DE GRUYTER Adv. Calc. Var. 2020; 13(1): 33–52

Research Article

Pascal Gussmann and Alexander Mielke*

Linearized elasticity as Mosco limit of finite elasticity in the presence of cracks

https://doi.org/10.1515/acv-2017-0010
Received March 2, 2017; revised July 13, 2017; accepted September 21, 2017

Abstract: The small-deformation limit of finite elasticity is considered in presence of a given crack. The rescaled finite energies with the constraint of global injectivity are shown to Γ-converge to the linearized elastic energy with a local constraint of non-interpenetration along the crack.

Keywords: Gamma convergence, Mosco convergence, finite-strain elasticity, rigidity estimate for crack domains, contact, global injectivity, local non-interpenetration condition

MSC 2010: 49J45, 74M15, 35J85, 49J40, 74B20

Communicated by: Gianni Dal Maso

1 Introduction

In [3] Dal Maso, Negri, and Percivale showed that finite-strain elasticity Γ-converges to small-strain linearized elasticity under the assumptions of small loadings. Later, this result was extended to different settings, e.g., to situations with much weaker coercivity conditions Agostiniani, Dal Maso, and DeSimone [1], to multi-well energies by Schmidt [17], or to materials with residual stresses by Paroni and Tomassetti [15, 16]. Also evolutionary problems were treated, e.g., in elastoplasticity by the second author and Stefanelli [13] and in crack propagation by Negri and Zanini [14]. In this contribution we discuss an extension of the results in [3] to a setting where the reference domain has a crack Γ_{cr} of a certain class including cracks with kinks, see Section 2 for details.

The presence of the crack destroys the Lipschitz property of the cracked domain $\Omega_{\mathrm{cr}} := \Omega \setminus \Gamma_{\mathrm{cr}}$, and therefore crucial tools, such as the well-known rigidity estimate from [4], have to be adapted to the setting of cracked domains, see Proposition 3.2. More importantly, the setting of domains with cracks requires to introduce an additional constraint of global injectivity of the deformations $y : \Omega \subset \mathbb{R}^d \to \mathbb{R}^d$. A crucial step for the small-deformation Γ-limit is to show that this particular *global injectivity condition* leads to a *local non-interpenetration condition* along the crack $\Gamma_{\mathrm{cr}} \subset \Omega$.

In [2] Ciarlet and Nečas proposed the condition

$$\int_\Omega \det \nabla y(x)\,\mathrm{d}x \le \mathrm{vol}(y(\Omega)),$$

where $\mathrm{vol}(A)$ denotes the d-dimensional volume. This condition has been used in various applications, e.g., by Giacomini and Ponsiglione [5] in the SBV-theory for brittle materials or by Mariano and Modica [12] in the theory of weak diffeomorphisms to describe deformations in “complex bodies”. In [6, Proposition 3.2.1],

Pascal Gussmann, Weierstrass-Institut, Mohrenstrasse 39, 10117 Berlin, Germany, e-mail: pascal@gussmann.org
***Corresponding author: Alexander Mielke,** Weierstrass-Institut, Mohrenstrasse 39, 10117 Berlin, Germany;
and Institut für Mathematik, Humboldt-Universität zu Berlin, Berlin, Germany, e-mail: alexander.mielke@wias-berlin.de.
https://orcid.org/0000-0002-4583-3888

Giaquinta, Modica, and Souček showed that the above condition is equivalent to the condition

$$\int_{\Omega} \varphi(y(x))|\det \nabla y(x)|\,\mathrm{d}x \le \int_{\mathbb{R}^d} \varphi(z)\,\mathrm{d}z \quad \text{for all } \varphi \in \mathrm{C}_{\mathrm{c}}(\mathbb{R}^d, \mathbb{R}) \text{ with } \varphi \ge 0, \tag{1.1}$$

which we will simply call GMS condition.

This latter condition turns out to be an appropriate formulation for our purpose. In particular, assuming that $y_\varepsilon : \Omega \to \mathbb{R}^d$ satisfy (1.1), we will deduce that a weak limit $u : \Omega \to \mathbb{R}^d$ for $\varepsilon \to 0$ of the rescaled displacements

$$u_\varepsilon : x \mapsto \frac{1}{\varepsilon}(y_\varepsilon(x) - x)$$

satisfies the following local jump condition on the crack:

$$0 \le [\![u(x)]\!]_\nu := (u^+(x) - u^-(x)) \cdot \nu(x), \tag{1.2}$$

where u^+ and u^- are the traces of u on Γ_{cr} from the upper and the lower side, respectively, see Theorem 4.1.

Our analysis is based on elastic energies of integral type, i.e.

$$\mathcal{E}(y) = \int_{\Omega} W(\nabla y(x))\,\mathrm{d}x.$$

In finite-strain elasticity, the classical assumptions for W are coercivity, i.e. p-growth from below as in Assumption 1.1 (c), and the determinant constraint giving local orientation preservation, see Assumption 1.1 (a). For the derivation of the linearized theory, we need to impose conditions on the quadratic behavior of W near the identity matrix $F = I$.

Assumption 1.1. With $\mathrm{GL}_+(d) := \{A \in \mathbb{R}^{d\times d} \mid \det A > 0\}$ and $\mathrm{SO}(d) := \{R \in \mathbb{R}^{d\times d} \mid R^\top R = I,\ \det(R) = 1\}$ we pose the following conditions on the stored-energy density $W : \mathbb{R}^{d\times d} \to [0, \infty]$:

(a) $W(F) = \infty$ for all $F \in \mathbb{R}^{d\times d} \setminus \mathrm{GL}_+(d)$.

(b) $W(RF) = W(F)$ for all $F \in \mathbb{R}^{d\times d}$, $R \in \mathrm{SO}(d)$.

(c) There exist $p > d$ and $c_W, C_W > 0$ such that for all $F \in \mathbb{R}^{d\times d}$: $W(F) \ge c_W \max\{\mathrm{dist}(F, \mathrm{SO}(d))^2, |F|^p - C_W\}$.

(d) There exists a constant $\mathbb{C} \ge 0$ with $\mathbb{C}^\top = \mathbb{C}$ such that for all $\delta > 0$ there exists an $r_\delta > 0$ such that for all $A \in B_{r_\delta}(0) \subset \mathbb{R}^{d\times d}$: $|W(I + A) - \frac{1}{2}\langle A, \mathbb{C}A\rangle| \le \delta\langle A, \mathbb{C}A\rangle$.

In particular, condition (d) states that $A \mapsto \frac{1}{2}\langle A, \mathbb{C}A\rangle$ is the second order Taylor expansion of W around I. It implies $W(I) = 0$, $\partial_F W(I) = 0$ and $\partial_F^2 W(I) = \mathbb{C}$, where the second part yields that the material is stress free and, if W would be C^2 in a neighborhood of I, from the third part the assumed symmetry of $\mathbb{C}$ could be deduced. Moreover, the seminorm given by $|A|_{\mathbb{C}}^2 := \frac{1}{2}\langle A, \mathbb{C}A\rangle$ is equivalent to the norm $A \mapsto |A^{\mathrm{sym}}|$ as on the one hand the frame indifference (b) implies $\mathbb{C}A = \mathbb{C}A^{\mathrm{sym}}$ for every $A \in \mathbb{R}^{d\times d}$ and on the other hand the first part of assumption (c) being $W(F) \ge c_W \,\mathrm{dist}^2(F, \mathrm{SO}(d))$ and assumption (d) imply $c_W|A^{\mathrm{sym}}| \le |A|_{\mathbb{C}}^2$ (see [13] for the details).

To take the small-deformation limit, one considers *small deformations* of the form $y_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ for small parameters $\varepsilon > 0$, where u_ε remains bounded in a suitable function space. As the above discussed quadratic behavior of W around I suggests, the scaling of $W(\nabla y_\varepsilon) = W(I + \varepsilon\nabla u)$ by $\frac{1}{\varepsilon^2}$ will be appropriate to obtain linearized elasticity in the bulk, namely

$$\overline{W}_\varepsilon(\,\cdot\,) := \frac{1}{\varepsilon^2} W(I + \varepsilon\,\cdot\,) \xrightarrow{\mathrm{M}} \frac{1}{2}|\cdot|_{\mathbb{C}}^2. \tag{1.3}$$

The correspondingly rescaled elastic energies (cf. [3]) without GMS condition reads

$$\widetilde{\mathcal{F}}_\varepsilon(u) := \int_{\Omega} \frac{1}{\varepsilon^2} W(x, I + \varepsilon\nabla u(x))\,\mathrm{d}x$$

while we are interested in the elastic energy with the GMS condition (1.1), namely

$$\mathcal{F}_\varepsilon : \mathrm{H}^1_{g,\mathrm{Dir}} \to \mathbb{R} \cup \{\infty\}, \quad u \mapsto \begin{cases} \widetilde{\mathcal{F}}_\varepsilon(u) & \text{if } \mathrm{id} + \varepsilon u \text{ satisfies (1.1)}, \\ \infty & \text{else}, \end{cases} \tag{1.4}$$

where Γ_{Dir} and $\mathrm{H}^1_{g,\mathrm{Dir}}$ are specified in (2.4) such that $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ implies $(u-g)|_{\Gamma_{\mathrm{Dir}}} = 0$. The functional $\widetilde{\mathcal{F}}_\varepsilon$ is the one considered in [3], and it is shown to Γ-converge to

$$\widetilde{\mathcal{F}}(u) = \int_{\Omega_{\mathrm{cr}}} \frac{1}{2}\langle e(u), \mathbb{C}e(u)\rangle \,\mathrm{d}x, \quad \text{where } e(u) := (\nabla u)^{\mathrm{sym}} := \frac{1}{2}(\nabla u + (\nabla u)^\top).$$

The main result of this work is the Mosco convergence (i.e. Γ-convergence with respect to both weak and strong H^1-topology) of $\mathcal{F}_\varepsilon$ to the functional $\mathcal{F}$, which is obtained from $\widetilde{\mathcal{F}}$ by adding the local non-interpenetration condition (1.2), namely

$$\mathcal{F} : \mathrm{H}^1_{g,\mathrm{Dir}} \to \mathbb{R} \cup \{\infty\}, \quad u \mapsto \begin{cases} \widetilde{\mathcal{F}}(u) & \text{if } u \text{ satisfies (1.2)}, \\ \infty & \text{else}. \end{cases} \tag{1.5}$$

The equi-coercivity of the functionals $\mathcal{F}_\varepsilon$ is directly implied by the equi-coercivity of $\widetilde{\mathcal{F}}_\varepsilon$, once the rigidity result of [4] has been generalized to our class of crack domains $\Omega_{\mathrm{cr}} := \Omega \setminus \Gamma_{\mathrm{cr}}$ as specified in Section 2. Thus, the coercivity in Assumption 1.1 (c) and the energy bound $\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon) \le C < \infty$ imply $\|u_\varepsilon\|_{\mathrm{H}^1} \le C$ and $\|\varepsilon u_\varepsilon\|_{\mathrm{L}^p} \le C$, which gives $\|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty} \le C\varepsilon^r$ for some $r > 0$, see Proposition 3.6. Our main Theorem 2.4 states the following Γ-convergence:

$$\mathcal{F}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{F} \quad \text{in } \mathrm{H}^1_{g,\mathrm{Dir}}, \quad \text{i.e.} \quad \begin{cases} \text{for all } u_\varepsilon \rightharpoonup u \text{ in } \mathrm{H}^1_{g,\mathrm{Dir}}: & \mathcal{F}(u) \le \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon), \\ \text{for all } \tilde{u} \in \mathrm{H}^1_{g,\mathrm{Dir}} \text{ there exists } \tilde{u}_\varepsilon \to \tilde{u}: & \mathcal{F}(\tilde{u}) \ge \limsup_{\varepsilon\to 0} \mathcal{F}_\varepsilon(\tilde{u}_\varepsilon). \end{cases} \tag{1.6}$$

In Section 4 we provide the liminf estimate (in the weak topology of $\mathrm{H}^1(\Omega; \mathbb{R}^d)$), where because of the result in [3] it remains to establish the local non-interpenetration condition (1.2) as a limit of the global condition (1.1), which is not too difficult, see Theorem 4.1. The construction of recovery sequences for the limsup estimate (now in the strong topology of H^1) is more delicate, as in general (even for very smooth) displacements $u \in \mathrm{H}^1(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$ satisfying the local non-interpenetration condition (1.2) the associated close-to-identity deformation $y_\varepsilon = \mathrm{id} + \varepsilon u$ does not satisfy the GMS condition (1.1) for global injectivity, see Example 5.1. On the one hand, our construction of recovery sequences invokes an approximation of functions in $\mathrm{H}^1(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$ satisfying (1.2) by functions in $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$ still satisfying (1.2), which is reminiscent to the density results in Proposition 5.4 for convex constraints derived in [7, 8]. On the other hand, we have to use an artificial forcing apart of the two crack sides to be able to guarantee (1.1), see Proposition 5.2.

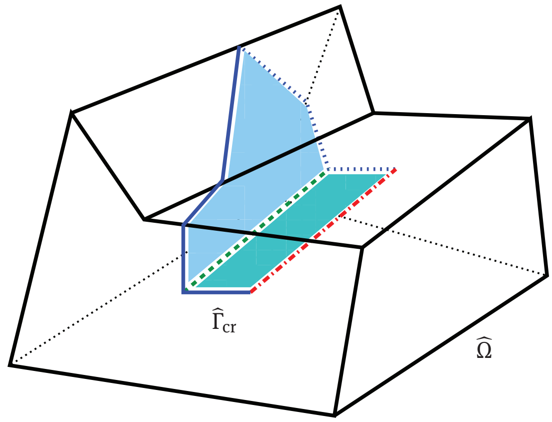
In the present work, we are only able to treat the static situation as in [3], which is in contrast to [13, 14], where the passage from finite-strain to linearized elasticity is handled in the rate-independent setting. However, the treatment of the contact problem in finite-strain seems still to be too difficult. In [11] the quasistatic evolution of fracture in linearized elasticity is developed, where cracks may occur along arbitrary paths that have $\mathrm{C}^{1,\mathrm{Lip}}$ regularity, which is the same regularity needed for our analysis.

2 Transformation and main result

Throughout this paper we consider a reference configuration with a Lipschitz domain Ω and a given crack Γ_{cr} on which the displacements $u \in \mathrm{H}^1(\Omega_{\mathrm{cr}}, \mathbb{R}^d)$ may have jumps. We expect that our theory works for general domains Ω and cracks Γ_{cr} that are piecewise $C^{1,\mathrm{Lip}}$, if all the edges and corners are non-degenerate. However, to avoid an overload of technicalities, we concentrate on the essential difficulties that arise (i) by smooth pieces of the crack, (ii) by the edge of the crack, (iii) by kinks inside a crack, and (iv) through the intersection of the crack with the boundary $\partial\Omega$.

Thus, we define a model domain $\widehat{\Omega}$ with a model crack $\widehat{\Gamma}_{\mathrm{cr}}$ that displays all these difficulties and then consider all domains Ω with cracks Γ_{cr} that are obtained by a bi-Lipschitz mapping $T : \Omega \to \widehat{\Omega}$ such that $\widehat{\Gamma}_{\mathrm{cr}} = T(\Gamma_{\mathrm{cr}})$.

Conditions on the model pair $(\widehat{\Omega}, \widehat{\Gamma}_{\mathrm{cr}})$. Our conditions essentially say that $\widehat{\Omega}_{\mathrm{cr}} = \widehat{\Omega} \setminus \widehat{\Gamma}_{\mathrm{cr}}$ can be written as the union of two Lipschitz domains A_+ and A_- that have a nontrivial intersection $A_+ \cap A_-$, which is a Lipschitz set again, and that define $\widehat{\Gamma}_{\mathrm{cr}}$ as the intersection of the boundaries ∂A_+ and ∂A_-.



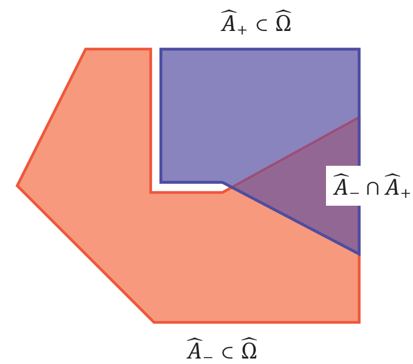


Figure 1: Left: Crack $\widehat{\Gamma}_{\rm cr}$ (areas shaded in light blue) inside the domain $\widehat{\Omega}$, the crack edge $\widehat{\Gamma}_{\rm edge}$ is red, the crack kink $\widehat{\Gamma}_{\rm kink}$ is green lying between the two shaded areas, and $\partial\Omega \cap \widehat{\Gamma}_{\rm cr}$ is blue. Right: Decomposition of a planar $\widehat{\Omega}$ into overlapping Lipschitz domains $\widehat{A}_+$ and $\widehat{A}_-$ according to Assumption 2.1 (c).

Assumption 2.1. By using the normal vector $\widehat{\nu} \in \mathbb{S}^{d-1}$ of the crack $\widehat{\Gamma}_{\rm cr}$ and the outward normal vector $\widehat{n} \in \mathbb{S}^{d-1}$ on $\partial\Omega$, our precise assumptions are the following:
(a) $\widehat{\Omega} \subset \mathbb{R}^d$ is a bounded Lipschitz domain.
(b) $\widehat{\Gamma}_{\rm cr} := (([0,1] \times \{0\} \times \mathbb{R}^{d-2}) \cup (\{0\} \times [0,\infty] \times \mathbb{R}^{d-2}))$, $\widehat{\Gamma}_{\rm edge} := \{(1,0)\} \times \mathbb{R}^{d-2}$, $\widehat{\Gamma}_{\rm kink} := \{(0,0)\} \times \mathbb{R}^{d-2}$.
(c) The sets $\widehat{A}_+ := \{\widehat{x} \in \widehat{\Omega} \mid (\widehat{x}_1 > 0,\ \widehat{x}_2 > 0) \text{ or } \widehat{x}_1 > 1\}$ and $\widehat{A}_- := \{\widehat{x} \in \widehat{\Omega} \mid \widehat{x}_1 < 0 \text{ or } \widehat{x}_1 > 1 \text{ or } \widehat{x}_2 < 0\}$ as well as $\widehat{A}_+ \cap \widehat{A}_-$ and $\widehat{A}_- \setminus \widehat{A}_+$ have Lipschitz boundary.
(d) *Transversality of* $\widehat{\Gamma}_{\rm cr}$: The sets $\partial\widehat{\Omega}$ and $\widehat{\Gamma}_{\rm cr}$ intersect transversally, i.e.

$$\exists \delta > 0\ \forall \widehat{x}_0 \in \partial\widehat{\Omega} \cap \widehat{\Gamma}_{\rm cr} \setminus (\widehat{\Gamma}_{\rm edge} \cup \widehat{\Gamma}_{\rm kink})\ \exists \varrho > 0: \quad (\widehat{n}(\widehat{x}) \cdot \widehat{\nu}(\widehat{x}_0))^2 \le 1 - \delta \text{ for } \mathcal{H}^{d-1}\text{-a.e. } \widehat{x} \in \partial\widehat{\Omega} \cap B_\varrho(\widehat{x}_0).$$

(e) *Transversality of* $\widehat{\Gamma}_{\rm edge}$ *and* $\widehat{\Gamma}_{\rm kink}$: The sets $\widehat{\Gamma}_{\rm edge}$ and $\widehat{\Gamma}_{\rm kink}$ intersect with $\partial\widehat{\Omega}$ transversally, i.e.

$$\exists \delta > 0\ \forall \widehat{x}_0 \in (\widehat{\Gamma}_{\rm edge} \cup \widehat{\Gamma}_{\rm kink}) \cap \partial\widehat{\Omega}\ \exists \varrho > 0: \quad (\widehat{n}(\widehat{x}) \cdot e_1)^2 + (\widehat{n}(\widehat{x}) \cdot e_2)^2 \le 1 - \delta \text{ for } \mathcal{H}^{d-2}\text{-a.e. } \widehat{x} \in \partial\widehat{\Omega} \cap B_\varrho(\widehat{x}_0).$$

The conditions on $(\widehat{\Omega}, \widehat{\Gamma}_{\rm cr})$ are illustrated in Figure 1. The model crack $\widehat{\Gamma}_{\rm cr}$ defined in (b) contains two special subsets, namely (i) the crack edge $\widehat{\Gamma}_{\rm edge}$ and (ii) the crack kink $\widehat{\Gamma}_{\rm kink}$. For all other points we have the well-defined crack normal $\nu(\widehat{x}) = (1, 0, \ldots, 0)^\top \in \mathbb{R}^d$ or $(0, 1, 0, \ldots, 0)^\top$, respectively. Conditions (d) and (e) ask that the crack $\widehat{\Gamma}_{\rm cr}$ and its edge $\widehat{\Gamma}_{\rm edge}$ and kink $\widehat{\Gamma}_{\rm kink}$ to not meet the boundary $\partial\widehat{\Omega}$ tangentially.

The decomposition $\widehat{\Omega}_{\rm cr} \subset \widehat{A}_+ \cup \widehat{A}_-$ in (c) will be used for three purposes, namely (i) for the derivation of a rigidity result for the cracked domain, (ii) to construct enough good test functions for deriving the jump condition in Theorem 4.1, and (iii) for distinction of different cases in Proposition 5.2.

The domains Ω and the cracks $\Gamma \subset \Omega$ for which we will formulate our theory are now obtained by a bi-Lipschitz mapping $T : \mathbb{R}^d \to \mathbb{R}^d$ that is additionally $\mathrm{C}^{1,\mathrm{Lip}} = \mathrm{W}^{2,\infty}$. Thus, the conditions on the pair (Ω, Γ) or the cracked domain $\Omega_{\rm cr} := \Omega \setminus \Gamma$ are the following:

Assumption 2.2 (Assumptions on (Ω, Γ)). The pair of sets $(\widehat{\Omega}, \widehat{\Gamma}_{\rm cr})$ satisfies Assumption 2.1 and there exists a bi-Lipschitz map $T : \mathbb{R}^d \to \mathbb{R}^d$ such that $\widehat{\Omega} = T(\Omega)$, $\widehat{\Gamma}_{\rm cr} = T(\Gamma_{\rm cr})$, and $T \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d; \mathbb{R}^d)$.

Note that the true crack $\Gamma_{\rm cr}$ will be piecewise $\mathrm{C}^{1,\mathrm{Lip}}$, since we allowed for a kink in $\widehat{\Gamma}_{\rm cr}$.

As a first consequence of this assumption we see that $\Omega_{\rm cr}$ can also be decomposed similarly to $\widehat{\Omega}_{\rm cr}$ in Assumption 2.1 (c). Defining $A_\pm := T^{-1}(\widehat{A}_\pm)$ with $\widehat{A}_\pm$ from Assumption 2.1 (c) we have that

$$\begin{aligned} &A_+, A_- \subset \Omega \text{ are Lipschitz domains with } A_+ \cup A_- = \Omega_{\rm cr} \\ &\text{such that } A_+ \cap A_- \text{ and } A_- \setminus A_+ \text{ are also Lipschitz domains.} \end{aligned} \tag{2.1}$$

This overlapping covering of $\Omega_{\rm cr}$ in assumption (2.1) is used for three different purposes. First, it allows us to extend the rigidity result from Lipschitz domains to our crack domains $\Omega_{\rm cr}$, see Corollary 3.3. Second, it allows us to derive the jump condition (1.2) in Theorem 4.1 by applying the divergence theorem on a disjoint cover given by A_+ and $A_- \setminus A_+$. Finally, and third, we use it in Proposition 5.2 for the construction of injective close-to-identity deformations.

The assumption that $T : \mathbb{R}^d \to \mathbb{R}^d$ is a bi-Lipschitz mapping means that it is bijective and that both T and T^{-1} are Lipschitz continuous. The additional condition $T \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d;\mathbb{R}^d)$ then implies $T^{-1} \in \mathrm{C}^{1,\mathrm{Lip}}(\mathbb{R}^d;\mathbb{R}^d)$. A diffeomorphism $y : \Omega \to \mathbb{R}^d$ can be transformed to a mapping on $\widehat{\Omega}$ via the transform

$$\widehat{y}(\widehat{x}) = T(y(T^{-1}(\widehat{x}))) \quad \text{or} \quad y(x) = T^{-1}(\widehat{y}(T(x))).$$

In particular, for $\widehat{y}_{\varepsilon,\widehat{u}} := \mathrm{id} + \varepsilon\widehat{u} : \widehat{\Omega} \to \mathbb{R}^d$ we find the expansion

$$y_\varepsilon(x) = T^{-1}(\widehat{y}_{\varepsilon,\widehat{u}}(T(x))) = x + \varepsilon\nabla T(x)^{-1}\widehat{u}(T(x)) + O(\varepsilon^2).$$

The mapping from $\widehat{u}$ to the corresponding term in y_ε is called the *Piola transform P_T for vector fields*, cf. also [9, 10]. Under the assumption (2.2) the mapping

$$P_T : \mathrm{H}^1(\widehat{\Omega}) \to \mathrm{H}^1(\Omega), \qquad \widehat{u} \mapsto (u : x \mapsto \nabla T(x)^{-1}\widehat{u}(T(x))) \tag{2.2}$$

is a bijective bounded linear mapping as well as its inverse $P_{T^{-1}} : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\widehat{\Omega})$.

The Piola transform is especially useful for us, as it also transforms the local non-interpenetration condition in the correct way, see, e.g., [9, 10]. If $\widehat{\nu}(\widehat{x})$ is the normal vector at $\widehat{x} \in \widehat{\Gamma}_{\mathrm{cr}}$, then it is related to the normal vector $\nu(x)$ at $x = T^{-1}(\widehat{x}) \in \Gamma$ via

$$\nu(x) = \frac{1}{|\nabla T(x)^\top\widehat{\nu}(T(x))|}\nabla T(x)^\top\widehat{\nu}(T(x)) \quad \text{or} \quad \widehat{\nu}(T(x)) = \frac{1}{|\nabla T(x)^{-\top}\nu(x)|}\nabla T(x)^{-\top}\nu(x).$$

Thus, for the jump over the crack we obtain the relation

$$\begin{aligned}
[\![u]\!]_\nu(x) &= (u^+(x) - u^-(x)) \cdot \nu(x)\\
&= (\nabla T(x)^{-1}\widehat{u}^+(T(x)) - \nabla T(x)^{-1}\widehat{u}^-(T(x))) \cdot \nu(x)\\
&= (\widehat{u}^+(T(x)) - \widehat{u}^-(T(x))) \cdot \nabla T(x)^{-\top}\nu(x)\\
&= |\nabla T(x)^{-\top}\nu(x)|[\![\widehat{u}]\!]_{\widehat{\nu}}(T(x)).
\end{aligned} \tag{2.3}$$

Thus, the jumps translate correctly if we take into account the prefactor that associates with the stretching of surface elements.

Consequence 2.3. For future use of the above assumptions on $(\Omega, \Gamma_{\mathrm{cr}})$ we derive the following well-known consequences, which will be employed below in our theory of Γ-convergence:

(a) Ω is a Lipschitz domain, and for all $x_0 \in \partial\Omega$ there exist an open neighborhood $U \subset \mathbb{R}^d$ of x_0 and a bi-Lipschitz $\Psi_{x_0} : U \to V \subset \mathbb{R}^d$ such that $U \cap \Omega \subset \Psi_{x_0}^{-1}(\{v_d > 0\})$ and $U \cap \partial\Omega \subset \Psi_{x_0}^{-1}(\{v_d = 0\})$.

(b) *Transversality of* Γ_{cr} *and* $\partial\Omega$: for all $x_0 \in \Gamma_{\mathrm{cr}} \cap \partial\Omega$ there exist $\widehat{\eta}_{x_0} \in \mathbb{S}^{d-1}$, $\kappa > 0$, and U and Ψ_{x_0} as in (a) such that
  (i) $\nabla\Psi_{x_0}(x)^\top e_d \cdot \nabla T(x)^{-1}\widehat{\eta}_{x_0} \geq \kappa\, \mathcal{L}^d$-a.e. in $U \cap \Omega$,
  (ii) $\widehat{\eta}_{x_0} \cdot \nabla T(x)^{-\top}\nu(x) = 0\ \mathcal{H}^{d-1}$-a.e. in $U \cap \Gamma_{\mathrm{cr}}$,
  (iii) $\widehat{\eta}_{x_0} \in \{(0,0)\} \times \mathbb{R}^{d-2}$ if $x_0 \in \partial\Omega \cap \Gamma_{\mathrm{edge}}$,
  where $\Gamma_{\mathrm{edge}} := T^{-1}(\widehat{\Gamma}_{\mathrm{edge}})$ with $\widehat{\Gamma}_{\mathrm{edge}} := \{(1,0)\} \times \mathbb{R}^{d-2}$.

Note that condition (ii) in (b) simply means $\widehat{\eta}_{x_0} \cdot \widehat{\nu}(T(x)) = 0$, where $\widehat{\nu}$ takes one of the values $e_1, e_2 \in \mathbb{R}^d$, or even both values if $T(x_0) \in \widehat{\Gamma}_{\mathrm{kink}}$. Hence, this condition follows directly from Assumption 2.1 (d), but we will use the form as given in (b) for a full neighborhood. Similarly, condition (iii) in (b) is a direct consequence of Assumption 2.1 (e).

Note that the angle of $\frac{\pi}{2}$ at the kink of $\widehat{\Gamma}_{\mathrm{cr}}$ is not essential and will be varied by the mapping $\nabla T^{-1}(y)$ for $y \in \widehat{\Gamma}_{\mathrm{cr}} \cap \widehat{\Omega}_{\mathrm{cr}}$. Furthermore, the choice of $\widehat{\Gamma}_{\mathrm{cr}} = T(\Gamma_{\mathrm{cr}}) \subset \widehat{\Omega}$ in (2.2) is just an example as easy as possible while still showing the crucial difficulties. We expect that the theory works for any Lipschitz surface that is piecewise $C^{1,\mathrm{Lip}}$. The proofs and constructions are made with the intention to be adaptable to other special situations.

The transversality condition in Consequence 2.3 (b) requires the crack Γ_{cr} and the boundary $\partial\Omega$ to intersect transversally. Technically it enables us to use the following implicit function theorem for Lipschitz maps to conclude $\partial\widehat{\Omega}$ being a graph in the direction η, which is parallel to $\widehat{\Gamma}_{\mathrm{cr}}$ in a whole open neighborhood of $T(x_0)$. You can interpret this graphically when having in mind the fact, that normal vectors transform by the cofactor of the gradient. Then equation (i) of Consequence 2.3 (b) can be read as the vector field $\eta_{x_0} = \nabla T(x)^{-\top}\widehat{\eta}_{x_0}$, which is constant on the flat configuration $\widehat{\Omega} \setminus \widehat{\Gamma}_{\mathrm{cr}}$ having an angle bounded away from $\frac{\pi}{2}$ to the normal on the boundary, which is given by $\nabla\Psi_{x_0}(x)e_d = \nabla\Psi_{x_0}(x)(0,\dots,0,1)^\top$. The last two requirements specify that for $x_0 \in \Gamma_{\mathrm{cr}}$ or $x_0 \in \Gamma_{\mathrm{edge}}$ the vector $\widehat{\eta}_{x_0}$ is tangential to $\widehat{\Gamma}_{\mathrm{cr}}$ or $\widehat{\Gamma}_{\mathrm{edge}}$, respectively.

To collect all the assumptions, we now specify the boundary conditions in terms of the part $\Gamma_{\mathrm{Dir}} \subset \partial\Omega$, where the Dirichlet boundary conditions $(u-g)|_{\Gamma_{\mathrm{Dir}}} = 0$ are imposed:

$$\begin{aligned} &\overline{\Gamma}_{\mathrm{Dir}} \cap \overline{\Gamma}_{\mathrm{cr}} = \emptyset, \quad \mathcal{H}^{d-1}(\Gamma_{\mathrm{Dir}}) > 0, \quad g \in \mathrm{W}^{1,\infty}(\Omega;\mathbb{R}^d), \\ &\mathrm{H}^1_{g,\mathrm{Dir}} := \mathrm{clos}_{\mathrm{H}^1(\Omega_{\mathrm{cr}})}(\{u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}};\mathbb{R}^d) \mid (u-g)|_{\Gamma_{\mathrm{Dir}}} = 0\}). \end{aligned} \tag{2.4}$$

Theorem 2.4 (Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{F}$). *Let Assumptions 1.1, 2.2, and* (2.4) *be satisfied and $\mathcal{F}_\varepsilon$ and $\mathcal{F}$ defined as in* (1.4) *and* (1.5). *Then $\mathcal{F}_\varepsilon$ Mosco-converges to $\mathcal{F}$ in $\mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$.*

The proof of this result is the content of the following sections. In particular, the liminf estimate is established in Proposition 4.3, and the limsup estimate in Theorem 5.5.

The following result is a weak version of the implicit function theorem that will be needed to represent the boundary $\partial\Omega$ near a point $x_0 \in \partial\Omega \cap \Gamma_{\mathrm{cr}}$, see Corollary 2.6.

Theorem 2.5 (Special version of Implicit Function Theorem). *Let $U_m \subset \mathbb{R}^m$, $U_n \subset \mathbb{R}^n$ be open sets, $a \in U_m$, $b \in U_n$ and let $F : U_m \times U_n \to \mathbb{R}^n$ be a Lipschitz map with $F(a,b) = 0$. Suppose there exists a constant $K > 0$ such that for all $x \in U_m$ and $y_1, y_2 \in U_n$ it holds*

$$|F(x,y_1) - F(x,y_2)| \geq K|y_1 - y_2|. \tag{2.5}$$

Then there exist an open neighborhood V_m of a, $V_m \subset U_m$, and a Lipschitz map $\varphi : V_m \to \mathbb{R}^n$ such that $\varphi(a) = b$ and

$$F^{-1}(0) = \{(x,\varphi(x)) \mid x \in V_m\}.$$

Proof. We will sketch the proof briefly.

By (2.5), which is a Lipschitz analog of the invertibility of $\nabla_y F$ in the differentiable version of the inverse function theorem, the map $f : U_m \times U_n \supset \mathbb{R}^{m+n} \to \mathbb{R}^{m+n}$, $(x,y) \mapsto (x, \delta F(x,y))$ is bi-Lipschitz for $0 < \delta < \|\nabla F\|^{-1}_{L^\infty}$. In particular, f is continuous, injective and maps an open subset of $\mathbb{R}^{m+n}$ to $\mathbb{R}^{m+n}$, thus by Brouwer's invariance of domain theorem f is an open map, i.e. $f(U_m \times U_n)$ is open in $\mathbb{R}^{m+n}$ and f^{-1} is continuous. Consider the embedding $e_m : \mathbb{R}^m \to \mathbb{R}^{m+n}$, $x \mapsto (x,0)$ and the projection $p_n : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$, $(x,y) \mapsto y$. Both e_m and p_n are Lipschitz continuous, thus $\varphi := p_m \circ f^{-1} \circ e_m$ defines a Lipschitz map on $V_m := e_m^{-1}(f(U_m \times U_n))$, which is open by continuity of e_m and f^{-1}. Because of the assumption $F(a,b) = 0$ we have $a \in V_m$ and $\varphi(a) = b$. Regarding the claimed equality $F^{-1}(0) = \{(x,\varphi(x)) \mid x \in V_m\}$ we get on the one hand the inclusion "$\supset$" from $F(x,\varphi(x)) = 0$, which follows by construction of φ. On the other hand for every $(x,y) \in U_m \times U_n$ with $F(x,y) = 0$ we have $f(x,y) = (x,0)$ such that x lies in the domain V_m of φ by construction of V_m, which gives the other inclusion "$\subset$". □

We are now able to write the boundary $\partial\widehat{\Omega}$ near $\widehat{x}_0 \in \partial\widehat{\Omega} \cap \widehat{\Gamma}_{\mathrm{cr}}$ as a Lipschitz graph over the plane $\widehat{P}_{\widehat{x}_0}$ through $\widehat{x}_0 = T(x_0)$ that is normal to $\widehat{\eta}_{\widehat{x}_0}$. This construction will be needed in the proof of Proposition 5.4.

Corollary 2.6. *Let $\widehat{x}_0 = T(x_0) \in \widehat{\Gamma}_{\mathrm{cr}} \cap \partial\widehat{\Omega}$ and let U and $\widehat{\eta}_{x_0}$ be as in the transversality condition of Consequence 2.3* (b). *Set $\widehat{P}_{x_0} := \{\widehat{x} \in \mathbb{R}^d \mid (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0} = 0\}$. Then there exist an open neighborhood $\widehat{V}$ of $T(x_0)$ and a Lipschitz continuous function $\varphi_{x_0} : \widehat{V} \cap \widehat{P}_{x_0} \to \mathbb{R}$ such that the function*

$$\widehat{g} : \widehat{V} \to \mathbb{R}, \quad \widehat{g}(\widehat{x}) := \varphi_{x_0}(\widehat{x} - [(\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}]\widehat{\eta}_{x_0}) - (\widehat{x} - T(x_0)) \cdot \widehat{\eta}_{x_0}$$

characterizes $\partial\widehat{\Omega}$ locally via $\widehat{g}(\widehat{x}) > 0$ for $\widehat{x} \in \widehat{\Omega}$, $\widehat{g}(\widehat{x}) = 0$ for $\widehat{x} \in \partial\widehat{\Omega}$, and $\widehat{g}(\widehat{x}) < 0$ for $\widehat{x} \in \mathbb{R}^d \setminus \mathrm{clos}(\widehat{\Omega})$.

Similarly, the boundary $\partial\Omega$ near a point $x_0 \in \Gamma_{\mathrm{cr}} \cap \partial\Omega$ can be characterized by the function $g = \widehat{g} \circ T^{-1}$, where $\widehat{g}$ is obtained as above for $\widehat{x}_0 = T(x_0)$.

Proof. Take Ψ_{x_0} as in the transversality condition of Consequence 2.3 (b) and introduce local coordinates $z \in \widehat{P}_{x_0}$ and $y \in \mathbb{R}$ providing a unique representation of $\hat{x} \in \mathbb{R}^d$ via $\hat{x} = z + y\hat{\eta}_{x_0}$. The map

$$F : U \cap \widehat{P}_{x_0} \times \mathbb{R} \to \mathbb{R}, \quad F(z,y) := e_d \cdot \Psi_{x_0}(T^{-1}(z + y\hat{\eta}_{x_0})),$$

is Lipschitz and satisfies $F^{-1}(0) \subset \partial\Omega$. Moreover, applying the chain rule, we obtain the transversality condition $\frac{\partial}{\partial y}F(z,y) \geq \kappa$. As $\widehat{P}_{x_0}$ can be identified with $\mathbb{R}^{d-1}$, the special version of the Implicit Function Theorem (Theorem 2.5) is applicable and we obtain the Lipschitz function φ_{x_0} such that $F(z,y) = 0$ can locally be expressed as $y = \varphi_{x_0}(z)$. The remaining assertions follow by simple computations. □

3 Coercivity of $\mathcal{F}_\varepsilon$ via rigidity

The equi-coercivity of the $\mathcal{F}_\varepsilon$ is directly implied by the equi-coercivity of the $\widetilde{\mathcal{F}}_\varepsilon$, since $\mathcal{F}_\varepsilon \geq \widetilde{\mathcal{F}}_\varepsilon$ holds. For extending the proof of the equi-coercivity of $\widetilde{\mathcal{F}}_\varepsilon$ from [3] we have to generalize the rigidity estimate from [4] from Lipschitz domains to domains with cracks. For this we will use the overlapping decomposition $\Omega_{\mathrm{cr}} = A_+ \cup A_-$ from (2.1).

Definition 3.1 (Rigidity domains). A domain $\widetilde{\Omega} \subset \mathbb{R}^d$ is called a *rigidity domain* if there exists a constant $C > 0$ such that for all $v \in \mathrm{H}^1(\widetilde{\Omega}, \mathbb{R}^d)$ we have

$$\inf_{R \in \mathrm{SO}(d)} \|\nabla v - R\|^2_{L^2(\widetilde{\Omega})} \leq C\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\|^2_{L^2(\widetilde{\Omega})}. \tag{3.1}$$

The smallest such constant we call *rigidity constant* $\mathcal{R}(\widetilde{\Omega})$.

In [4] it is proved that every bounded Lipschitz domain is a rigidity domain. Furthermore, a doubling argument can be found therein similar to the one used in the following proof.

Proposition 3.2. *Let $A, B \subset \mathbb{R}^d$ be bounded rigidity domains such that $A \cap B$ is a rigidity domain with positive volume. Then $A \cup B$ is a rigidity domain, and we have*

$$\mathcal{R}(A \cup B) \leq (2 + 4\mu_A)\mathcal{R}(A) + (2 + 4\mu_B)\mathcal{R}(b) + 4(\mu_A + \mu_B)\mathcal{R}(A \cap B),$$

where $\mu_A = \mathrm{vol}(A)/\mathrm{vol}(A \cap B) \geq 1$ and $\mu_B = \mathrm{vol}(B)/\mathrm{vol}(A \cap B) \geq 1$.

Proof. We fix $v \in \mathrm{H}^1(A \cup B, \mathbb{R}^d)$ and denote by $R_A, R_B, R_{A\cap B} \in \mathrm{SO}(d)$ the minimizers $R \in \mathrm{SO}(d)$ in (3.1) on the corresponding domains. Hence on $A \cup B$ we obtain the estimate

$$\int_{A\cup B} |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x \leq I_A + I_B, \quad \text{where } I_D := \int_D |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x.$$

Writing briefly $\delta(F) := \mathrm{dist}(F, \mathrm{SO}(d))^2$ we can estimate

$$\begin{aligned} I_A &\leq 2\int_A |\nabla v(x) - R_A|^2 \,\mathrm{d}x + 2\int_A |R_A - R_{A\cap B}|^2 \,\mathrm{d}x \\ &\leq 2\mathcal{R}(A)\int_A \delta(\nabla v(x))\,\mathrm{d}x + 2\mu_A \int_{A\cap B} |R_A - R_{A\cap B}|^2 \,\mathrm{d}x, \end{aligned}$$

where we used that R_A is the minimizer for the set A, that $|R_A - R_{A\cap B}|$ is constant and the definition of μ_A. For the second term of I_A we have

$$\begin{aligned} \int_{A\cap B} |R_A - R_{A\cap B}|^2 \,\mathrm{d}x &\leq 2\int_{A\cap B} |R_A - \nabla v(x)|^2 \,\mathrm{d}x + 2\int_{A\cap B} |\nabla v(x) - R_{A\cap B}|^2 \,\mathrm{d}x \\ &\leq 2\mathcal{R}(A)\int_A \delta(\nabla v(x))\,\mathrm{d}x + 2\mathcal{R}(A\cap B)\int_{A\cap B} \delta(\nabla v(x))\,\mathrm{d}x. \end{aligned}$$

Together we find $I_A \leq ((2 + 4\mu_A)\mathcal{R}(A) + 4\mu_A\mathcal{R}(A \cap B)) \int_{A\cup B} \delta(\nabla v(x))\,\mathrm{d}x$.

Interchanging A and B, we find the analogous estimate for I_B, and the result follows. □

In this form, the rigidity estimate applies to our situation by our Assumption 2.1 (c) on the decomposition of Ω in two overlapping Lipschitz domains. We simply apply the above proposition to $\Omega_{\mathrm{cr}} = A \cup B$ with $A = A_+$ and $B = A_-$, see (2.1).

Corollary 3.3 (Ω_{cr} is a rigidity domain). *Let* $(\Omega, \Gamma_{\mathrm{cr}})$ *satisfy* (2.2). *Then* $\Omega_{\mathrm{cr}} = \Omega \setminus \Gamma_{\mathrm{cr}}$ *is a rigidity domain, i.e. there is a constant* $C > 0$ *such that for all* $v \in \mathrm{H}^1(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$ *there exists an* $R \in \mathrm{SO}(d)$ *such that*

$$\|\nabla v - R\|_{L^2(\Omega_{\mathrm{cr}})} \le C\|\mathrm{dist}(\nabla v, \mathrm{SO}(d))\|_{L^2(\Omega_{\mathrm{cr}})}.$$

Before proving coercivity, let us note the following quantitative statement on the rotations showing up when applying the rigidity estimate to small deformations $y_\varepsilon = \mathrm{id} + \varepsilon u$. In [3] as well as for us, it is a main step in the proof of the equi-coercivity. Moreover, we will need it for proving Theorem 4.1 on the local non-interpenetration in the next chapter. The main point is to show that for mappings $y_\varepsilon = \mathrm{id} + \varepsilon u$ the corresponding rotation matrices $R_{\mathrm{id}+\varepsilon u}$ that are minimizers in the rigidity estimate are also close to the identity matrix $I \in \mathbb{R}^{d\times d}$. For this we use the boundary conditions $u|_{\Gamma_{\mathrm{Dir}}} = g$.

Lemma 3.4. *Let* Ω, Γ_{cr}, *and* W *satisfy Assumptions 1.1 and 2.2 and fix* $g \in \mathrm{W}^{1,\infty}(\Omega)$. *Then there exist constants* $C_{\mathcal{F}}, C_{\mathcal{R}} > 0$ *such that for all* $\varepsilon \in]0, 1[$ *and all* $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ *the following holds:*

$$\int_{\Omega_{\mathrm{cr}}} |I + \varepsilon\nabla u(x) - R_{\mathrm{id}+\varepsilon u}|^2\,\mathrm{d}x \le C_F \varepsilon^2 \widetilde{\mathcal{F}}_\varepsilon(u), \tag{3.2a}$$

$$|I - R_{\mathrm{id}+\varepsilon u}|^2 \le C_R \varepsilon^2 \Big(\widetilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\mathrm{Dir}}} |g|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big), \tag{3.2b}$$

where R_v *denotes the minimizer* $R \in \mathrm{SO}(d)$ *in* (3.1) *for fixed* $v \in \mathrm{H}^1(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$.

Proof. Combining the coercivity of W in Assumption 1.1 (c) with the rigidity constant from Corollary 3.3, we obtain (3.2a) with $C_F = \mathcal{R}(\Omega_{\mathrm{cr}})/c_W$.

To derive the second estimate, we set $R_\varepsilon := R_{\mathrm{id}+\varepsilon u}$ and $\zeta_\varepsilon := \fint_{\Omega_{\mathrm{cr}}} (x + \varepsilon u(x) - R_\varepsilon x)\,\mathrm{d}x$. By continuity of the traces and Poincaré's inequality we find

$$\begin{aligned}\int_{\Gamma_{\mathrm{Dir}}} |(x + \varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon|^2\,\mathrm{d}\mathcal{H}^{d-1} &\le C_2\|(x + \varepsilon u(x)) - R_\varepsilon x - \zeta_\varepsilon\|_{H^1(\Omega_{\mathrm{cr}},\mathbb{R}^d)}\\ &\le C_3 \int_{\Omega_{\mathrm{cr}}} |(I + \varepsilon\nabla u(x)) - R_\varepsilon|^2\,\mathrm{d}x \le C_4\varepsilon^2\widetilde{\mathcal{F}}_\varepsilon(u)\end{aligned}$$

with $C_4 = C_F C_3$. Exploiting $u|_{\Gamma_{\mathrm{Dir}}} = g$ and the prefactor ε, we obtain

$$\int_{\Gamma_{\mathrm{Dir}}} |(I - R_\varepsilon)x - \zeta_\varepsilon|^2\,\mathrm{d}\mathcal{H}^{d-1} \le C_5\varepsilon^2\Big(\widetilde{\mathcal{F}}_\varepsilon(u) + \int_{\Gamma_{\mathrm{Dir}}} |g|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big).$$

Note that $R_\varepsilon - I$ is an element of the closed cone K generated by $\mathrm{SO}(d) - I$, on which [3, Lemma 3.3] applies (see the derivation of (3.14) therein). Thus

$$|I - R_\varepsilon|^2 \le C_6 \min_{\zeta\in\mathbb{R}^d} \int_{\Gamma_{\mathrm{Dir}}} |(I - R_\varepsilon)x - \zeta|^2\,\mathrm{d}\mathcal{H}^{d-1},$$

and estimate (3.2b) follows with $C_R = C_6 C_5$. ☐

Now we can proof the equi-coercivity of $\widetilde{\mathcal{F}}_\varepsilon$ on $\mathrm{H}^1_{g,\mathrm{Dir}}$.

Proposition 3.5 (First a priori bound). *Assume that* Ω, Γ_{cr}, *and* W *satisfy Assumptions 1.1 and 2.2. Then there exist* $c_{\mathcal{F}}, C_{\mathcal{F}} > 0$ *such that for all* $\varepsilon \in]0, 1[$ *and all* $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ *we have*

$$\widetilde{\mathcal{F}}_\varepsilon(u) \ge c_{\mathcal{F}}\|u\|^2_{\mathrm{H}^1} - C_{\mathcal{F}}.$$

Proof. By the first part of Assumption 1.1 (c) on W and Corollary 3.3 we have

$$\begin{aligned}\|(I+\varepsilon\nabla u)-R_\varepsilon\|^2_{L^2} &\le C_1\int_{\Omega_{\mathrm{cr}}}\mathrm{dist}^2(I+\varepsilon\nabla u(x),\mathrm{SO}(d))\,\mathrm{d}x\\ &\le C_2\int_{\Omega_{\mathrm{cr}}}W(I+\varepsilon\nabla u(x))\,\mathrm{d}x\le C_2\varepsilon^2\mathcal{F}_\varepsilon(u).\end{aligned}$$

Using both estimates from Lemma 3.4, we proceed to obtain

$$\varepsilon^2\|\nabla u\|^2_{L^2}\le 2(\|I-R_\varepsilon\|^2_{L^2}+\|I+\varepsilon\nabla u-R_\varepsilon\|^2_{L^2})\le\varepsilon^2C_3\Big(\mathcal{F}_\varepsilon(u)+\int_{\Gamma_{\mathrm{Dir}}}|g|^2\,\mathrm{d}\mathcal{H}^{d-1}\Big)$$

with $C_3=2C_F+2C_R$. Dividing by ε^2 and exploiting the boundary conditions in $\mathrm{H}^1_{g,\mathrm{Dir}}$ as well as Poincaré's inequality, we arrive at the desired result. □

The above result shows that sequences $(u_\varepsilon)_\varepsilon$ with bounded energy $\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon)\le C<\infty$ are bounded in $\mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$. The next results provides a weaker, but still useful a priori bound, which implies that $\varepsilon u_\varepsilon$ converges to 0 in $\mathrm{L}^\infty(\Omega;\mathbb{R}^d)$ for energy bounded sequences.

Proposition 3.6 (Second a priori bound). *Let W satisfy Assumption 1.1. Consider a sequence $(u_\varepsilon)_{\varepsilon>0}$ with $\sup_{\varepsilon>0}\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon)\le C_*<\infty$. Then there exists a constant $C>0$ such that*

$$\|\varepsilon u_\varepsilon\|_{\mathrm{W}^{1,p}}\le C\quad\textit{and}\quad\|\varepsilon u_\varepsilon\|_{\mathrm{L}^\infty}\le C\varepsilon^r\tag{3.3}$$

with $r\in\,]0,1[$ arbitrary for $d=2$, and $r=\frac{2(p-d)}{d(p-2)}\in\,]0,1[$ for $d\ge3$.

Proof. The first estimate in (3.3) follows directly from the coercivity in Assumption 1.1 (c) for W:

$$\varepsilon^2C_*\ge\varepsilon^2\widetilde{\mathcal{F}}_\varepsilon(u_\varepsilon)\ge\int_{\Omega_{\mathrm{cr}}}c_W(|I+\varepsilon\nabla u_\varepsilon(x)|^p-C_W)\,\mathrm{d}x\ge\frac{c_W}{2}\|\varepsilon\nabla u_\varepsilon\|^p_{\mathrm{L}^p}-\overline{C}.$$

Using Poincaré's inequality for $u_\varepsilon\in\mathrm{H}^1_{g,\mathrm{Dir}}$, we obtain a uniform bound in $\mathrm{W}^{1,p}(\Omega;\mathbb{R}^d)$.

For the second estimate in (3.3) we use the Gagliardo–Nirenberg interpolation estimate for $f=\varepsilon u_\varepsilon$, where we crucially exploit $p>d$ as provided in Assumption 1.1 (c):

$$\|f\|_{\mathrm{L}^\infty}\le C\|f\|^\theta_{\mathrm{W}^{1,p}}\|f\|^{1-\theta}_{\mathrm{H}^1}.$$

For $d=1$ we can take $\theta=0$ because $\mathrm{H}^1\subset\mathrm{L}^\infty$, and for $d=2$ any $\theta\in\,]0,1]$ is sufficient. For $d\ge3$ we can choose $\theta=\frac{d-2}{d}\frac{p-2}{p}\in\,]0,1[$, and the result follows by using Proposition 3.5, which gives $\|f\|^{1-\theta}_{\mathrm{H}^1}\le\varepsilon^{1-\theta}C$. □

4 The liminf estimate

In contrast to the equi-coercivity the Γ-$\liminf$ estimate for $\mathcal{F}_\varepsilon$ does not follow directly from the Γ-$\liminf$ estimate for $\widetilde{\mathcal{F}}_\varepsilon$, since we have to consider the case $\mathcal{F}(u)=\infty$ carefully, i.e. we have to show that the global injectivity condition (1.1) generates the local non-interpenetration condition (1.2) in the limit $\varepsilon\to0$. This is the content of the following result.

Theorem 4.1 (Local non-interpenetration). *Consider functions $u_\varepsilon,u\in\mathrm{H}^1(\Omega_{\mathrm{cr}},\mathbb{R}^d)$ and assume that $u_\varepsilon\overset{H^1}{\rightharpoonup}u$ and $\liminf_{\varepsilon\to0}\mathcal{F}_\varepsilon(u_\varepsilon)<\infty$; then $[\![u]\!]_\nu\ge0$ holds $\mathcal{H}^{d-1}$-a.e. on Γ_{cr}.*

To prove this theorem, we will first prove the following linearization result concerning the determinant of $I+\varepsilon\nabla u$:

Lemma 4.2. *There exist $C_{\det}>0$ depending on $\Omega,\Gamma_{\mathrm{Dir}},\Gamma_{\mathrm{cr}}$ and the exponent $p>d$ and constants from Assumption 1.1* (c) *such that for all $\varepsilon\in\,]0,1[$ and all $u\in\mathrm{H}^1_{g,\mathrm{Dir}}$ we have*

$$\int_{\Omega_{\mathrm{cr}}}|\det(I+\varepsilon\nabla u(x))-1-\varepsilon\,\mathrm{div}\,u(x)|\,\mathrm{d}x\le\varepsilon^2C_{\det}(\widetilde{\mathcal{F}}_\varepsilon(u)+C_{\det}).$$

Proof. For matrices $A \in \mathbb{R}^{d\times d}$ we have

$$|\det(I + A) - (1 + \operatorname{tr} A)| \le C_d(|A|^2 + |A|^d),$$

where we will insert $A = \varepsilon\nabla u(x)$. To control the term $|A|^d$, we will use $W(I + A)$ and $|I + A|^p \ge \frac{1}{2}|A|^p - C_1$, which yields

$$W(I + A) \ge c_W(|I + A|^p - C_W) \ge \frac{c_W}{2}(|A|^p - C_2).$$

Using $W(F) \ge 0$, we even have $W(I + A) \ge \frac{c_W}{2}[|A|^p - C_2]_+$, where $[a]_+ := \max\{a, 0\}$. Because of $p > d \ge 2$ there exists $C_* > 0$ such that $t^d \le C_*(t^2 + (t^p - C_2)_+)$ for all $t \ge 0$. Thus, inserting $t = |A| = |\varepsilon\nabla u(x)|$, setting $C_3 = C_d(C_* + 1)$, and integrating over Ω_{cr} results in

$$\begin{aligned}\int_{\Omega_{\mathrm{cr}}} |\det(I + \varepsilon\nabla u(x)) - (1 + \varepsilon \operatorname{div} u(x))|\,\mathrm{d}x &\le \int_{\Omega_{\mathrm{cr}}} C_d(|\varepsilon\nabla u|^2 + |\varepsilon\nabla u|^d)\,\mathrm{d}x\\ &\le \int_{\Omega_{\mathrm{cr}}} C_3(|\varepsilon\nabla u|^2 + [|\varepsilon\nabla u|^p - C_2]_+)\,\mathrm{d}x\\ &\le \varepsilon^2 C_3\|\nabla u\|^2_{\mathrm{L}^2} + \varepsilon^2\frac{2C_3}{c_W}\widetilde{\mathcal{F}}_\varepsilon(u).\end{aligned}$$

Together with Proposition 3.5 we see that the assertion holds with $C_{\det}$ chosen as the maximum of $\frac{C_3}{c_{\mathcal{F}}} + \frac{2C_3}{c_W}$ and $\frac{C_3 C_{\mathcal{F}}}{c_{\mathcal{F}}}$. □

With this lemma at hand, we are now able to complete the proof of the main theorem of this section. The idea is to consider the GMS condition (1.1) for global injectivity for $y_\varepsilon = \mathrm{id} + \varepsilon u_\varepsilon$ with nonnegative test functions $\varphi \in \mathrm{C}^\infty(\Omega)$. Dividing by ε and passing to the limit with the help of the above lemma, one can derive the relation $\int_{\Omega_{\mathrm{cr}}} \operatorname{div}(\varphi u)\,\mathrm{d}x \ge 0$, which provides the local non-interpenetration condition (1.2).

Proof of Theorem 4.1. As $\alpha := \liminf_{\varepsilon\to 0} \mathcal{F}_\varepsilon(u_\varepsilon) < \infty$ there is a subsequence (ε_j, u_j) such that $\mathrm{id} + \varepsilon_j u_j$ fulfills the GMS-condition (1.1) and $\det(I + \varepsilon_j\nabla u_j) > 0$ a.e. on Ω. Hence, by rearranging (1.1), for every $\varphi \in \mathrm{C}^\infty_{\mathrm{c}}(\mathbb{R}^d)$ with $\varphi \ge 0$ we have

$$\begin{aligned}0 &\ge \frac{1}{\varepsilon_j}\int_{\Omega_{\mathrm{cr}}} (\varphi(x + \varepsilon_j u_j(x))\det(I + \varepsilon_j\nabla u_j(x)) - \varphi(x))\,\mathrm{d}x\\ &= \frac{1}{\varepsilon_j}\int_{\Omega_{\mathrm{cr}}} \varphi(x + \varepsilon_j u_j(x))(\det(I + \varepsilon_j\nabla u_j(x)) - (1 + \varepsilon_j \operatorname{div} u_j(x)))\,\mathrm{d}x\\ &\quad + \int_{\Omega_{\mathrm{cr}}} \varphi(x + \varepsilon_j u_j(x))\operatorname{div} u_j(x)\,\mathrm{d}x + \int_{\Omega_{\mathrm{cr}}} \frac{1}{\varepsilon_j}(\varphi(x + \varepsilon_j u_j(x)) - \varphi(x))\,\mathrm{d}x.\end{aligned}$$

It follows from Lemma 4.2 and Hölder's inequality that the first summand on the right-hand side is bounded by $\varepsilon_j\|\varphi\|_{\mathrm{L}^\infty} C_{\det}(\alpha + C_{\det})$ and thus converges to 0 for $j \to \infty$. For the latter two summands we use Proposition 3.6, namely

$$\|\varepsilon_j u_j\|_{\mathrm{L}^\infty} \to 0.$$

The second summand converges to $\int_{\Omega_{\mathrm{cr}}} \varphi(x)\operatorname{div} u(x)\,\mathrm{d}x$, because $\operatorname{div} u_j \overset{L^2}{\rightharpoonup} \operatorname{div} u$ weakly and $\varphi\circ(\mathrm{id}+\varepsilon_j u_j) \to \varphi$ strongly in $\mathrm{L}^2(\Omega)$. Finally, the third term can be treated by using the relation

$$\frac{1}{\varepsilon_j}(\varphi(x + \varepsilon_j u_j(x)) - \varphi(x)) = \int_{s=0}^{1} \nabla\varphi(x + s\varepsilon_j u_j(x)) \cdot u_j(x)\,\mathrm{d}s$$

such that weak convergence $u_j \rightharpoonup u$ shows convergence to $\int_{\Omega_{\mathrm{cr}}} \nabla\varphi(x) \cdot u(x)\,\mathrm{d}x$. Altogether the limit $\varepsilon \to 0$ provides three limit values on the right-hand side, namely

$$\begin{aligned}0 &\ge 0 + \int_{\Omega_{\mathrm{cr}}} \varphi(x)\operatorname{div} u(x)\,\mathrm{d}x + \int_{\Omega_{\mathrm{cr}}} \nabla\varphi(x) \cdot u(x)\,\mathrm{d}x\\ &= \int_{\Omega_{\mathrm{cr}}} \operatorname{div}(\varphi u)(x)\,\mathrm{d}x = -\int_{\Gamma_{\mathrm{cr}}} \varphi(x)[\![u]\!]_\nu(x)\mathrm{d}a(x).\end{aligned}$$

For the last identity we now restricted to $\varphi \in \mathrm{C_c}(\Omega)$ such that no boundary terms on $\partial\Omega$ are present. Moreover, we have to recall that u lies in $\mathrm{H}^1_{g,\mathrm{Dir}} \subset \mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$ such that the upper and lower traces at the crack Γ_{cr} may be different. By applying the divergence theorem on the Lipschitz sets A_+ and $A_- \setminus A_+$ (see (2.1)) separately, all terms cancel except for the jump along Γ_{cr}. As $\varphi \geq 0$ was arbitrary, we conclude $[\![u]\!]_\nu \geq 0\,\mathcal{H}^{d-1}$-a.e. on Γ_{cr}. □

We are now ready for deriving the liminf part for our Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{F}$.

Proposition 4.3 (Liminf estimate). *For every sequence $\varepsilon_j \to 0$ and $u_j, u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ with $u_j \rightharpoonup u$ in $\mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$ we have*

$$\mathcal{F}(u) \leq \liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j).$$

Proof. We can assume that $\alpha := \liminf_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) < \infty$, since otherwise the inequality holds trivially. Thus, there is a subsequence (ε_j, u_j) such that $\mathrm{id} + \varepsilon_j u_j$ is globally injective and that $\mathcal{F}_{\varepsilon_j}(u_j) = \widetilde{\mathcal{F}}_{\varepsilon_j}(u_j) \to \alpha$. By Theorem 4.1 we conclude $[\![u]\!]_\nu \geq 0$. Consequently, the liminf estimate above reduces to the liminf estimate for $\widetilde{\mathcal{F}}_\varepsilon$:

$$\mathcal{F}(u) = \widetilde{\mathcal{F}}(u) \leq \alpha = \lim_{j\to\infty} \widetilde{\mathcal{F}}_{\varepsilon_j}(u_j) = \lim_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j).$$

Because $\widetilde{\mathcal{F}}$ is convex, by [13, Lemma 4.2] it suffices to show the pointwise liminf estimate of the respective densities. From Assumption 1.1 (d) we even obtain pointwise equality using

$$\Big|\frac{1}{\varepsilon^2} W(I+\varepsilon G) - \frac{1}{2}\langle G, \mathbb{C}G\rangle\Big| \leq \frac{\delta}{2}\langle G, \mathbb{C}G\rangle \leq \delta\frac{|\mathbb{C}|}{2}|G|^2 \quad \text{for } G \in B_{r_\delta/\varepsilon}(0).$$

Since $\delta > 0$ is arbitrary, for each fixed G we have $\lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2} W(I+\varepsilon G) = \frac{1}{2}\langle G, \mathbb{C}G\rangle$. □

5 The limsup estimate

Showing the limsup estimate in (1.6) amounts in the construction of a recovery sequence $u_\varepsilon \to u$ converging strongly in $\mathrm{H}^1_{g,\mathrm{Dir}} \subset \mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$. In the case without constraints (1.1) or (1.2) the limsup estimate for the Γ-convergence $\widetilde{\mathcal{F}}_\varepsilon \xrightarrow{\Gamma} \widetilde{\mathcal{F}}$ is much simpler since for $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}};\mathbb{R}^d)$ we can take the constant recovery sequence $u_j = u$. Then the extension to general $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ follows by density and the strong continuity of $\widetilde{\mathcal{F}}$, see [3, Proposition 4.1].

Due to the constraints (1.1) and (1.2) in the functionals $\mathcal{F}_\varepsilon$ and $\mathcal{F}$, respectively, we have to do some extra work. First, setting

$$\mathbf{C}_g := \{u \in \mathrm{H}^1_{g,\mathrm{Dir}} \mid [\![u]\!]_\nu \geq 0\},$$

we have to show that $\mathrm{W}^{1,\infty} \cap \mathbf{C}_g$ is dense in $\mathbf{C}_g$ with respect to the H^1 norm. Second we would like to use that $u \in \mathrm{W}^{1,\infty} \cap \mathbf{C}_g$ implies that the close-to-identity deformations $\mathrm{id} + \varepsilon u$ are globally injective for sufficiently small $\varepsilon > 0$. The following example shows that this cannot be expected in general.

Example 5.1 (Non-injectivity). Consider the domain $\widetilde{\Omega} =]-1,1[^2 \subset \mathbb{R}^2$, the crack $\widetilde{\Gamma}_{\mathrm{cr}} = \{0\} \times [0,\infty[$, the cracked domain $\widetilde{\Omega}_{\mathrm{cr}} := \widetilde{\Omega} \setminus \widetilde{\Gamma}_{\mathrm{cr}}$ and the displacement

$$u : \widetilde{\Omega}_{\mathrm{cr}} \to \mathbb{R}^2, \quad u(x_1,x_2) = \begin{cases} (0,0)^\top & \text{for } x_2 < 0, \\ (x_2 + (x_2)^2, x_2)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 > 0, \\ (x_2, 0)^\top & \text{for } x_2 \geq 0 \text{ and } x_1 < 0. \end{cases}$$

Then $u \in \mathrm{W}^{1,\infty}(\widetilde{\Omega}_{\mathrm{cr}};\mathbb{R}^2)$, and along the crack we have $\widetilde{\nu}(0,x_2) = e_1 = (1,0)^\top$ and the jump

$$[\![u]\!]_{\widetilde{\nu}}(0,x_2) = (x_2)^2 > 0,$$

except on the crack tip $\widetilde{\Gamma}_{\mathrm{edge}} = (0,0)^\top$.

However, $y_\varepsilon := \mathrm{id} + \varepsilon u$ is not injective for any $\varepsilon > 0$ near the crack tip. To see this, we set $x_\varepsilon^+ = ((\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2})^\top$ and $x_\varepsilon^- = (-(\frac{\varepsilon}{2})^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2})^\top$ which lie in the first and second quadrant, respectively. We have

$$y_\varepsilon(x_\varepsilon^+) = \Big(\frac{\varepsilon^2}{2} + 3\Big(\frac{\varepsilon}{2}\Big)^3, \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2}\Big)^\top = y_\varepsilon(x_\varepsilon^-),$$

which violates injectivity. Even more, we see that the second quadrant is mapped to $\{y \in \mathbb{R}^2 \mid y_2 \geq 0,\ y_1 < \varepsilon y_2\}$ while the first quadrant is mapped to $\{y \in \mathbb{R}^2 \mid y_2 \geq 0,\ y_1 > h_\varepsilon(y_2)\}$ with $h_\varepsilon(z) = \varepsilon z(1+\varepsilon+z)/(1+\varepsilon)^2$. Thus, each point in the area

$$\{(y_1, y_2) \mid 0 < y_2 < \varepsilon(1+\varepsilon),\ \varepsilon y_2 > y_1 > h_\varepsilon(y_2)\}$$

has two preimages.

The main problem in handling domains with cracks is that of the missing Lipschitz property. For Lipschitz domains Ω we have $\mathrm{C}^{\mathrm{Lip}}(\Omega) = \mathrm{W}^{1,\infty}(\Omega)$ with an estimate

$$\mathrm{Lip}_\Omega(u) \leq C_\Omega \|\nabla u\|_{\mathrm{L}^\infty(\Omega)}. \tag{5.1}$$

For convex domains one has $C_\Omega = 1$ but for general domains the constant depends on the relation between Euclidean distance and the inner distance

$$d_\Omega : \Omega \times \Omega \to \mathbb{R}, \quad d_\Omega(x, \tilde{x}) = \inf\{\mathrm{Length}(\gamma) \mid \gamma \text{ connects } x \text{ with } \tilde{x} \text{ inside } \Omega\}.$$

Then the chain rule guarantees $|u(x) - u(\tilde{x})| \leq \|\nabla u\|_\infty d_\Omega(x, \tilde{x})$. Thus, we can choose

$$C_\Omega = \sup\Big\{\frac{d_\Omega(x, \tilde{x})}{|x - \tilde{x}|} \;\Big|\; x, \tilde{x} \in \Omega,\ x \neq \tilde{x}\Big\}$$

in (5.1).

In a domain Ω_{cr} with a crack, we obviously have $C_{\Omega_{\mathrm{cr}}} = \infty$, since points x^+ and x^- on two opposite sides may have arbitrary small Euclidean distance $|x^+ - x^-|$ but large inner distance $d_{\Omega_{\mathrm{cr}}}(x^+, x^-)$. This explains the difficulty in proving global injectivity, since for a close-to-identity mapping $y_\varepsilon = \mathrm{id} + \varepsilon u$ we have

$$|y_\varepsilon(x^+) - y_\varepsilon(x^-)| \geq |x^+ - x^-| - \varepsilon|u(x^+) - u(x^-)| \geq |x^+ - x^-| - \varepsilon\|\nabla u\|_{\mathrm{L}^\infty(\Omega_{\mathrm{cr}})} d_{\Omega_{\mathrm{cr}}}(x^+, x^-).$$

Thus, for Lipschitz domains Ω with $C_\Omega < \infty$ the global injectivity follows easily if $\varepsilon\|\nabla u\|_{\mathrm{L}^\infty(\Omega)} C_\Omega \leq \frac{1}{2}$, but for cracked domains Ω_{cr} we have to be much more careful. Indeed, we have to require that our functions $u \in \mathbf{C}_g \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$ also have a crack opening that is bounded from below linearly by the distance of the points on the crack from the edge Γ_{edge}. In the next result, we will show that we can achieve this by a suitable forcing apart.

Proposition 5.2. *Let $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}, \mathbb{R}^d) \cap \mathbf{C}_g$. Then there is a sequence $u_k \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}, \mathbb{R}^d) \cap \mathrm{H}^1_{g,\mathrm{Dir}}$ satisfying $u_k \xrightarrow{H^1} u$ and for all $k \in \mathbb{N}$ there exists an $\varepsilon_k > 0$ such that for all $\varepsilon \in\,]0, \varepsilon_k[$, $\mathrm{id} + \varepsilon u_k$ satisfies (1.1).*

Proof. Motivated by the above example we will use the displacement $\widehat{\varphi}_{\delta,\eta} : \widehat{\Omega}_{\mathrm{cr}} \to \mathbb{R}^d$, which forces to two sides of the crack $\widehat{\Gamma}_{\mathrm{cr}}$ apart. For two small parameters $\delta, \eta > 0$ we set $\widehat{\varphi}_{\delta,\eta}(\hat{x}) = \delta\lambda_\eta(\hat{x})\hat{n} \in \mathrm{H}^1(\widehat{\Omega}_{\mathrm{cr}}, \mathbb{R}^d)$ with $\hat{n} = (1, 1, 0, \ldots, 0)^\top \in \mathbb{R}^d$. The scalar function $\lambda_\eta \in \mathrm{W}^{1,\infty}(\widehat{\Omega}_{\mathrm{cr}})$ is given by

$$\lambda_\eta(x_1, x_2, \ldots, x_d) = \begin{cases} 0 & \text{if } x_1 > 1, \\ \min\{1, \frac{1}{\eta}(1 - x_1)\} & \text{for } x_1 \in\,]0, 1] \text{ and } x_2 > 0, \\ -\min\{1, \frac{1}{\eta}(1 - x_1)\} & \text{for } x_1 \in\,]0, 1] \text{ and } x_2 < 0, \\ -1 & \text{for } x_2 \leq 0. \end{cases}$$

Hence the jump of λ_η grows linearly with slope $\frac{1}{\eta}$ with the distance from $\widehat{\Gamma}_{\mathrm{edge}}$ and then saturates at the values ± 1.

We now choose an exponent $\alpha \in\,]1, 2[$ and a positive sequence $\delta_k \to 0$ and set $\eta_k = \delta_k^\alpha$. With this we define $\widehat{\varphi}_k := \widehat{\varphi}_{\delta_k,\eta_k}$ on $\widehat{\Omega}_{\mathrm{cr}}$ and use the pullback of $\widehat{\varphi}_k$ to the reference configuration Ω via the Piola transform $\varphi_k(x) = \nabla T(x)^{-1}\widehat{\varphi}_k(T(x))$, see (2.2). Moreover, using (2.4) we can choose a cut-off function $\gamma \in \mathrm{W}^{1,\infty}(\Omega; [0, 1])$ that is 1 on a neighborhood of Γ_{cr} and vanishes on Γ_{Dir}. With this we define the required sequence

$$u_k \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}, \mathbb{R}^d), \quad x \mapsto u_k(x) = u(x) + \gamma(x)\varphi_k(x).$$

Note that the boundary value on Γ_{Dir} is not changed, i.e. $u_k \in \mathrm{H}^1_{g,\mathrm{Dir}}$.

To show the convergence $u_k = u + \gamma\varphi_k \overset{\mathrm{H}^1}{\to} u$, we need the smallness of $\gamma\varphi_k$. Using

$$\|\gamma\varphi_k\|_{\mathrm{H}^1(\Omega_{\mathrm{cr}})} \le \|\gamma\|_{\mathrm{W}^{1,\infty}(\Omega)}\|\nabla T^{-1}\|_{\mathrm{W}^{1,\infty}(\widehat{\Omega})}\|\widehat{\varphi}_k\|_{\mathrm{H}^1(\widehat{\Omega}_{\mathrm{cr}})}$$

will give the first condition for α

$$\begin{aligned}\|\widehat{\varphi}_k\|^2_{\mathrm{L}^2(\widehat{\Omega})} &\le \mathrm{vol}(\widehat{\Omega})|\widehat{n}|^2\delta_k^2,\\ \|\nabla\widehat{\varphi}_k\|^2_{\mathrm{L}^2(\widehat{\Omega}_{\mathrm{cr}})} &\le \int\limits_{\widehat{\Omega}\cap\{1-\eta_k\le x_1\le 1\}} \Big(\frac{\delta_k}{\eta_k}\Big)^2 \,\mathrm{d}x \le \mathrm{diam}(\widehat{\Omega})^{d-1}\delta_k^{2-\alpha},\end{aligned}$$

where we used $\eta_k = \delta_k^\alpha$. Because of $\alpha < 2$ we have $\|u_k - u\|_{\mathrm{H}^1} \to 0$ as desired.

Let us now come to the global invertibility. We establish the existence of $\varepsilon_k > 0$ by a contradiction argument. For this, we can keep k fixed for most parts of the proof (namely up to and including (5.8)) and assume there is a sequence $\varepsilon_j \to 0$ such that $\mathrm{id} + \varepsilon_j u_k$ is not globally invertible for all $j \in \mathbb{N}$. Thus, there exist $x_j, y_j \in \Omega_{\mathrm{cr}}$ with $(\mathrm{id} + \varepsilon_j u_k)(x_j) = (\mathrm{id} + \varepsilon_j u_k)(y_j)$, i.e.

$$0 \ne x_j - y_j = \varepsilon_j(u_k(y_j) - u_k(x_j)). \tag{5.2}$$

By boundedness of Ω there is a (not relabeled) subsequence such that x_j and y_j both converge. Since (5.2) gives $|x_j - y_j| \le \varepsilon_j\|u_k\|_{\mathrm{L}^\infty(\Omega_{\mathrm{cr}})} \le \varepsilon_j(\|u\|_{L^\infty(\Omega_{\mathrm{cr}})} + 3\delta_k)$, these two limits are the same, from now denoted by z_∞. We next establish the following claim:

Claim. *The point z_∞ lies in the crack edge $\Gamma_{\mathrm{edge}} = T^{-1}(\{(1,0)\}\times\mathbb{R}^{d-2})$, and the convergence gives a very specific picture, i.e. $T(x_j)\cdot e_2 > 0$, $T(y_j)\cdot e_2 < 0$, $T(x_j)\cdot e_1 < 1$, $T(y_j)\cdot e_1 < 1$, and*

$$\frac{|x_j - y_j|}{(1 - T(x_j)\cdot e_1) + (1 - T(y_j)\cdot e_1)} \to 0 \quad \text{as } j \to \infty. \tag{5.3}$$

That means that x_j and y_j converge to z_∞ by approaching the crack asymptotically from left above and from left below, respectively.

A major part of the proof of the claim is due to Lipschitz continuity. If both, x_j and y_j, are in A_+ or both are in A_-, then with $L_k := \mathrm{Lip}_{A_\pm}(u_k)$ we would obtain

$$|x_j - y_j| = \varepsilon_j|u_k(y_j) - u_k(x_j)| \le \varepsilon_j L_k d_U(x_j, y_j) \le \varepsilon_j L_k C_U|x_j - y_j|.$$

For $\varepsilon_j L_k C_U < 1$ this implies $x_j = y_j$, which contradicts (5.2). Thus, we have $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ or vice versa. Using $x_j, y_j \to z_\infty$, we conclude $z_\infty \in \Gamma_{\mathrm{cr}}$.

For the subsequent arguments we choose the notation such that always $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$. If $\widehat{z}_\infty := T(z_\infty) \in \widehat{\Gamma}_{\mathrm{cr}} \setminus \widehat{\Gamma}_{\mathrm{kink}}$, we have a normal vector to $\widehat{\Gamma}_{\mathrm{cr}}$ given by

$$\widehat{\nu} = \begin{cases} e_1 := (1, 0, \dots, 0) & \text{for } e_1\cdot z_\infty = 0,\\ e_2 := (0, 1, 0, \dots, 0) & \text{for } e_2\cdot z_\infty = 0.\end{cases}$$

By the above choice $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ we obtain

$$(T(x_j) - T(y_j))\cdot\widehat{\nu} > 0 \tag{5.4}$$

for sufficiently big $j \in \mathbb{N}$. Thus, exploiting the smoothness of T across the crack and relation (5.2) again, we obtain

$$\begin{aligned}0 < \frac{1}{\varepsilon_j}(T(x_j) - T(y_j))\cdot\widehat{\nu} &= \int_0^1 \nabla T(x_j + t(y_j - x_j))\,\mathrm{d}t\,\frac{1}{\varepsilon_j}(x_j - y_j)\cdot\widehat{\nu}\\ &\overset{(5.2)}{=} \int_0^1 \nabla T(x_j + t(y_j - x_j))\,\mathrm{d}t\,(u_k(y_j) - u_k(x_j))\cdot\widehat{\nu}.\end{aligned}$$

Passing to the limit $j \to \infty$, we find the jump condition

$$0 \le \nabla T(z_\infty)(u_k^-(z_\infty) - u_k^+(z_\infty)) \cdot \widehat{\nu} = (u_k^-(z_\infty) - u_k^+(z_\infty)) \cdot \nabla T(z_\infty)^\top \widehat{\nu}.$$

However, because of the non-interpenetration condition $[\![u_k]\!]_\nu = [\![u]\!]_\nu + [\![\varphi_k]\!]_\nu \ge 0$, where $[\![\varphi_k]\!]_\nu > 0$ except on the crack edge, we have

$$(u_k^+(z_\infty) - u_k^-(z_\infty)) \cdot \nabla T(z_\infty)^\top \widehat{\nu} \ge 0, \quad \text{where equality holds if and only if } z_\infty \in \Gamma_{\mathrm{edge}}. \tag{5.5}$$

Thus, we conclude that z_∞ cannot lie in $\Gamma_{\mathrm{cr}} \setminus (\Gamma_{\mathrm{kink}} \cup \Gamma_{\mathrm{edge}})$.

It remains to exclude $z_\infty \in \Gamma_{\mathrm{kink}}$. If this would be the case the case, then both (5.4) and (5.5) still hold for some $\widehat{\nu}$ but for different reasons. One the one hand, using $x_j \in A_+$ and $y_j \in A_-$ for all j, there is a subsequence such that condition (5.4) holds for either $\widehat{\nu} = e_1$ or $\widehat{\nu} = e_2$. On the other hand, (5.5) holds for both $\widehat{\nu} = e_1$ and $\widehat{\nu} = e_2$ by continuity of u_k. Thus, we similarly conclude $z_\infty \notin \Gamma_{\mathrm{kink}}$, and $z_\infty \in \Gamma_{\mathrm{edge}}$, which is the first part of the above claim.

From here on let $\widehat{U} := B_\varrho(T(z_\infty)) \subset \widehat{\Omega}$ with $\varrho < 1$ such that $\widehat{U}$ does cannot touch Γ_{kink}. Then $T(x_j), T(y_j) \in \widehat{U}$ for j big enough, and $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$ gives

$$T(x_j) \cdot e_1 < 1, \quad T(x_j) \cdot e_2 > 0, \quad T(y_j) \cdot e_1 < 1, \quad T(y_j) \cdot e_2 < 0,$$

which is the second part of the above claim.

To see the last part of the claim, note that we have either (5.3) as claimed or there is a subsequence (not relabeled) such that

$$(1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1) \le C|x_j - y_j| \tag{5.6}$$

with some positive constant C independent of j. We assume now (5.6) in order to generate a contradiction. Indeed, the smallness of the quantities on the left-hand side allow us to exploit the Lipschitz continuity of u_k on $T^{-1}(\{\widehat{x} \in \widehat{U} \mid \widehat{x}_1 \ge 1\})$, which is the domain to the right of the crack edge containing the intersection $A_+ \cap A_-$. Introducing the projections

$$x_j' := T^{-1}(T(x_j) + (1 - T(x_j) \cdot e_1)e_1) \quad \text{and} \quad y_j' := T^{-1}(T(y_j) + (1 - T(y_j) \cdot e_1)e_1),$$

we can compare them with x_j and y_j, respectively, as well as x_j' and y_j' to each other:

$$\begin{aligned}
\frac{1}{\varepsilon_j}|x_j - y_j| &= |u_k(x_j) - u_k(y_j)| \\
&\le |u_k(x_j) - u_k(x_j')| + |u_k(x_j') - u_k(y_j')| + |u_k(y_j') - u_k(y_j)| \\
&\le L_k(|x_j - x_j'| + |x_j' - y_j'| + |y_j' - y_j|) \\
&\le L_k(2|x_j - x_j'| + |x_j - y_j| + 2|y_j' - y_j|) \\
&\le L_k\big(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty}(|T(x_j) - T(x_j')| + |T(y_j') - T(y_j)|)\big) \\
&\le L\big(|x_j - y_j| + 2\|\nabla T^{-1}\|_{L^\infty}((1 - T(x_j) \cdot e_1) + (1 - T(y_j) \cdot e_1))\big) \\
&\overset{(5.6)}{\le} L|x_j - y_j|(1 + 2\|\nabla T^{-1}\|_{L^\infty} C).
\end{aligned}$$

After dividing by $|x_j - y_j| \ne 0$, we see that this contradicts $\varepsilon_j \to 0$, such that (5.6) must be false, and hence (5.3) and the whole above claim is established.

We still have to produce a contradiction to show that (5.2) is false. But now we can use the relations in the above claim, in particular the convergence (5.3). To this end, we will use the assumption $\alpha > 1$ in the definition $\eta_k = \delta_k^\alpha$.

In the following calculation we use the abbreviation

$$A_j := \int_0^1 \nabla T(x_j + t(y_j - x_j))\,\mathrm{d}t \in \mathbb{R}^{d\times d}$$

and insert relation (5.2) (recall $u_k = u + \gamma\varphi_k$ with $\gamma \equiv 1$ in a neighborhood of Γ_{cr}):

$$\begin{aligned}
0 &\le \frac{1}{\varepsilon_j}(T(x_j) - T(y_j)) \cdot e_2 = \frac{1}{\varepsilon_j} A_j(x_j - y_j) \cdot e_2 \overset{(5.2)}{=} A_j(u_k(y_j) - u_k(x_j)) \cdot e_2 \qquad (5.7)\\
&= A_j\big((u(y_j) - u(y_j')) + (u(y_j') - u(x_j')) + (u(x_j') - u(x_j)) + (\varphi_k(y_j) - \varphi_k(x_j))\big) \cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty} \|\nabla u\|_{\mathrm{L}^\infty} (|y_j - y_j'| + |y_j' - x_j'| + |x_j' - x_j|) + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty} \|\nabla u\|_{\mathrm{L}^\infty} \big(|x_j - y_j| + 2(|x_j - x_j'| + |y_j - y_j'|)\big) + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2\\
&\le \|\nabla T\|_{\mathrm{L}^\infty} \|\nabla u\|_{\mathrm{L}^\infty} \big(|x_j - y_j| + 2\|\nabla T^{-1}\|_{\mathrm{L}^\infty}((1 - T(x_j)\cdot e_1) + (1 - T(y_j)\cdot e_1))\big) + A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2.
\end{aligned}$$

Dividing by $(1 - T(x_j)\cdot e_1) + (1 - T(y_j)\cdot e_1)$ and taking the limit $j \to \infty$, the assumed convergence (5.3) implies that the first summand of the right-hand side converges to the constant $C_u := 2\|\nabla T\|_{\mathrm{L}^\infty}\|\nabla u\|_{\mathrm{L}^\infty}\|\nabla T^{-1}\|_{\mathrm{L}^\infty}$, which is independent of k. The idea is now to show that for our choice of $\alpha > 1$ the second summand makes the right-hand side negative for sufficiently small δ_k, which then produces a contradiction.

For this, we exploit the definition of φ_k via the function λ_{η_k} and the choices $x_j \in A_+ \setminus A_-$ and $y_j \in A_- \setminus A_+$. Since x_j and y_j are near Γ_{edge}, we obtain

$$\lambda_\eta(T(x_j)) = \frac{1}{\eta}(1 - T(x_j)\cdot e_1) \quad \text{and} \quad \lambda_\eta(T(y_j)) = -\frac{1}{\eta}(1 - T(y_j)\cdot e_1).$$

Inserting this with $\eta = \delta_k^\alpha$, we find

$$\begin{aligned}
A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2 &= \delta_k A_j\big(\nabla T(y_j)^{-1}\lambda_{\delta_k^\alpha}(T(y_j))\widehat{n} - \nabla T(x_j)^{-1}\lambda_{\delta_k^\alpha}(T(x_j))\widehat{n}\big) \cdot e_2\\
&= \delta_k A_j\Big(-\frac{1}{\delta_k^\alpha}\nabla T(y_j)^{-1}(1 - T(y_j)\cdot e_1)\widehat{n} - \frac{1}{\delta_k^\alpha}\nabla T(x_j)^{-1}(1 - T(x_j)\cdot e_1)\widehat{n}\Big) \cdot e_2\\
&= -\delta_k^{1-\alpha}\big((1 - T(y_j)\cdot e_1)e_2 \cdot A_j\nabla T(y_j)^{-1}\widehat{n} + (1 - T(x_j)\cdot e_1)e_2 \cdot A_j\nabla T(x_j)^{-1}\widehat{n}\big).
\end{aligned}$$

The matrices $A_j\nabla T(y_j)^{-1}$ and $A_j\nabla T(x_j)^{-1}$ converge to $I \in \mathbb{R}^{d\times d}$ by dominated convergence and continuity of ∇T, thus we have $e_2 \cdot A_j\nabla T(x_j)^{-1}\widehat{n} \to e_2 \cdot \widehat{n} = 1$ and similarly for y_j. Because both $(1 - T(x_j)\cdot e_1)$ and $(1 - T(y_j)\cdot e_1)$ are positive, this implies the convergence

$$\delta_k^{\alpha-1}\frac{A_j(\varphi_k(y_j) - \varphi_k(x_j)) \cdot e_2}{(1 - T(x_j)\cdot e_1) + (1 - T(y_j)\cdot e_1)} \to -1 \quad \text{for } j \to \infty. \qquad (5.8)$$

Inserting this into (5.7) divided by $(1 - T(x_j)\cdot e_1) + (1 - T(y_j)\cdot e_1) > 0$, we obtain $0 \le 2C_u - \frac{1}{2}\delta_k^{1-\alpha}$ for each fixed k in the limit $j \to \infty$. Thus, making δ_k smaller if necessary, we arrive at a contradiction, because $\delta_k \to 0$ and $\alpha > 1$.

This shows that (5.2) cannot hold for $\varepsilon_j \to 0$. Thus, the existence of $\varepsilon_k > 0$ is established, and Proposition 5.2 is proved. □

To extend the achieved knowledge from the dense set $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}; \mathbb{R}^d) \cap \mathbf{C}_g$ to the general case $u \in \mathbf{C}_g$, we have to show that all functions $u \in \mathbf{C}_g$ can be approximated by $u_k \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}; \mathbb{R}^d) \cap \mathbf{C}_g$, i.e. we have to approximate under the convex constraint of local non-interpenetration. Similar approximation results for more classical state constraints are contained in [7, 8].

To handle our conditions of non-negativity of jumps over the crack, we can use a reflection and decomposition into odd and even parts. To simplify the reading of the following proof, we illustrate this idea by a simple scalar two-dimensional problem.

Example 5.3 (Straight crack in $\mathbb{R}^2$). We consider $\Omega = \mathbb{R}^2$, $\Gamma_{\mathrm{cr}} = \mathbb{R} \times \{0\}$, and a function $u \in \mathrm{H}^1(\Omega \setminus \Gamma_{\mathrm{cr}})$ with $[\![u]\!]_\nu \ge 0$. To find a smooth approximation, we define

$$u^{\mathrm{even}}(x) = \frac{1}{2}(u(x_1, x_2) + u(x_1, -x_2)) \quad \text{and} \quad u^{\mathrm{odd}}(x) = \frac{1}{2}(u(x_1, x_2) - u(x_1, -x_2))$$

such that $u = u^{\mathrm{even}} + u^{\mathrm{odd}}$, $[\![u^{\mathrm{even}}]\!]_\nu = 0$, and $[\![u^{\mathrm{odd}}]\!]_\nu = 2u^{\mathrm{odd}}(\cdot, 0^+) = [\![u]\!]_\nu$.

We can easily approximate u^{even} by $v_k \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2)$, since it lies in $\mathrm{H}^1(\mathbb{R}^2)$. For u^{odd} we do not want to smoothen the jump along Γ. Hence, we define a “positive extension via reflection” as follows:

$$\widetilde{u}(x_1, x_2) = \begin{cases} u^{\mathrm{odd}}(x_1, x_2) & \text{for } x_2 > 0,\\ \max\{0, u^{\mathrm{odd}}(x_1, -x_2)\} & \text{for } x_2 < 0.\end{cases}$$

Since $\widetilde{u}(\cdot, 0^+) = u^{\mathrm{odd}}(\cdot, 0^+) = \frac{1}{2}[\![u]\!]_\nu \geq 0$, we conclude that $[\![\widetilde{u}]\!]_\nu = 0$, which implies $\widetilde{u} \in \mathrm{H}^1(\mathbb{R}^2)$. Defining convolution kernels $\psi_k \in \mathrm{C}_\mathrm{c}^\infty(\mathbb{R}^2)$ with $\psi_k \geq 0$, $\int_{\mathbb{R}^2} \psi_k \,\mathrm{d}y = 1$, and $\mathrm{supp}(\psi_k) \subset B_{1/k}((0, -\frac{1}{k})) \subset \mathbb{R} \times]-\infty, 0[$, we can define $\widetilde{v}_k = \psi_k * \widetilde{u} \in \mathrm{C}^\infty(\mathbb{R}^2)$ and check that $\widetilde{v}_k \to \widetilde{u}$ in $\mathrm{H}^1(\mathbb{R}^2)$ and that $\widetilde{v}_k(x_1, 0) \geq 0$, because $\widetilde{u}(x_1, x_2) \geq 0$ for $x_2 \leq 0$ and the kernel ψ_k also has its support in $\mathbb{R} \times]-\infty, 0[$. Thus, setting

$$u_k(x_1, x_2) = v_k(x_1, x_2) + \mathrm{sign}(x_2)\widetilde{v}_k(x_1, |x_2|),$$

we obtain $u_k \in \mathrm{C}^\infty(\Omega \setminus \Gamma_{\mathrm{cr}})$ with $u_k \to u$ in $\mathrm{H}^1(\Omega \setminus \Gamma_{\mathrm{cr}})$ and $[\![u_k]\!]_\nu \geq 0$.

The analogous construction for our general crack $\Gamma_{\mathrm{cr}} \subset \Omega$ works similarly by mapping the displacements $u : \Omega_{\mathrm{cr}} \to \mathbb{R}^d$ via the Piola transform onto displacements $\widehat{u} : \widehat{\Omega}_{\mathrm{cr}} \to \mathbb{R}^d$, where the positivity of the jumps is preserved, see (2.3). To simplify the proof, we introduce some notation for mollifiers and shifts. We choose a fixed convolution kernel $\psi \in \mathrm{C}_\mathrm{c}(\mathbb{R}^d)$ with $\mathrm{supp}\,\psi \subset B_1(0)$, $\psi \geq 0$, $\int_{\mathbb{R}^d} \psi \,\mathrm{d}x = 1$, and $\psi(x) = \psi(\widetilde{x})$ if $|x| = |\widetilde{x}|$. With this we define the mollifier M_k^a with shift vector $a \in \mathbb{R}^d$ via

$$(M_k^a u)(x) = \int\limits_{|z|\leq 1} \psi(z) u\Big(x - \frac{1}{k}(z - a)\Big) \,\mathrm{d}z = \int\limits_{|y-x|<\frac{1}{k}} k^d \psi(k(x - y)) u\Big(y + \frac{1}{k} a\Big) \,\mathrm{d}y.$$

The shift vector a will be chosen differently above and below a crack to avoid intersecting the crack, see, e.g., (5.9).

Of course, we can take full advantage that the crack $\widehat{\Gamma}_{\mathrm{cr}}$ is piecewise flat. The only point that is more delicate arises for points in the intersection of Γ_{cr} and $\partial\Omega$.

Proposition 5.4. *Let $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ with $[\![u]\!]_\nu \geq 0$. Then there is a sequence $u_k \in \mathrm{H}^1_{g,\mathrm{Dir}} \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}, \mathbb{R}^d)$ with $[\![u_k]\!]_\nu \geq 0$ such that $u_k \to u$ in $\mathrm{H}^1(\Omega_{\mathrm{cr}}; \mathbb{R}^d)$.*

Proof. First, we show that it suffices to consider the case $(\widehat{\Omega}, \widehat{\Gamma}_{\mathrm{cr}})$ instead of the more general $(\Omega, \Gamma_{\mathrm{cr}})$. For this we can use the Piola transforms $P_T : \mathrm{H}^1(\widehat{\Omega}) \to \mathrm{H}^1(\Omega)$ from (2.2). With the inverse mapping $(P_T)^{-1} = P_{T^{-1}}$. As T and T^{-1} lie in $\mathrm{W}^{2,\infty}$, we see that P_T is also a linear bounded map from $\mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}}, \mathbb{R}^d)$ into $\mathrm{W}^{1,\infty}(\widehat{\Omega}_{\mathrm{cr}}, \mathbb{R}^d)$ with linear bounded inverse $P_{T^{-1}}$. Thus, for the given $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ with $[\![u]\!]_\nu \geq 0$ we may consider $\widehat{u} := P_{T^{-1}} u \in \mathrm{H}^1(\widehat{\Omega})$ with $[\![\widehat{u}]\!]_{\widehat{\nu}} \geq 0$. If we find approximations $\widehat{u}_k$, then $u_k = P_T \widehat{u}_k$ provides the desired sequence.

Second, we observe that it suffices to show the assertion locally in a neighborhood U of each point $x^* \in \mathrm{clos}(\widehat{\Omega})$ because by compactness we have a finite cover of such neighborhoods and recombination by partition of unity gives the result. In all cases we consider $U = B_\delta(x^*) \cap \widehat{\Omega}$ and may consider $\widehat{u}$ with $\mathrm{supp}(\widehat{u}) \subset B_\varepsilon(x^*)$ for some $\varepsilon \in]0, \delta[$. Thus, convolutions $M_k^a \widehat{u}$ will be well defined for k sufficiently large, as long as $\mathrm{supp}(\widehat{u}) + B_{1/k}(a)$ stays inside of $B_\delta(x^*) \cap \widehat{\Omega}_{\mathrm{cr}}$.

We now discuss the occurring different cases.

Bulk points in Ω_{cr}. For $x^* \in \widehat{\Omega}_{\mathrm{cr}}$ and a ball $B_\delta(x^*) \Subset \widehat{\Omega}_{\mathrm{cr}}$ the convolution $\widehat{u}_k = M_k^0 \widehat{u}$ is smooth and converges in $\mathrm{H}^1(B_\delta(x^*), \mathbb{R}^d)$ to $\widehat{u}$.

Free boundary. For the case $x^* \in \partial\widehat{\Omega} \setminus (\widehat{\Gamma}_{\mathrm{cr}} \cup \widehat{\Gamma}_{\mathrm{Dir}})$ we extend $\widehat{u}$ to the outside of $\widehat{\Omega}$ first. For this we take a ball $B_\delta(x^*)$ with $B_{2\delta}(x^*) \cap \widehat{\Gamma}_{\mathrm{cr}} = \emptyset$ and by the Lipschitz property of $\partial\widehat{\Omega}$ there is a bi-Lipschitz chart $\Psi : B_\delta(x^*) \to \mathbb{R}^d$ with $\widehat{\Omega} \cap B_\delta(x^*) \subset \Psi^{-1}(\{y_d > 0\})$, $\partial\widehat{\Omega} \cap B_\delta(x^*) \subset \Psi^{-1}(\{y_d = 0\})$, and $B_\delta(x^*) \setminus \mathrm{clos}(\widehat{\Omega}) \subset \Psi^{-1}(\{y_d < 0\})$. Now an $\mathrm{H}^1(B_\delta(x^*), \mathbb{R}^d)$-extension $\widetilde{u}$ of $\widehat{u}$ is given by $\widetilde{u}(x) = u(\Psi^{-1}(R(\Psi(x))))$, where $R(y) = (y_1, \ldots, y_{d-1}, |y_d|)$. The desired approximations are then given by $\widehat{u}_k = M_k^0 \widetilde{u}|_{B_\delta(x^*) \cap \widehat{\Omega}}$.

Dirichlet part of the boundary. For $x^* \in \Gamma_{\mathrm{Dir}}$ there exists $B_\delta(x^*)$ disjoint from the crack Γ_{cr}, and by the definition of $\mathrm{H}^1_{g,\mathrm{Dir}}$ there is a $\mathrm{W}^{1,\infty}$-sequence coinciding with g on $\widehat{\Gamma}_{\mathrm{Dir}}$.

Flat parts of the crack. For $x^* \in \widehat{\Gamma}_{\mathrm{cr}} \setminus (\widehat{\Gamma}_{\mathrm{edge}} \cup \widehat{\Gamma}_{\mathrm{kink}} \cup \partial\widehat{\Omega})$ we proceed similarly as in Example 5.3. Since x^* is neither a point in $\partial\widehat{\Omega}$ nor in the crack kink $\widehat{\Gamma}_{\mathrm{kink}}$ or the crack edge $\widehat{\Gamma}_{\mathrm{edge}}$, we can assume, without loss of generality, that $x^* \in \{0\} \times]0, \infty[\times \mathbb{R}^{d-2}$ with $\widehat{\nu} = e_1$, the case $x^* \in]0, 1[\times \{0\} \times \mathbb{R}^{d-2}$ with $\widehat{\nu} = e_2$ is analogous. We take $B_\delta(x^*)$ that touches neither of the critical parts.

For a fixed $n \in \{2, \ldots, d\}$ we approximate the component $v = \widehat{u}^{[n]}$ of $\widehat{u} = (\widehat{u}^{[1]}, \ldots, \widehat{u}^{[d]})$ simply via

$$v_k(x) = M_k^{\mathrm{sign}(x_1)e_1} v \quad \text{for } x \in B_\delta(x^*) \setminus \widehat{\Gamma}_{\mathrm{cr}}, \tag{5.9}$$

where we can use that the parts left and right of the crack at $x_1 = 0$ are independent (no jump conditions. The shift vectors $\pm e_1$ take care that mollifications never touch the crack.

For $n = 1$ we need to be more careful since $v = \widehat{u}^{[1]}$ has to have a positive jump over the crack, namely $v(0^+, \cdot) - v(0^-, \cdot) = [\![\widehat{u}]\!]_{\widehat{\nu}} \geq 0$. We define the odd and even parts via

$$v^{(i)}(x_1, \ldots, x_d) = \frac{1}{2}\big(v(x_1, \ldots, x_d) + (-1)^i v(-x_1, x_2, \ldots, x_d)\big).$$

The even function $v^{(0)}$ lies in $\mathrm{H}^1(B_\delta(x^*))$, because it has no jump, thus we can approximate $v^{(0)}$ by the even functions $v_k^{(0)} = M_k^0 v^{(0)}$.

The odd function $v^{(1)}$ has a positive jump which needs to be preserved. Hence we restrict it to the semi-ball with $x_1 > 0$ and use the nonnegative extension of Example 5.3, namely $x \mapsto \max\{0, v^{(1)}(-x_1, x_2, \ldots)\}$ for $x_1 < 0$. This leads to $\widetilde{v} \in \mathrm{H}^1(B_\delta(x^*))$, which is nonnegative for $x_1 < 0$. Thus, the mollifications $\widetilde{v}_k = M_k^{-e_1}\widetilde{k}$ converge to $\widetilde{v}$, and the shift vector $-e_1$ guarantees that $\widetilde{v}_k$ is nonnegative for $x_1 \leq 0$, which implies that the trace of $\widetilde{v}_k$ on $B_\delta(x^*) \cap \{x_1 = 0\}$ is nonnegative.

The desired approximations for $v = u^{[1]}$ are then given by

$$u_k^{[1]}(x) = v_k(x) = M_k^0 v^{(0)}(x) + \operatorname{sign}(x_1) v_k^{(1)}(|x_1|, x_2, \ldots, x_d).$$

Crack edge. For a point $x^* \in \widehat{\Gamma}_{\text{edge}} \setminus \partial\widehat{\Omega}$ we have $\widehat{\nu} = e_2$. For a $\delta \in \,]0, 1[$ with $B_{2\delta}(x^*) \cap \partial\widehat{\Omega} = \emptyset$ we proceed similarly. For $n \neq 2$ we consider the component $v = \widehat{u}^{[n]}$, which may have an arbitrary jump along the intersection $B_\delta(x^*) \cap \{x_1 < 1 \text{ and } x_2 = 0\}$ but has no jump along $\{x_1 > 1 \text{ and } x_2 = 0\}$.

To handle this case, we work with a continuously varying shift vector $a_k(x)$ as follows. Let

$$h(x) = \max\{0, \min\{x, 1\}\}$$

and set

$$a_k : B_\delta(x^*) \setminus \widehat{\Gamma}_{\text{cr}} \to \mathbb{R}^d, \quad x \mapsto \operatorname{sign}(x_2) h\Big(\sqrt{k}(1 - x_1)\Big) e_2 + \sqrt{k}\, e_1.$$

The main observation is that $x \mapsto \frac{1}{k} a_k(x)$ is a function in $\mathrm{W}^{1,\infty}(B_\delta(x^*) \setminus \widehat{\Gamma}_{\text{cr}}; \mathbb{R}^d)$ with norm bounded by $C/\sqrt{k}$. Moreover, for all $x \in B_\delta(x^*) \setminus \widehat{\Gamma}_{\text{cr}}$ the convolution integration domain $x + B_{1/k}(a_k(x))$ does not intersect $\widehat{\Gamma}_{\text{cr}}$. Thus, $v_k = M_k^{a_k(x)} v$ is well-defined, smooth on $B_\delta(x^*) \setminus \widehat{\Gamma}_{\text{cr}}$, and converges to v.

The case $v = \widehat{u}^{[2]}$ is more difficult, since we need to maintain the non-negativity of the jump. Using the even and odd parts

$$v^{(i)}(x_1, \ldots, x_d) = \frac{1}{2}\big(u(x_1, \ldots, x_d) + (-1)^i v(x_1, -x_2, x_3, \ldots, x_d)\big),$$

we see that the even part $v^{(0)}$ lies in $\mathrm{H}^1(B_\delta(x^*))$, so we use the mollifications $v_k^{(0)} = M_k^0 v^{(0)}$.

The odd part $v^{(1)}$ is delicate, since we need non-negativity of the jump for $x_1 < 1$ and no jump for $x_1 > 1$. For this we restrict $v^{(1)}$ to the upper semi-ball $B_\delta(x^*) \cap \{x_2 > 0\}$ and extend it to a function $w \in \mathrm{H}^1(B_\delta(x^*))$ which is 0 in $\{x_1 > 1 \text{ and } x_2 < 0\}$. For this, we define a piecewise affine bi-Lipschitz S map between the triangle $\{x \in \mathbb{R}^2 \mid 0 \leq x_2 \leq 1 - |x_1 - 1|\}$ and the square $[0, 1] \times [-1, 0]$ via

$$S(x_1, x_2) = \big(\min\{1, x_1\} - x_2, \min\{0, 1 - x_1\} - x_2\big).$$

This mapping keeps $(1, 0)$ fixed, is the identity on the line $L_1 := [0, 1] \times \{0\}$, and maps the line $L_2 := [1, 2] \times \{0\}$ to the line $L_3 := \{1\} \times [-1, 0]$. Thus, setting

$$w(x) = \begin{cases} v^{(1)}(x) & \text{for } x_2 > 0, \\ \max\{0, v^{(1)}(S^{-1}(x_1, x_2), x_3, \ldots)\} & \text{for } x_2 < 0 \text{ and } x_1 < 1, \\ 0 & \text{for } x_2 < 0 \text{ and } x_1 > 1, \end{cases}$$

we find that $w \in \mathrm{H}^1(B_\delta(x^*))$, since the traces on L_1, L_2, and L_3 match by construction. Thus, as w is nonnegative for $x_2 < 0$ and even 0 if additionally $x_1 > 1$, we see that the approximation $w_k = M_k^{e_1 - e_2} w$ satisfies $w_k \to w \in \mathrm{H}^1(B_\delta(x^*))$ and is still nonnegative for $x_2 < 0$ and even 0 if additionally $x_1 > 1$.

As above we conclude that $v_k = v_k^{(0)} + \operatorname{sign}(x_2) w_k$ lies in $\mathrm{W}^{1,\infty}(B_\delta(x^*) \setminus \widehat{\Gamma}_{\text{cr}})$ and converges to $v = \widehat{u}^{[2]}$.

Crack kink. Let us come to $x^* \in \widehat{\Gamma}_{\rm kink}$ with $B_{2\delta}(x^*) \cap (\partial\widehat{\Omega} \cup \widehat{\Gamma}_{\rm edge}) = \emptyset$. We again decompose the components $v = \widehat{u}^{[n]}$ in odd and even parts, but now we have two hyperplanes, so we need four parts with evenness and oddness in x_1 and x_2, respectively. For $i, j \in \{0, 1\}$ we set

$$\begin{aligned} v^{(i,j)}(x) = \frac{1}{4}\big(&v(x_1, x_2, x_3, \dots, x_d) + (-1)^i v(-x_1, x_2, x_3, \dots, x_d) \\ &+ (-1)^j v(x_1, -x_2, x_3, \dots, x_d) + (-1)^{i+j} v(-x_1, -x_2, x_3, \dots, x_d)\big). \end{aligned}$$

Thus, each function $v^{(i,j)}$ is completely determined by its value in the positive quadrant

$$Q_+ := \{x \in \mathbb{R}^d \mid x_1, x_2 > 0\},$$

namely

$$v^{(i,j)}(x_1, x_2, x_3, \dots) = \operatorname{sign}(x_1^i x_2^j) v^{(i,j)}(N(x)), \quad \text{where } N(x) = (|x_1|, |x_2|, x_3, \dots, x_d).$$

Each component will be approximated by functions $v_k^{(i,j)} \in \mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ such that the desired full approximations v_k of v take the form

$$v_k(x) = \sum_{i,j=0}^{1} \operatorname{sign}(x_1^i x_2^j) v_k^{(i,j)}(N(x)). \tag{5.10}$$

However, to guarantee that v_k lies in $\mathrm{W}^{1,\infty}(B_\delta(x^*)\backslash\widehat{\Gamma}_{\rm cr})$ we have to show that there are no jumps at (i) $\Sigma_1 := \{x_1 = 0 \text{ and } x_2 < 0\}$ and at (ii) $\Sigma_2 := \{x_1 < 0 \text{ and } x_2 = 0\}$. Moreover, for $n \in \{1, 2\}$ we need a non-negativity condition on the jump along $C_n := \{x_n = 0 \text{ and } x_{3-n} > 0\}$:

(i) $d_k^{(1)} := v_k^{(1,0)} - v_k^{(1,1)}$ has trace 0 on $-\Sigma_1 = C_1$,
(ii) $d_k^{(2)} := v_k^{(0,1)} - v_k^{(1,1)}$ has trace 0 on $-\Sigma_2 = C_2$,
(iii) if $n = 1$, then $s_k^{(1)} := v_k^{(1,0)} + v_k^{(1,1)}$ has a nonnegative trace on C_1,
(iv) if $n = 2$, then $s_k^{(2)} := v_k^{(0,1)} + v_k^{(1,1)}$ has a nonnegative trace on C_2.

We only explain the case $n = 1$, since the case $n = 2$ is analogous when interchanging x_1 and x_2. The cases $n \geq 3$ are even simpler, since only (i) and (ii) are needed.

The idea is to start from the corresponding $d^{(i)}$ and $s^{(1)}$ for the desired limits $v^{(i,j)}$ and approximate those. The differences $d^{(m)} \in \mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ can be extended by 0 across the plane $C_m = -\Sigma_m \subset \partial Q_+$ such that

$$d_k^{(m)} = M_k^{e_{3-m}-e_m} d^{(m)} \to d^{(m)} \quad \text{in } \mathrm{H}^1(B_\delta(x^*) \cap Q_+) \qquad \text{and} \qquad d^{(m)}|_{C_m} = 0.$$

Here the shift vector $-e_m$ guarantees the vanishing trace, while e_{3-m} is used to avoid the other crack part C_{3-m}.

Finally, a positivity preserving extension $\widetilde{s}$ of $s^{(1)}$ across C_1 via $\max\{0, s^{(1)}(-x_1, x_2, \dots)\}$ gives

$$s_k^{(1)} = M_k^{-e_1+e_2} \widetilde{s}|_{B_\delta(x^*)\cap Q_+}.$$

Thus, we have $s_k^{(1)} \to s^{(1)}$ in $\mathrm{H}^1(B_\delta(x^*) \cap Q_+)$ and $s_k^{(1)}|_{C_1} \geq 0$. With this, $v_k^{(i,j)}$ for $i + j \geq 1$ can be uniquely calculated from $d_k^{(1)}$, $d_k^{(2)}$, and $s_k^{(1)}$, while the even-even function $v^{(0,0)}$ can be approximated arbitrarily. This results in

$$v_k^{(0,0)} = M_k^{e_1+e_2} v^{(0,0)}, \quad v_k^{(1,1)} = \frac{1}{2}(s_k^{(1)} - d_k^{(1)}), \quad v_k^{(1,0)} = d_k^{(1)} + v_k^{(1,1)}, \quad v_k^{(0,1)} = d_k^{(2)} + v_k^{(1,1)}.$$

With this construction, v_k defined in (5.10) gives the desired approximations.

Crack and boundary. For $x^* \in \partial\widehat{\Omega} \cap \widehat{\Gamma}_{\rm cr}$ we again use reflection to extend $\widehat{u}$ from $\widehat{\Omega} \cap B_\delta(x^*)$ to the outside but this time specialized by using Corollary 2.6. With U, φ_{x^*}, and η_{x^*} from there, we define the map $R : B_\delta(x^*) \to \widehat{\Omega}$ with

$$R(x) = x - 2\max\{0, (x - x^*) \cdot \eta_{x^*} - \varphi_{x^*}(x - \eta_{x^*} \cdot (x - x^*)\eta_{x^*})\}\eta_{x^*},$$

which is Lipschitz continuous and satisfies the property $R^{-1}(U \cap \widehat{\Gamma}_{\rm cr}) \subset \widehat{\Gamma}_{\rm cr}$ and if $x^* \in \widehat{\Gamma}_{\rm edge}$, we also have $R^{-1}(U \cap \widehat{\Gamma}_{\rm edge}) \subset \widehat{\Gamma}_{\rm edge}$. Thus, we can extend $\widehat{u}$ by $\widehat{u} \circ R \in \mathrm{H}^1(V \setminus \widehat{\Gamma}_{\rm cr}, \mathbb{R}^d)$, where $V = R^{-1}(\widehat{\Omega} \cap U)$ is an open neighborhood of x^*. Now one can proceed as in the case $x^* \in \widehat{\Omega} \cap \widehat{\Gamma}_{\rm cr}$ above.

Thus Proposition 5.4 is established. □

We are now ready to proof the desired limsup estimate by constructing a recovery sequence $(u_\varepsilon)_\varepsilon$ that converges strongly in $\mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d)$. This result also provides the final part of the proof of the main Theorem 2.4 on the Mosco convergence $\mathcal{F}_\varepsilon \xrightarrow{\mathrm{M}} \mathcal{F}$.

Theorem 5.5 (Limsup estimate). *For every $u \in \mathrm{H}^1_{g,\mathrm{Dir}}$ there exists a sequence (ε_j, u_j) with*

$$\varepsilon_j \to 0, \qquad u_j \to u \quad \text{in } \mathrm{H}^1_{g,\mathrm{Dir}} \subset \mathrm{H}^1(\Omega_{\mathrm{cr}};\mathbb{R}^d), \qquad \text{and} \qquad \limsup_{j\to\infty} \mathcal{F}_{\varepsilon_j}(u_j) \le \mathcal{F}(u).$$

Proof. For $\mathcal{F}(u) = \infty$ there is nothing to show, so we restrict to the case $\mathcal{F}(u) < \infty$ which implies $[\![u]\!]_\nu \ge 0$.

Case $u \in \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}},\mathbb{R}^d)$. Applying Proposition 5.2, we obtain a sequence (ε_k, u_k) with $u_k \to u$ in $\mathrm{H}^1(\Omega_{\mathrm{cr}},\mathbb{R}^d)$ such that $y_k = \mathrm{id} + \varepsilon_k u_k$ satisfies the GMS condition (1.1), which implies

$$\mathcal{F}_{\varepsilon_k}(u_k) = \widetilde{\mathcal{F}}_{\varepsilon_k}(u_k) = \int_{\Omega_{\mathrm{cr}}} \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x))\,\mathrm{d}x = \int_{\Omega_{\mathrm{cr}}} \overline{W}_\varepsilon(\nabla u_k(x))\,\mathrm{d}x.$$

Since all u_k lie in $\mathrm{W}^{1,\infty}$, we may assume that $\varepsilon_k\|\nabla u_k\|_{\mathrm{L}^\infty} \le r_{1/2}$ with $r_\delta > 0$ from Assumption 1.1 (d) for $\delta = \frac{1}{2}$. Thus, we have

$$\overline{W}_\varepsilon(\nabla u_k(x)) = \frac{1}{\varepsilon_k^2} W(I + \varepsilon_k \nabla u_k(x)) \le \Big(\frac{1}{2} + \frac{1}{2}\Big)|\nabla u_k(x)|^2_{\mathbb{C}} \le |\mathbb{C}||\nabla u_k(x)|^2 =: g_k(x).$$

Using $\nabla u_k \to \nabla u$ in $\mathrm{L}^2(\Omega,\mathbb{R}^{d\times d})$, we conclude $g_k \to g$ in $\mathrm{L}^1(\Omega)$, where $g(x) = |\mathbb{C}||\nabla u(x)|^2$. Moreover, we may choose a subsequence (not relabeled) such that $\nabla u_k(x) \xrightarrow{k\to\infty} \nabla u(x)$ a.e. in Ω_{cr}. Using Assumption 1.1 (d), we obtain $\overline{W}_{\varepsilon_k}(\nabla u_k(x)) \to \frac{1}{2}|\nabla u(x)|^2_{\mathbb{C}}$ a.e. in Ω. Now the generalized Lebesgue Dominated Convergence Theorem provides the desired limit, namely

$$\lim_{k\to\infty} \mathcal{F}_{\varepsilon_k}(u_k) = \lim_{k\to\infty} \int_{\Omega_{\mathrm{cr}}} \overline{W}_\varepsilon(\nabla u_k(x))\,\mathrm{d}x = \int_{\Omega_{\mathrm{cr}}} \frac{1}{2}\langle \nabla u(x), \mathbb{C}\nabla u(x)\rangle\,\mathrm{d}x = \mathcal{F}(u).$$

General $u \in \mathbf{C}_g$. For a general $u \in \mathbf{C}_g$ Proposition 5.4 guarantees the existence of an approximating sequence $u_j \in \mathbf{C}_g \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}};\mathbb{R}^d)$. By the first case there are for each j sequences $(\varepsilon_{j,k}, u_{j,k})_{k\in\mathbb{N}}$ with

$$u_{j,k} \in \mathbf{C}_g \cap \mathrm{W}^{1,\infty}(\Omega_{\mathrm{cr}};\mathbb{R}^d), \qquad \varepsilon_{j,k} \to 0, \quad u_{j,k} \to u_j, \quad \text{and} \quad \mathcal{F}_{\varepsilon_{j,k}}(u_{j,k}) \to \mathcal{F}(u_j) \quad \text{as } k \to \infty.$$

To construct a diagonal sequence, we use the strong continuity of $\mathcal{F}$ restricted to the convex set $\mathbf{C}_g$, namely there exists a constant $C_F > 0$ such that for all $v \in \mathbf{C}_g$ with $\|v - u\|_{\mathrm{H}^1} \le 1$,

$$|\mathcal{F}(v) - \mathcal{F}(u)| \le C_F\|v - u\|_{\mathrm{H}^1}.$$

With this we can construct a diagonal sequence as follows. For $n \in \mathbb{N}$ we choose $j_n \ge n$ with $\|u - u_{j_n}\|_{\mathrm{H}^1} < \frac{1}{n}$. Next we choose $k_n \ge n$ with

$$\varepsilon_{j_n,k_n} < \frac{1}{n}, \quad \|u_{j_n,k_n} - u_{j_n}\|_{\mathrm{H}^1} < \frac{1}{n}, \quad \text{and} \quad |\mathcal{F}_{\varepsilon_{j_n,k_n}}(u_{j_n,k_n}) - \mathcal{F}(u_{j_n})| < \frac{1}{n}.$$

Setting $\widetilde{\varepsilon}_n = \varepsilon_{j_n,k_n}$ and $\widetilde{u}_n = u_{j_n,k_n}$, we obtain $\widetilde{\varepsilon}_n < \frac{1}{n}$, $\|\widetilde{u}_n - u\|_{\mathrm{H}^1} < \frac{2}{n}$, and

$$|\mathcal{F}_{\widetilde{\varepsilon}_n}(\widetilde{u}_n) - \mathcal{F}(u)| \le |\mathcal{F}_{\varepsilon_{k_n}}(u_{j_n,k_n}) - \mathcal{F}(u_{j_n})| + |\mathcal{F}(u_{j_n}) - \mathcal{F}(u)| \le \frac{1}{n} + \frac{C_F}{n} \to 0.$$

Thus, $(\widetilde{\varepsilon}_n, \widetilde{u}_n)_{n\in\mathbb{N}}$ is a strongly converging recovery sequence for $u \in \mathbf{C}_g$. □

Acknowledgment: We thank the unknown referee for a very careful reading, which helped us to improve and correct some of our arguments.

Funding: The research was partially supported by Deutsche Forschungsgemeinschaft (DFG) via MATHEON project C18 (Analysis and Numerics of Multidimensional Models for Elastic Phase Transformations in Shape Memory Alloys) and via the project C05 (Effective Models for Interfaces with Many Scales) in the collaborative research center SFB 1114 *Scaling Cascades in Complex Systems*.

References

[1] V. Agostiniani, G. Dal Maso and A. DeSimone, Linear elasticity obtained from finite elasticity by Γ-convergence under weak coerciveness conditions, *Ann. Inst. H. Poincaré Anal. Non Linéaire* **29** (2012), no. 5, 715–735.
[2] P. G. Ciarlet and J. Nečas, Injectivity and self-contact in nonlinear elasticity, *Arch. Ration. Mech. Anal.* **97** (1987), no. 3, 171–188.
[3] G. Dal Maso, M. Negri and D. Percivale, Linearized elasticity as Γ-limit of finite elasticity, *Set-Valued Anal.* **10** (2002), no. 2–3, 165–183.
[4] G. Friesecke, R. D. James and S. Müller, A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity, *Comm. Pure Appl. Math.* **55** (2002), no. 11, 1461–1506.
[5] A. Giacomini and M. Ponsiglione, Non-interpenetration of matter for SBV deformations of hyperelastic brittle materials, *Proc. Roy. Soc. Edinburgh Sect. A* **138** (2008), no. 5, 1019–1041.
[6] M. Giaquinta, G. Modica and J. Souček, *Cartesian Currents in the Calculus of Variations. II*, Ergeb. Math. Grenzgeb. (3) 38, Springer, Berlin, 1998.
[7] M. Hintermüller and C. N. Rautenberg, On the density of classes of closed convex sets with pointwise constraints in Sobolev spaces, *J. Math. Anal. Appl.* **426** (2015), no. 1, 585–593.
[8] M. Hintermüller, C. N. Rautenberg and S. Rösel, Density of convex intersections and applications, *Proc. Royal Soc. A* **473** (2017), Article ID 20160919.
[9] D. Knees, A. Mielke and C. Zanini, On the inviscid limit of a model for crack propagation, *Math. Models Methods Appl. Sci.* **18** (2008), no. 9, 1529–1569.
[10] D. Knees and A. Schröder, Global spatial regularity for elasticity models with cracks, contact and other nonsmooth constraints, *Math. Methods Appl. Sci.* **35** (2012), no. 15, 1859–1884.
[11] G. Lazzaroni and R. Toader, A model for crack propagation based on viscous approximation, *Math. Models Methods Appl. Sci.* **21** (2011), no. 10, 2019–2047.
[12] P. M. Mariano and G. Modica, Ground states in complex bodies, *ESAIM Control Optim. Calc. Var.* **15** (2009), no. 2, 377–402.
[13] A. Mielke and U. Stefanelli, Linearized plasticity is the evolutionary Γ-limit of finite plasticity, *J. Eur. Math. Soc. (JEMS)* **15** (2013), no. 3, 923–948.
[14] M. Negri and C. Zanini, From finite to linear elastic fracture mechanics by scaling, *Calc. Var. Partial Differential Equations* **50** (2014), no. 3–4, 525–548.
[15] R. Paroni and G. Tomassetti, A variational justification of linear elasticity with residual stress, *J. Elasticity* **97** (2009), no. 2, 189–206.
[16] R. Paroni and G. Tomassetti, From non-linear elasticity to linear elasticity with initial stress via Γ-convergence, *Contin. Mech. Thermodyn.* **23** (2011), no. 4, 347–361.
[17] B. Schmidt, Linear Γ-limits of multiwell energies in nonlinear elasticity theory, *Contin. Mech. Thermodyn.* **20** (2008), no. 6, 375–396.