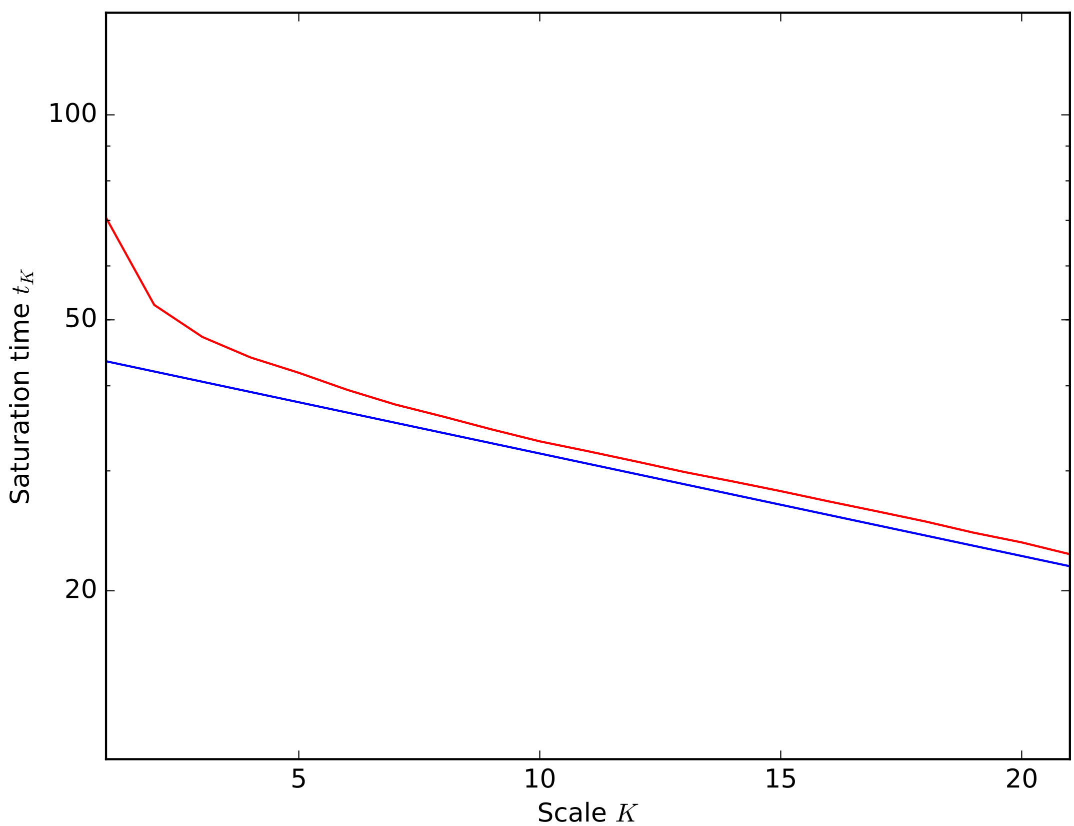


FIG. 1. Saturation times of various scales in the Lorenz (1969) model (red), with a −3-slope background KE spectrum as in Rotunno and Snyder (2008). The initial condition for this run is of a tiny magnitude at the second smallest of the 21 scales available. The blue curve shows a line of fit with $\beta = 0.05$.

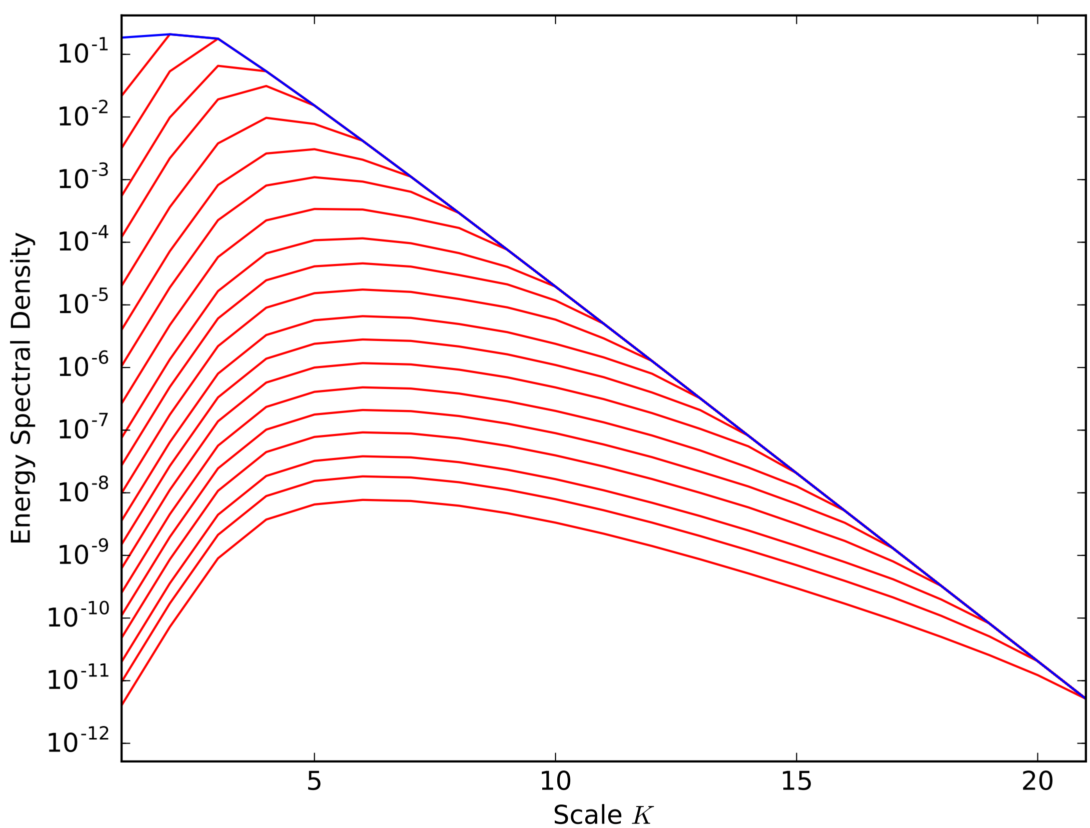
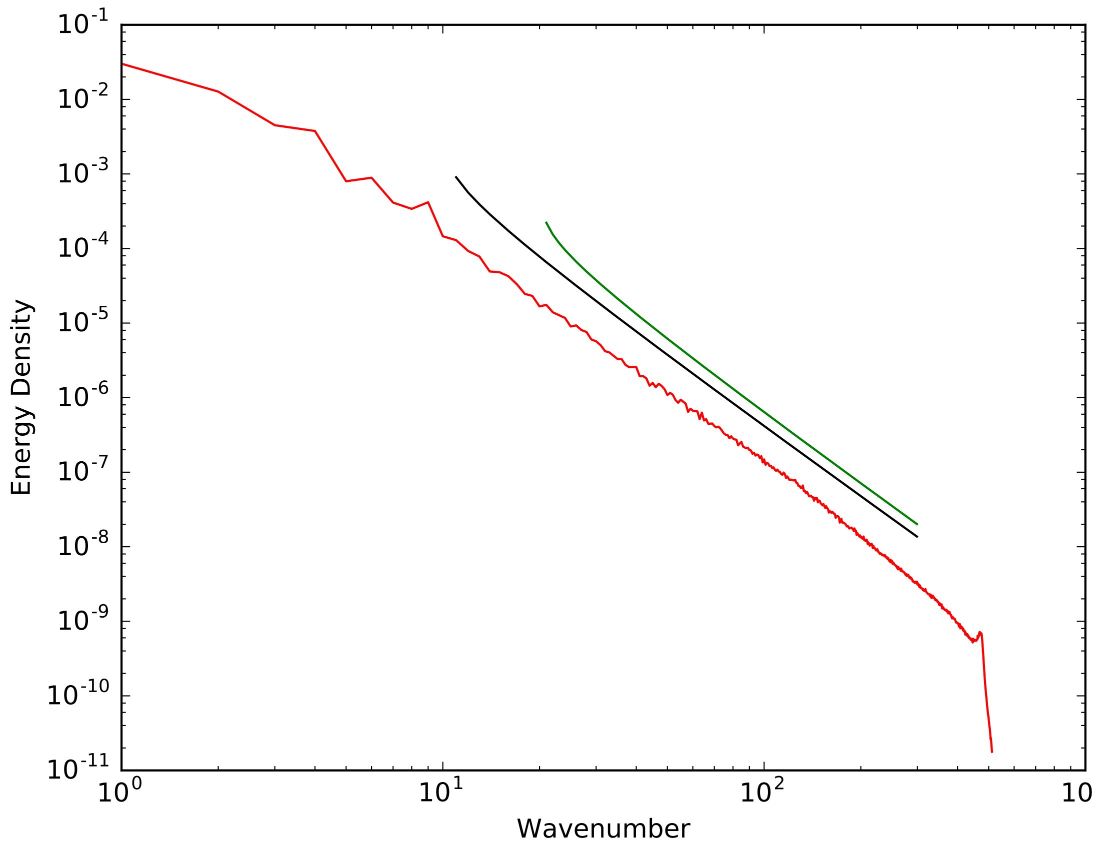


FIG. 2. The evolution of error KE spectra (red, from bottom to top) and the background spectrum (blue) for the same model run as Figure 1.



391 FIG. 3. KE spectrum (averaged over the five cases) of the initial condition (red), and logarithmically corrected
392 -3 reference spectra $E(k) \sim k^{-3}\left[\log\left(\frac{k}{k_r}\right)\right]^{-\frac{1}{3}}$ ($k_r = 10$ in black, $k_r = 20$ in green), where $E(\cdot)$ is the one-
393 dimensional KE spectral density.

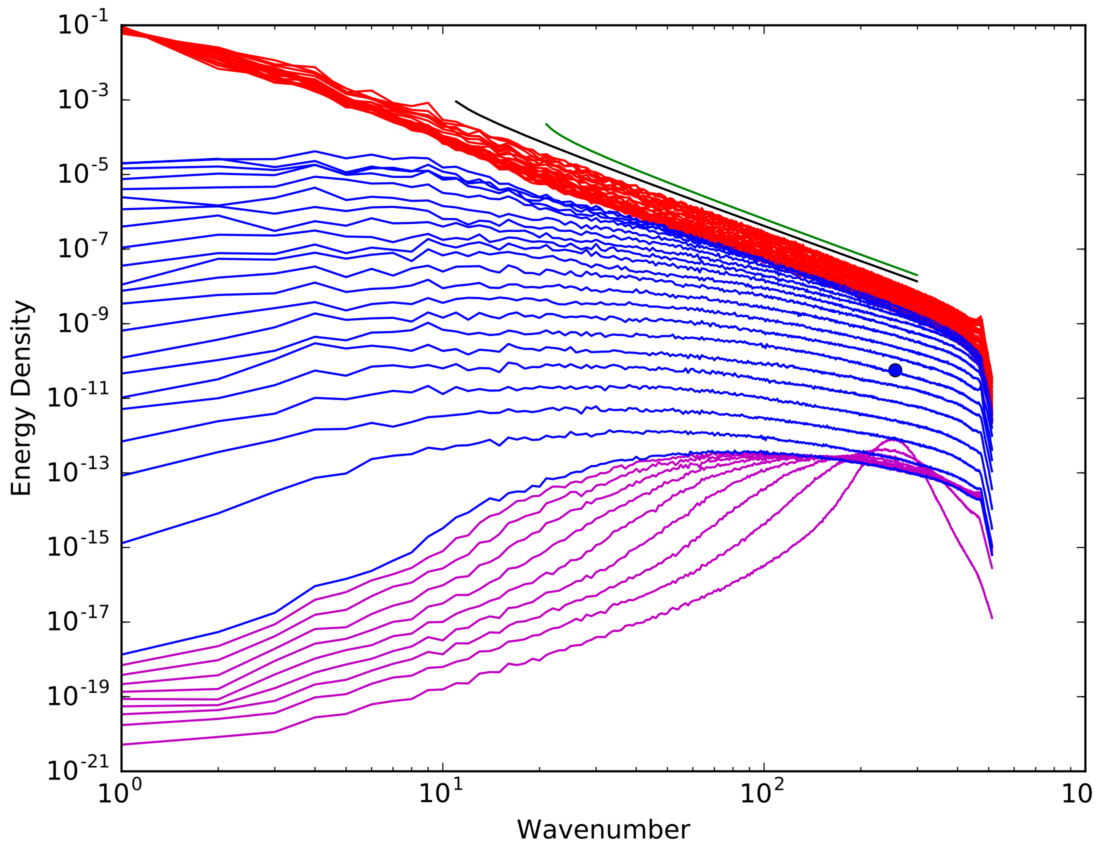
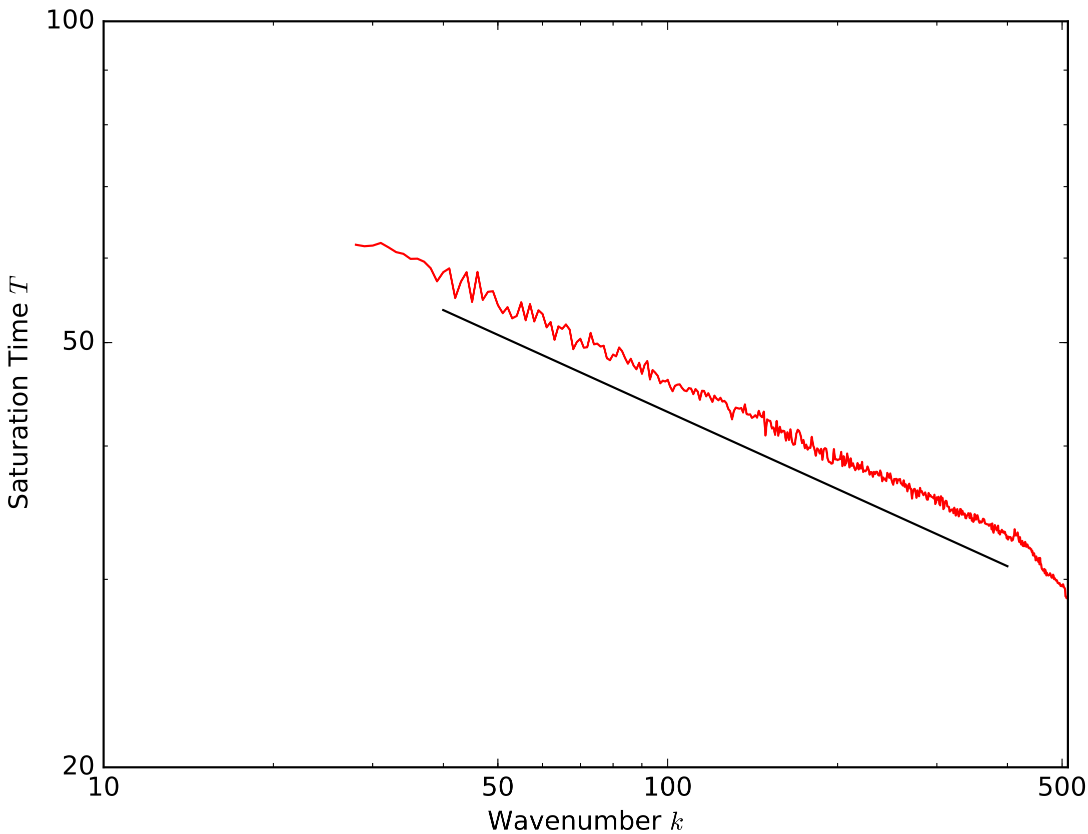


FIG. 4. Evolution of the error KE spectrum (magenta and blue, bottom to top) for an initial perturbation (blue dot) at $k_p = 256$. The magenta curves are for $t = 0.3, 0.6, \ldots, 2.7$ and the blue curves are for $t = 3, 6, \ldots, 66$. The background KE spectra at $t = 0, 3, 6, \ldots, 66$, scaled by a factor of 2, are shown in red (top to bottom), with the reference spectra in black and green as in Figure 3. The spectra are averaged over the five cases.



398 FIG. 5. Saturation times T at different wavenumbers k (red) for an initial error at wavenumber $k_0 = 256$,
399 averaged over the five cases. The black curve shows a line of fit with $\beta = 0.24$.