The dynamic state index with moisture and phase changes

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The dynamic state index (DSI) is a scalar field that combines variational information on the total energy and enstrophy of a flow field with the second law of thermodynamics. Its magnitude is a combined local measure for non-stationarity, diabaticity, and dissipation in the flow, and it has been shown to provide good qualitative indications for the onset and presence of precipitation and the organization of storms. The index has been derived thus far for ideal fluid models only, however, so that one may expect more detailed insights from a revised definition of the quantity that includes more complex aerothermodynamics. The present paper suggests definitions of DSI-like indicators for flows of moist air with phase changes and precipitation. In this way, the DSI is generalized to signal deviations from avariety of different types of balanced states. A comparison of these indices evaluated with respect to one and the same flow field enables the user to test whether the flow internally balances any combination of the physical processes encoded in the generalized DSI-indices.

I. INTRODUCTION

A. The original DSI: Deviations from stationary, adiabatic, inviscid states in dry air

The Dynamic State Index (DSI) is a parameter based on first principles of fluid mechanics that indicates local deviations of the atmospheric flow field from a stationary, adiabatic, and inviscid solution of the non-hydrostatic compressible governing equations [10]. In this way, the DSI can be evaluated on a given atmospheric flow field to detect atmospheric developments such as fronts or hurricanes. Atmospheric processes involve the interaction of energetic, thermodynamic, and vortex-related quantities. The (scalar) Dynamic State Index combines this information in a particular way.

For dry conservative systems, the DSI is given by the Jacobi-determinant of three constitutive quantities: an advected scalar ψ, an energy variable B, here given by the Bernoulli stream function, and the potential vorticity (PV) Π:

$$\mathrm{DSI} = \frac{\partial(\psi, B, \Pi)}{\partial(a,b,c)} = \frac{1}{\rho}\frac{\partial(\psi, B, \Pi)}{\partial(x,y,z)}\,, \tag{1}$$

where x, y and z denote the Cartesian coordinates and $dm = da\;db\;dc$, where a, b, c are the Lagrangian mass coordinates. Thus, integrating the conservation of mass in the Lagrangian sense with

$$\rho = \frac{\partial(a,b,c)}{\partial(x,y,z)}\,, \tag{2}$$

leads to the right hand side representation of the DSI in (1) (See eq. (8.2) in [10] with $\psi = s$ denoting the specific entropy). The precise definitions of ψ, B and Π depend on the underlying equations of motion. For more complex conservative systems involving moisture DSI-like indicators become sums of Jacobi-determinants as shown below. In the presence of irreversible processes, such as precipitation, such a compact representation turns out not to be available, however.

The formulation of the DSI in (1) is equivalent to its representation as the mass flux divergence of the "steady wind" $\mathbf{v}_{st} = -(\nabla B \times \nabla\Psi)/\rho\Pi$ (c.f. Schär [13]):

$$\mathrm{DSI} = -\frac{\Pi^2}{\rho}\nabla\cdot(\rho\mathbf{v}_{st}) = 0\,. \tag{3}$$

The field $\mathbf{v}_{st}$ may be interpreted as the local basic state wind that would have to prevail for given fields (ψ, B, Π) for the flow to be stationary. Again, the precise definition of the basic state wind depends on the model equations. In the following $\mathbf{v}_{st}$ will simply be called steady wind. This principal definition of the DSI via the steady wind will be used below as a basis for the derivations of DSI-like indices for more complex systems involving moisture and precipitation.

The basic state is characterized by DSI=0, and for dry dynamics this amounts to vanishing advection tendencies of the three constitutive quantities in the determinant (1), see, e.g., Névir and Sommer [11] or Müller and Névir [8]. This property can be traced back to the Lagrangian conservation of the three constitutive quantities under the assumption of stationarity. The originally introduced DSI by Névir [10] is based on the adiabatic non-hydrostatic compressible governing equations for dry air without consideration of thermodynamical sources and sinks, such as solar forcing, in the basic state. In this case, regarding (1), ψ corresponds to the potential temperature, B is the Bernoulli function (or total enthalpy) [see 16, section 1.10] and Π is Ertel's potential vorticity formed with the potential temperature as the advected scalar [see 16, section 4.5].

This DSI concept can be applied to indicate non-steady, diabatic and frictional atmospheric processes across all scales: Weber and Névir [17] show how the characteristic dipole structure of the Dynamic State Index can be used to diagnose the evolution of high- and low-pressure areas on the synoptic scale, or hurricanes on the meso-scale. On the convective scale, several authors have shown that the DSI is strongly correlated with

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intensive precipitation processes, see e.g. Claussnitzer et al. [2], Gaßmann [5], Weijenborg et al. [18].

B. DSI-like indicators for generalized balances and fluids

As stated above, the original Dynamic State Index generates a non-zero signal when the underlying flow is non-stationary or when it involves diabatic or dissipative processes. In other words, the DSI indicates deviations from a particular kind of balanced state. This perspective gives rise to generalizations of the DSI as indicators for deviations from different types of balance.

One prominent example in question is the geostropic and hydrostatic balance that forms the basis of quasi-geostrophic theory and is considered approximately valid on synoptic length and time scales. To accomodate such scale-dependent aspects in the DSI framework, Müller et al. [9] derived a Dynamic State Index, $\mathrm{DSI_{QG}}$, directly from the quasi-geostrophic model. In this case, the Bernoulli function reduces to $B = \frac{1}{f_0}\phi$, where ϕ is Earth's gravitational potential, f_0 denotes the Coriolis parameter and Π is the quasi-geostrophic, rather than the Ertel's, potential vorticity. By tracing back the original asymptotic derivation of the quasi-geostrophic model from the full compressible flow equations in [12], these authors also showed that the $\mathrm{DSI_{QG}}$ is the leading-order asymptotic approximation of the original DSI for compressible flows in this limit. As a consequence, by comparing the $\mathrm{DSI_{QG}}$ with the original DSI for one and the same flow field of a compressible fluid, one can ascertain whether a balanced flow in the sense of vanishing DSI is also in the quasi-geostrophic regime – in which case the $\mathrm{DSI_{QG}}$ should be comparably small.

In the same spirit Müller and Névir [8] recently introduced a DSI for the Rossby model and confirmed its high correlation to precipitation patterns and its applicability to the phenomenological concept of "Großwetterlagen". For the (two-dimensional) Rossby model on the β-plane, the related $\mathrm{DSI_{Ro}}$ is given by the Jacobi-determinant of just two quantities, namely of the geopotential height ($B = \frac{1}{f_0}\phi$) and the absolute vorticity ($\Pi = \zeta_a$). Utilizing the three indices DSI, $\mathrm{DSI_{QG}}$, and $\mathrm{DSI_{Ro}}$ available, one can now test whether a balanced flow ($\mathrm{DSI} \ll 1$) is also balanced on synoptic ($\mathrm{DSI_{QG}} \ll 1$), or even on the scale of the external Rossby radius ($\mathrm{DSI_{Ro}} \ll 1$) and beyond.

Another possible DSI generalization consists of replacing the reference thermodynamic state change underlying the construction of the index from the isentropic ($p \sim \rho^\gamma$) to some polytropic ($p \sim \rho^\kappa$) pressure-density relationship, where γ and κ are the isentropic and polytropic exponents, respectively. With a change of this type, a non-vanishing DSI^κ can indicate, e.g., deviations from an isothermal, isochoric, or isobaric state for $\kappa = \gamma$, 1, or ∞, respectively. Furthermore, minimization of DSI^κ with respect to κ for a given slowly varying flow field characterizes the diabatic effects in the flow in terms of the most similar polytropic process.

The derivation of a polytropic DSI-family will be the subject of a forthcoming paper, but analogous considerations motivate the present work: The comparison, for one and the same flow field, of DSI-like variables that encode balances under different prevailing moist processes provides diagnostic insights into the aerothermodynamic nature of the flow. Mathematically, in developing a DSI for moist processes, we consider a basis of constituting quantities that generalize the potential temperature, ψ, the Bernoulli function, B, and the potential vorticity, Π, for model equations that include the effects of moist processes.

After a brief summary of the mathematical formalism underlying the original DSI concept, a generalized derivation of the same quantity that directly invokes the second law of thermodynamics will be presented in section II. Section III then utilizes the generalized derivation to provide a hierarchy of three DSI-like indices relevant for moist air flows. These indices signal balances for moist flows with and without phase changes and for precipitating and non-precipitating states. Section IV summarizes our results and provides further conclusions.

II. GENERALIZED DERIVATION OF DSI-LIKE VARIABLES

A. Classical derivation of the DSI

Here we recall the derivation of the DSI for the equations for non-hydrostatic compressible flows of dry air [see 10, 15] for later reference. We start with the momentum equation

$$\partial_t \mathbf{v} + \boldsymbol{\xi}_a \times \mathbf{v} + \left(\frac{1}{2}\boldsymbol{\nabla}\mathbf{v}^2 + \frac{1}{\rho}\boldsymbol{\nabla} p + \boldsymbol{\nabla}\phi\right) = \mathbf{F}\,, \tag{4}$$

where $\mathbf{F}$ denotes frictional forces, ϕ the geopotential height field, p the pressure, ρ the density and $\boldsymbol{\Omega}$ the earth rotation, and where we have utilized the Weber-Transform

$$\mathbf{v}\cdot\nabla\mathbf{v} = \boldsymbol{\xi}\times\mathbf{v} + \frac{1}{2}\nabla\mathbf{v}^2 \tag{5}$$

where $\boldsymbol{\xi} = \nabla\times\mathbf{v}$ and $\boldsymbol{\xi}_a = \boldsymbol{\xi} + 2\boldsymbol{\Omega}$ denote the relative and absolute vorticities, respectively. Under adiabatic conditions the potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{R_d}{c_{pd}}} \tag{6}$$

is conserved along Lagrangian paths. Here R_d and c_{pd} are the ideal gas constant and the specific heat capacity at constant pressure for dry air, respectively. Thus, forming the cross product of (4) with $\nabla\theta$, we obtain

$$\begin{aligned}
&(\partial_t\mathbf{v} - \mathbf{F} + \nabla B + \frac{1}{\rho}\nabla p - \nabla H)\times\nabla\theta \\
&= -(\boldsymbol{\xi}_a\times\mathbf{v})\times\nabla\theta \;=\; \boldsymbol{\xi}_a\,\mathbf{v}\cdot\nabla\theta - \mathbf{v}\,\boldsymbol{\xi}_a\cdot\nabla\theta \\
&= -\boldsymbol{\xi}_a\partial_t\theta - \mathbf{v}\rho\Pi^\theta,
\end{aligned} \tag{7}$$

where

$$B = \frac{1}{2}\mathbf{v}^2 + H + \phi\,, \quad H = c_{pd}T\,, \quad \text{and} \quad \Pi^\theta = \frac{\boldsymbol{\xi}_a\cdot\nabla\theta}{\rho} \tag{8}$$

are the Bernoulli function, the enthalpy, and the potential vorticity, respectively. Using the ideal gas law $p = R_d\rho T$ we further obtain

$$\left(\frac{1}{\rho}\nabla p - \nabla H\right) \times \nabla\theta = T(R_d \nabla \ln p - c_{pd}\nabla \ln T) \times \nabla\theta \\ = -c_{pd} T \nabla \ln\theta \times \nabla\theta = 0\,. \tag{9}$$

Assuming stationarity and neglecting friction leads to the steady wind for adiabatic, inviscid, and steady flows [13]

$$\mathbf{v}_{st} = -\frac{1}{\rho \Pi^\theta}\left[\nabla B \times \nabla\theta\right]. \tag{10}$$

The DSI is designed to signal deviations from this steady wind, and its mathematical representation follows from the continuity equation based on the steady wind:

$$\mathrm{DSI} = -\frac{{\Pi^\theta}^2}{\rho}\nabla\cdot(\rho\mathbf{v}_{st}) = -\frac{1}{\rho}\nabla\Pi^\theta \cdot \left(\nabla B \times \nabla\theta\right) \\ = \frac{1}{\rho}\frac{\partial(\theta, B, \Pi^\theta)}{\partial(x,y,z)}\,. \tag{11}$$

Thus, the DSI indicates local deviations of an atmospheric flow field from a stationary, adiabatic, and inviscid state, i.e., the presence of instationary, viscous, or diabatic processes.

Interpreting the DSI geometrically and regarding isentropic surfaces, the steady wind based on the non-hydrostatic compressible governing equations blows along the isolines of the Bernoulli stream function as well as of the PV, and the DSI signal indicates non-alignment of these fields in the sense that for non-zero DSI the vectors $\nabla\theta, \nabla B$, and $\nabla\Pi^\theta$ are linearly independent. In turn, such non-alignment signals the presence of molecular transport or more general diabatic atmospheric processes.

Previous works corroborate, on the basis of meteorological observations, that the DSI as defined in (11) signals diabatic processes [2, 5, 18]. To obtain DSI variants that locate specific diabatic processes, for example the formation of clouds, or extreme precipitation, we will include successively more complex moist processes in the equations of motion and derive related new DSI-like indices in section III.

B. DSI and PV for a multi-component fluid

1. *A generalization of the* DSI

To generalize the derivation of the DSI for a multi-component fluid we proceed in analogy to the last section. Starting with the corresponding set of equations of motion, we derive the model-dependent steady wind leading to the scalar DSI-field that indicates deviations from this basic state. The point of departure for the derivations are again the momentum equations which we rewrite here as

$$\partial_t \mathbf{v} + \boldsymbol{\xi}_a \times \mathbf{v} + \nabla B = \mathbf{F} + \mathbf{G}\,. \tag{12}$$

where

$$B = \frac{1}{2}\mathbf{v}^2 + H + \phi \tag{13}$$

and

$$\mathbf{G} = \nabla H - \frac{1}{\rho}\nabla p = T\nabla S + \sum_{i=1}^{n_{\mathrm{sp}}} \mu_i \nabla Y_i \tag{14}$$

are again the Bernoulli function (or total enthalpy) from (8) with an appropriately scaled specific enthalpy H adapted to the system under consideration, G is the effective diabatic forcing term, S the specific entropy, and $\mu_i = H_i - T s_i$ and Y_i are the chemical potential and the mass fraction of the ith fluid constituent with $i \in \{1, \ldots, n_{\mathrm{sp}}\}$ and n_{sp} denoting the number of fluid components.

$$\Pi^s = \frac{\boldsymbol{\xi}_a \cdot \nabla s}{\rho}\,, \tag{15}$$

denote the Ertel-type potential vorticity based on the entropy s which satisfies the transport equation

$$\frac{ds}{dt} = \partial_t s + \mathbf{v}\cdot\nabla s = Q_s \tag{16}$$

with the generalized source term Q_s that subsumes all dissipative and external forcing processes that affect entropy evolution along fluid path lines. Taking the cross product of (12) with ∇s yields

$$(\partial_t \mathbf{v} + \nabla B - \mathbf{F} - \mathbf{G}) \times \nabla s = -(\boldsymbol{\xi}_a \times \mathbf{v}) \times \nabla s \\ = \boldsymbol{\xi}_a \, \mathbf{v}\cdot\nabla s - \mathbf{v}\, \boldsymbol{\xi}_a \cdot \nabla s \\ = -\boldsymbol{\xi}_a \partial_t s - \mathbf{v}\rho\Pi^s + \boldsymbol{\xi}_a Q_s\,. \tag{17}$$

Following Schär's procedure for the case of dry air [13], we now define a steady wind based on (17) by assuming stationarity ($\partial_t \equiv 0$). This yields

$$\mathbf{v}^s_{st} = -\frac{1}{\rho\Pi^s}\Big[(\nabla B - \mathbf{G} - \mathbf{F}) \times \nabla s - \boldsymbol{\xi}_a Q_s\Big]. \tag{18}$$

The definition of the DSI according to Névir [10], Névir and Sommer [11] then results from the continuity equation for the steady wind with a suitable normalization,

$$\mathrm{DSI}^s = -\frac{(\Pi^s)^2}{\rho}\nabla\cdot(\rho\mathbf{v}^s_{st}) \\ = \frac{(\Pi^s)^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi^s}\Big((\nabla B - \mathbf{G} - \mathbf{F}) \times \nabla s - \boldsymbol{\xi}_a Q_s\Big)\right]. \tag{19}$$

For a steady flow the DSI^s vanishes by definition of the steady wind, because under these conditions the steady wind coincides with the actual wind field and satisfies the continuity equation for steady conditions which amounts to

$$\nabla\cdot(\rho\mathbf{v}^s_{st}) = 0 \quad \Leftrightarrow \quad \mathrm{DSI}^s = 0\,. \tag{20}$$

Note that we have generalized the original DSI concept by allowing for both frictional forces, $\mathbf{F}$, and entropy production, Q_s. Of course, when $\mathbf{F} \equiv 0$ and $Q_s \equiv 0$ the indicator is tuned again to adiabatic, frictionless and steady flows as discussed in section II A.

The generalized DSI-concept as represented by (19) will be utilized below to suggest dynamic state indices for the equations of moist air flow. Entropy (or potential temperature), Bernoulli function, and PV will be adapted to the given model, leading to the formulation of the corresponding steady wind and of a related DSI.

2. Is there a generalization of the potential vorticity for inclusion in the DSI framework?

The vorticity transport equation, and with it the Lagrangian evolution of the entropy-based potential vorticity Π^s, can be derived as follows. Again, the point of departure are (12) and (14) which we combine to yield

$$\frac{\partial \mathbf{v}}{\partial t} + \boldsymbol{\xi}_a \times \mathbf{v} + \nabla B = T\nabla s + \sum_{i=1}^{n_{\rm sp}} \mu_i \nabla Y_i + \mathbf{F} \tag{21}$$

Taking the curl yields the vorticity transport equation

$$\frac{\partial \boldsymbol{\xi}}{\partial t} + \nabla \times (\boldsymbol{\xi}_a \times \mathbf{v}) = \nabla T \times \nabla s + \sum_{i=1}^{n_{\rm sp}} \nabla \mu_i \times \nabla Y_i + \nabla \times \mathbf{F} \,. \tag{22}$$

Incorporating the conservation of mass via the continuity equation and multiplying by ∇s, omitting the term $(\nabla T \times \nabla s) \cdot \nabla s$, and utilizing (16), leads to the evolution equation for the entropy-based potential vorticity $\Pi^s = (\boldsymbol{\xi}_a \cdot \nabla s)/\rho$,

$$\frac{d\Pi^s}{dt} = \frac{1}{\rho} \left(\sum_{i=1}^{n_{\rm sp}} \nabla \mu_i \times \nabla Y_i + \nabla \times \mathbf{F} \right) \cdot \nabla s + \frac{\boldsymbol{\xi}_a}{\rho} \cdot \nabla Q_s \,, \tag{23}$$

where, following Gassmann and Herzog [6],

$$\nabla \mu_i = \frac{1}{\rho_i} \nabla p_i - s_i \nabla T \,. \tag{24}$$

Clearly, for frictionless, $\mathbf{F} \equiv 0$, and adiabatic, $Q_s \equiv 0$, flow we must require in addition that the solenoidal first terms on the right hand side vanish as well [see e.g. 14].

With a more general advected scalar, ψ, used instead of entropy in forming the PV and DSI variables, Π^ψ and DSI$^\psi$, respectively, one would require

$$\mathbf{F} = 0 \,, \quad \frac{d\psi}{dt} = Q_\psi = 0 \,, \tag{25}$$

and

$$\nabla \psi \cdot \left(\nabla T \times \nabla s + \sum_{i=1}^{n_{\rm sp}} \nabla \mu_i \times \nabla Y_i \right) = 0 \tag{26}$$

for Π^ψ to be a Lagrangian conserved quantity.

Analogous potential vorticity equations can also be derived for models including moist processes. But, in general, retaining ρ as the total density and using the entropy S for the appropriate multi-species atmospheric models with moisture, we are not able to find a scalar ψ such that the solenoidal term vanishes in all cases. Schubert [14] discusses the evolution of potential vorticities, Π^{θ_x}, formulated in terms of different potential temperature-like variables θ_x. He demonstrates that the solenoidal term always vanishes when using the virtual potential temperature, θ_ρ, which is effectively a function of density and pressure only. This ansatz hides the influences of particular moist phase conversions, however, and this is why the Gibb's form involving $T\nabla S$ rather than $\nabla p/\rho$ is used in the momentum equation (21) here, and why the subsequent derivations of DSI-like quantities are based on entropy rather than on potential temperatures.

C. The DSI for dry air

In the following the steady wind and the DSI for the non-hydrostatic compressible governing equations for dry air are derived corroborating the DSI introduced by Névir [10]. For the derivations of the DSI for moist air we follow the same steps, but adapt the entropy, the Bernoulli function and the potential vorticity to the particular models.

Owing to the second law of thermodynamics, entropy is conserved along particle paths for adiabatic motions, and for dry air it satisfies the relation

$$\frac{ds_{(d)}}{dt} = c_{pd} \frac{d \ln T}{dt} - R_d \frac{d \ln p}{dt} \,. \tag{27}$$

Integrated, this yields the entropy equation of state,

$$s_{(d)} = c_{pd} \ln \frac{T}{T_0} - R_d \ln \frac{p}{p_0} \,, \tag{28}$$

where c_{pd} denotes the specific heat capacity at constant pressure for dry air, see also table I. We remark that the commonly used notation of S for the entropy is changed here to $s_{(d)}$ indicating that we consider the entropy for dry air. This will allow us to distinguish between the different entropies for dry air (d), moist air with vapor only (v), cloudy air (c) and precipitating air (r) later on. The other quantities will be indexed accordingly.

With the entropy being a conserved quantity for adiabatic motion, a natural choice for ψ is

$$\psi = s_{(d)} \,, \tag{29}$$

and, due to (25), the associated potential vorticity

$$\Pi^s_{(d)} = \frac{\boldsymbol{\xi}_a \cdot \nabla s_{(d)}}{\rho} \tag{30}$$

is also conserved during frictionless motion.

For the dry air case the enthalpy $h_{(d)}$ (up to integration constants) is defined by

$$h_{(d)} = c_{pd} T \,. \tag{31}$$

Using the ideal gas law

$$p = R_d \rho T \tag{32}$$

it follows that

$$dh_{(d)} = T ds_{(d)} + \frac{1}{\rho} dp \,. \tag{33}$$

Thus setting $H_{(d)} = h_{(d)}$ in the Bernoulli function in (13) we obtain for $\mathbf{G}_{(d)}$ according to (14) the identity

$$\mathbf{G}_{(d)} = T \nabla s_{(d)} \,, \tag{34}$$

which satisfies, obviously, $\mathbf{G}_{(d)} \times \nabla s_{(d)} = 0$ and the steady wind is thus given by

$$\mathbf{v}^s_{st(d)} = \frac{1}{\rho \Pi^s_{(d)}} \nabla s_{(d)} \times \nabla B_{(d)} \,. \tag{35}$$

TABLE I. Thermodynamic equation of state parameters for moist air at reference temperature $T_0 = 273.15$ K and reference pressure $p_0 = 10^5 Pa$ (see, e.g., [3, 4]):

c_{pd}	1005	J\kg\K	dry air specific heat capacity at constant pressure
R_d	287	J\kg\K	dry air gas constant
c_{pv}	1850	J\kg\K	water vapor specific heat capacity at constant pressure
R_v	462	J\kg\K	water vapor gas constant
c_l	4218	J\kg\K	liquid water specific heat capacity
$L_{\rm ref}$	$2.5 \cdot 10^6$	J\kg	latent heat of condensation at reference conditions

Then the DSI according to (19) reduces to

$$\mathrm{DSI}^s_{(d)} = \frac{1}{\rho}\nabla\Pi^s_{(d)} \cdot \left(\nabla s_{(d)} \times \nabla B_{(d)}\right), \tag{36}$$

where we have used the fact that $\nabla \cdot (\nabla s_{(d)} \times \nabla B_{(d)}) = 0$. Following Névir [10] the DSI can also be expressed as

$$\mathrm{DSI}^s_{(d)} = \frac{1}{\rho}\frac{\partial(s_{(d)}, B_{(d)}, \Pi^s_{(d)})}{\partial(x,y,z)}. \tag{37}$$

Remark 1 (The DSI in terms of the potential temperature)**.** *As explained in the introduction, the* DSI *has been defined by Névir [10] in terms of the potential temperature*

$$\psi = \theta = \theta_0 \exp\left(\frac{s_{(d)}}{c_{pd}}\right), \tag{38}$$

where θ_0 is the potential temperature at reference conditions, for which according to (6) *the identity $\theta_0 = T_0$ holds. This quantity is also conserved during adiabatic frictionless motion because θ_0 and c_{pd} are constants, and so is the related Ertel's potential vorticity*

$$\Pi^\theta_{(d)} = \frac{\xi_a \cdot \nabla\theta}{\rho}. \tag{39}$$

With these definitions, and according to (19)*, the* DSI *as introduced by Névir [10] reads*

$$\mathrm{DSI}^\theta_{(d)} = \frac{1}{\rho}\nabla\Pi^\theta_{(d)} \cdot \left(\nabla\theta \times \nabla B_{(d)}\right) = \frac{\theta^2}{c^2_{pd}}\mathrm{DSI}^s_{(d)}, \tag{40}$$

and it is proportional to the entropy-based version of the DSI *discussed above. This in particular means that from the perspective of applications, where the DSI is used to signal deviations from a balanced state, these two versions of the DSI are equally good.*

III. DSI-VARIANTS WITH MOISTURE AND PHASE CHANGES

A. The DSI for moist air

Here we derive the Dynamic State Index based on the equations of motion for moist air, but without phase changes, which will be incorporated in section III B below. Water components are introduced via the mixing ratios. For water vapor the latter is defined, e.g., as the ratio of the density of water vapor ρ_v over the density of dry air ρ_d,

$$q_v = \frac{\rho_v}{\rho_d} = E\frac{e}{p_d}, \qquad \text{where} \quad E = \frac{R_d}{R_v}, \tag{41}$$

and where e is the vapor pressure and R_v denotes the ideal gas constant for water vapor, see also table I. Here we have used that according to the ideal gas law for dry air and water vapor

$$p_d = \rho_d R_d T \qquad \text{and} \qquad e = p_v = \rho_v R_v T. \tag{42}$$

For the total pressure p of moist air (without liquid water) we thus have according to Dalton's law

$$p = p_d + e = \rho_d(R_d + q_v R_v)T =: \rho_d R' T, \tag{43}$$

or, equivalently,

$$p = R_d \rho T \frac{1 + \frac{q_v}{E}}{1 + q_v} =: R_d \rho T_v, \tag{44}$$

where ρ is the total density $\rho = \rho_d + \rho_v$ and T_v is referred to as the virtual temperature, see also [3, 4]. In a first step, we assume here that q_v is conserved, i.e., we do not allow for phase changes, and that there are no liquid water constituents. This amounts to

$$\frac{dq_v}{dt} = \partial_t q_v + \mathbf{v} \cdot \nabla q_v = 0. \tag{45}$$

Following Emanuel [4] the total entropy $s_{(v)}$ for the moist air with water vapor only is then given by

$$s_{(v)} = (c_{pd} + q_v c_{pv}) \ln\frac{T}{T_0} - \left(R_d \ln\frac{p_d}{p_0} + q_v R_v \ln\frac{e}{e_0}\right), \tag{46}$$

where c_{pv} denotes the specific heat capacity at constant pressure for water vapor, see also table I, and 0-indices denote reference values. Making use of (41), the entropy can alternatively be rewritten as

$$s_{(v)} = c'_p \ln\frac{T}{T_0} - R' \ln\frac{p}{p_0} + R' \ln\left(1 + \frac{q_v}{E}\right) - q_v R_v \ln\frac{q_v}{q_{v0}}, \tag{47}$$

where here and below we let

$$c'_p = c_{pd} + q_v c_{pv}, \qquad R' = R_d + q_v R_v. \tag{48}$$

The differential reads

$$ds_{(v)} = c'_p d\ln T - R' d\ln p + \left(c_{pv} \ln\frac{T}{T_0} - R_v \ln\frac{e}{e_0}\right) dq_v, \tag{49}$$

where we have used $\ln e/e_0 = \ln p/p_0 + \ln q_v/q_{v0} - \ln(1 + q_v/E)$. Notice that due to the conservation of q_v we find

$$\frac{ds_{(v)}}{dt} = c_p' \frac{d\ln T}{dt} - R' \frac{d\ln p}{dt} \,. \tag{50}$$

During isentropic motion, $s_{(v)}$ is conserved and we choose

$$\psi = s_{(v)} \qquad \text{and} \qquad \Pi^s_{(v)} = \frac{\boldsymbol{\xi}_a \cdot \nabla s_{(v)}}{\rho} \,. \tag{51}$$

To obtain the appropriate definition for the steady wind and the DSI for moist air with water vapor it now remains to determine the Bernoulli function $B_{(v)}$ for moist air and accordingly the remainder function $\mathbf{G}_{(v)}$. The total moist enthalpy is (up to constants) given by

$$h_{(v)} = c_p' T \,, \tag{52}$$

satisfying

$$dh_{(v)} = c_p' dT + c_{pv} T dq_v \,. \tag{53}$$

We set

$$H_{(v)} = \frac{h_{(v)}}{1+q_v} \,, \tag{54}$$

where we note that the normalization of $h_{(v)}$ by $(1+q_v) = \frac{\rho}{\rho_d}$ accounts for the fact that $\frac{1}{\rho_d}\nabla p$ arises in the gradient of $H_{(v)}$, or $T\nabla s_{(v)}$ respectively, instead of $\frac{1}{\rho}\nabla p$ appearing in the definition of $\mathbf{G}$, see also (58) below.

Using (49) and (43) one can compute

$$\begin{aligned} \frac{1}{\rho_d}\nabla p &= TR'\nabla \ln p && (55) \\ &= c_p'\nabla T - T\nabla s_{(v)} + T\left(c_{pv}\ln\frac{T}{T_0} - R_v \ln\frac{e}{e_0}\right)\nabla q_v \\ &= (1+q_v)\nabla H_{(v)} - T\nabla s_{(v)} + (1+q_v)\Lambda_{(v)}\nabla q_v \,, && (56) \end{aligned}$$

where we denote

$$\Lambda_{(v)} = \frac{1}{1+q_v}\left(H_{(v)} + c_{pv}T\left(\ln\frac{T}{T_0} - 1\right) - R_v T \ln\frac{e}{e_0}\right) , \tag{57}$$

such that by the definition of $\mathbf{G}$ in (14) we have

$$\mathbf{G}_{(v)} = \nabla H_{(v)} - \frac{\nabla p}{\rho} = \frac{1}{1+q_v} T\nabla s_{(v)} - \Lambda_{(v)}\nabla q_v \tag{58}$$

and for the cross product

$$\mathbf{G}_{(v)} \times \nabla s_{(v)} = -\Lambda_{(v)} \nabla q_v \times \nabla s_{(v)} \,. \tag{59}$$

The Bernoulli function for moist air without phase changes reads

$$B_{(v)} = \frac{1}{2}\mathbf{v}^2 + H_{(v)} + \phi \,. \tag{60}$$

Therefore the steady wind results in

$$\mathbf{v}_{st,(v)} = -\frac{1}{\rho \Pi_{(v)}} \Big((\nabla B_{(v)} + \Lambda_{(v)} \nabla q_v) \times \nabla s_{(v)} \Big) \,. \tag{61}$$

This steady wind describes a basic state that contains moist air without phase changes. Compared to the steady wind for dry air, this basic state $\mathbf{v}_{st,(v)}$ has an additional term that contains the water vapor mixing ratio. Deviations from this basic state are related to the generation of clouds and precipitation. These deviations are captured by the DSI for moist air

$$\mathrm{DSI}_{(v)} = \frac{(\Pi_{(v)})^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi_{(v)}} \Big(\nabla B_{(v)} + \Lambda_{(v)} \nabla q_v \Big) \times \nabla s_{(v)} \right] . \tag{62}$$

Comparing the $\mathrm{DSI}_{(v)}$ to $\mathrm{DSI}_{(d)}$ for dry air (36), the same additional term as in the steady wind representation is added. Noting that one velocity component of this steady wind is directed along the isosurfaces of the mixing ratio of water vapor, the DSI captures deviations from this alignment. The $\mathrm{DSI}_{(v)}$-signals are similar to the signals of the $\mathrm{DSI}_{(d)}$ for dry air, but indicating in more detail the process of moist air transport: The basic state is characterized by vanishing advection tendencies. The DSI captures deviations from the basic state and thus diagnoses the advection of moisture. Therefore, the $\mathrm{DSI}_{(v)}$ for moist air without phase changes captures the formation and dissolving of clouds. While the $\mathrm{DSI}_{(d)}$ for dry air signaled deviations from the adiabatic, inviscid and steady basic state, the difference $\mathrm{DSI}_{(d)} - \mathrm{DSI}_{(v)}$ can be used to locate local deviations from pure transport of moisture.

Remark 2 (The DSI based upon a modified potential temperature)**.** *In the moist air case the total derivative of entropy is expressed by that of a modified potential temperature θ':*

$$\frac{ds_{(v)}}{dt} = c_p' \frac{d\ln T}{dt} - R' \frac{d\ln p}{dt} = c_p' \frac{d\ln \theta'}{dt} \,, \tag{63}$$

where

$$\theta' = T \left(\frac{p_0}{p} \right)^{\frac{R'}{c_p'}} \,, \tag{64}$$

see also Emanuel [4]. Thus, during isentropic motion, θ' is conserved and we could therefore also choose

$$\psi = \theta' \qquad \text{and} \qquad \Pi^{\theta'} = \frac{\boldsymbol{\xi}_a \cdot \nabla \theta'}{\rho} \,. \tag{65}$$

We note however that the potential vorticity $\Pi^{\theta'}$ is not a conserved quantity anymore, since $s_{(v)} = F(\theta', q_v)$. The simple structure in (63) *is only obtained for the total derivative due to the conservation of q_v, but does not hold for the spatial gradient, which involves additional terms proportional to ∇q_v, see also* (49)*. Thus the solenoidal term in* (25) *does not vanish anymore. Nevertheless following the steps from above the* DSI *based upon θ' reads*

$$\mathrm{DSI}^{\theta'}_{(v)} = \frac{(\Pi^{\theta'}_{(v)})^2}{\rho} \nabla \cdot \left[\frac{1}{\Pi^{\theta'}_{(v)}} \Big((\nabla B_{(v)} + \Lambda^{\theta'}_{(v)} \nabla q_v) \times \nabla \theta' \Big) \right] \tag{66}$$

with

$$\Lambda^{\theta'}_{(v)} = \frac{1}{1+q_v} \left((c_{pv} - c_{pd})T + \frac{R_v c_{pd} - R_d c_{pv}}{c_p'} T \ln \frac{p_0}{p} \right) . \tag{67}$$

B. The DSI for cloudy air

Here we account for water vapor and cloud liquid water with mixing ratios

$$q_v = \frac{\rho_v}{\rho_d}\,, \qquad q_c = \frac{\rho_c}{\rho_d}\,, \tag{68}$$

respectively, and for their phase changes in the equations of motion. This leads us to expressions for the steady wind and DSI for cloudy air. Moreover, cloud liquid water, with density ρ_c, is assumed to be advected by the mean wind, *i.e.*, not to precipitate. Then the total water amount corresponds to

$$q_T = q_v + q_c\,, \tag{69}$$

and the total liquid water amount is

$$q_l = q_c\,. \tag{70}$$

For the moisture components we have the balance laws

$$\frac{dq_v}{dt} = -Q_{cd}\,, \qquad \frac{dq_c}{dt} = Q_{cd}\,, \tag{71}$$

where Q_{cd} denotes the condensation and evaporation rate. Obviously, the total amount of moisture is conserved, i.e.,

$$\frac{dq_T}{dt} = 0\,. \tag{72}$$

The total density ρ is given by $\rho = \rho_d + \rho_v + \rho_c$ and, following common approximations, the liquid water content is assumed to not exert any pressure on the air parcels, so that

$$p = p_d + p_v = p_d + e = \rho_d R' T = \rho R_d T \frac{1+\frac{q_v}{E}}{1+q_T}, \tag{73}$$

where $R' = R_d + q_v R_v$ as above, [3, 4]. The total enthalpy in the presence of liquid water reads (again up to constants)

$$h_{(c)} = c'_p T\,, \tag{74}$$

see, e.g., [4], where

$$c'_p = c_{pd} + q_v c_{pv} + q_l c_l \tag{75}$$

with c_l the specific heat capacity of liquid water, see table I. Neglecting the temperature dependence of the specific heat capacities provides a good approximation above the melting point, so that the latent heat of vaporization, L, which satisfies

$$dL = (c_{pv} - c_l)dT\,, \tag{76}$$

becomes linear in the temperature,

$$L = L_0 + (c_{pv} - c_l)T \quad \text{with} \quad L_0 = L(T_0) - (c_{pv} - c_l)T_0\,. \tag{77}$$

Further, the entropy in the case of cloudy air is given by

$$s_{(c)} = c'_p \ln\frac{T}{T_0} - R_d \ln\frac{p_d}{p_0} - q_v R_v \ln\frac{e}{e_0}. \tag{78}$$

As in previous steps, we additionally introduce

$$H_{(c)} = \frac{h_{(c)}}{1+q_T}\,.$$

For the gradients we then obtain

$$\nabla h_{(c)} = c'_p \nabla T + T\left(c_{pv}\nabla q_v + c_l \nabla q_l\right) \tag{79}$$

and

$$\nabla H_{(c)} = \frac{\nabla h_{(c)}}{1+q_T} - \frac{H_{(c)}}{1+q_T}\nabla q_T\,. \tag{80}$$

Making use of the ideal gas laws for dry air and water vapor, for the entropy accordingly, we obtain

$$\begin{aligned}\nabla s_{(c)} =& c'_p \nabla \ln T - \frac{1}{T}\frac{\nabla p}{\rho_d} \\ &+ \ln\frac{T}{T_0}(c_{pv}\nabla q_v + c_l \nabla q_l) - R_v \ln\frac{e}{e_0}\nabla q_v\,,\end{aligned} \tag{81}$$

and multiplication by $T\rho_d/\rho = T/(1+q_T)$ yields

$$\begin{aligned}\frac{\nabla p}{\rho} =& -\frac{T\nabla s_{(c)}}{1+q_T} + \frac{c'_p \nabla T}{1+q_T} \\ &+ \left(c_{pv} T \ln\frac{T}{T_0} - R_v T \ln\frac{e}{e_0}\right)\nabla q_v + c_l T \ln\frac{T}{T_0}\nabla q_c\,,\end{aligned} \tag{82}$$

such that for $\mathbf{G}_{(c)}$ we obtain

$$\begin{aligned}\mathbf{G}_{(c)} &= \nabla H_{(c)} - \frac{\nabla p}{\rho} \\ &= \frac{T\nabla s_{(c)}}{1+q_T} - \Lambda_{(c),1}\nabla q_v - \Lambda_{(c),2}\nabla q_l\,,\end{aligned} \tag{83}$$

where

$$\Lambda_{(c),1} = \frac{1}{1+q_T}\Big(H_{(c)} + c_{pv} T\big(\ln\frac{T}{T_0} - 1\big) - R_v T \ln\frac{e}{e_0}\Big), \tag{84}$$

$$\Lambda_{(c),2} = \frac{1}{1+q_T}\Big(H_{(c)} + c_l T\big(\ln\frac{T}{T_0} - 1\big)\Big)\,, \tag{85}$$

resemble (57). Again we choose

$$\psi = s_{(c)} \qquad \text{and} \qquad \Pi_{(c)} = \frac{\boldsymbol{\xi}_a \cdot \nabla s_{(c)}}{\rho}\,. \tag{86}$$

Furthermore,

$$\mathbf{G}_{(c)} \times \nabla s_{(c)} = -(\Lambda_{(c),1}\nabla q_v + \Lambda_{(c),2}\nabla q_l) \times \nabla s_{(c)}\,, \tag{87}$$

while the Bernoulli function for cloudy air reads

$$B_{(c)} = \frac{1}{2}\mathbf{v}^2 + H_{(c)} + \phi\,. \tag{88}$$

Finally, the steady wind results in

$$\mathbf{v}_{st,(c)} = \frac{1}{\rho\Pi_{(c)}}\Big[\nabla s_{(c)} \times (\nabla B_{(c)} + \Lambda_{(c),1}\nabla q_v + \Lambda_{(c),2}\nabla q_l)\Big]\,. \tag{89}$$

The steady wind for cloudy air describes an atmospheric basic state that includes water vapor, liquid water and phase changes, but no precipitation, and implies the

$$\begin{aligned}\mathrm{DSI}_{(c)} =& \frac{(\Pi_{(c)})^2}{\rho}\nabla\cdot\Big[\frac{1}{\Pi_{(c)}}\Big((\nabla B_{(c)} + \Lambda_{(c),1}\nabla q_v \\ &+ \Lambda_{(c),2}\nabla q_l) \times \nabla s_{(c)}\Big)\Big]\,.\end{aligned} \tag{90}$$

Thus, compared to the previously derived $\mathbf{v}_{st,(v)}$ and $\mathrm{DSI}_{(v)}$ for moist air without phase changes the $\mathrm{DSI}_{(c)}$ for cloudy air is extended by a term proportional to the gradient of the liquid water content, and latent heating appears explicitly through the definition of $s_{(c)}, B_{(c)}$ and $\Lambda_{(c),i}$, respectively. The utility of this new index is that the difference $\mathrm{DSI}_{(c)} - \mathrm{DSI}_{(v)}$ indicates processes associated with cloud generation or dissolution.

In the next section, we will further include the (vertical) transport of precipitation to derive a DSI variant that signals, e.g., extreme or less intense precipitation.

Remark 3 (Alternative expressions for the thermodynamic quantities)**.** *Often the enthaply in the case of liquid water being present is stated as (again up to constants)*

$$h_{(c)} = (c_{pd} + q_T c_l)T + Lq_v$$

see, e.g., [4]. Using (77)*, this coincides with* (74) *up to the constant value L_0. Accordingly, a common formulation of the entropy for cloudy air is*

$$s_{(c)} = (c_{pd} + q_T c_l)\ln\frac{T}{T_0} - R_d \ln\frac{p_d}{p_0} + \frac{Lq_v}{T} - q_v R_v \ln\frac{e}{e^*} \quad (91)$$

According to [4], the latent heat L satisfies

$$\frac{L}{T} = (c_{pv} - c_l)\ln\frac{T}{T_0} - R_v \ln\frac{e^*}{e_0}\,, \quad (92)$$

which is consistent with (76) *owing to the Clausius-Clapeyron relation for the saturation vapor pressure e^*,*

$$d\ln e^* = \frac{LdT}{R_v T^2}\,. \quad (93)$$

This verifies, in particular, the equivalence of (91) *and* (78)*.*

C. The DSI for precipitating air

To cover the precipitation of rain, the “rain amount”

$$q_r = \frac{\rho_r}{\rho_d}\,, \quad (94)$$

is introduced and its evolution includes vertical sedimentation with the terminal fall velocity, V_r. Then the total water amount and total density correspond to

$$q_T = q_v + q_c + q_r\,, \qquad \rho = \rho_d + \rho_v + \rho_c + \rho_r\,, \quad (95)$$

and the liquid water content is determined by

$$q_l = q_c + q_r\,. \quad (96)$$

The moisture quantities satisfy the balance laws

$$\frac{dq_v}{dt} = -Q_{cd} + Q_{ev}\,, \quad (97)$$

$$\frac{dq_c}{dt} = Q_{cd} - Q_{ac} - Q_{cr}\,, \quad (98)$$

$$\frac{dq_r}{dt} - \frac{1}{\rho_d}\partial_z(\rho_d V_r q_r) = -Q_{ev} + Q_{ac} + Q_{cr}\,, \quad (99)$$

where Q_{cd} again denotes condensation and evaporation rates, Q_{ev} the evaporation rate of rain, Q_{ac} the autoconversion rate of cloud water into rain once droplets grow big enough, and Q_{cr} is the collection rate of cloud water by the falling rain. The total amount of moisture is conserved up to the relative vertical transport of precipitation, so that

$$\frac{dq_T}{dt} = \frac{1}{\rho_d}\partial_z(\rho_d V_r q_r)\,. \quad (100)$$

The terminal fall velocity of precipitation affects also the momentum balance (12), which is extended by an additional term

$$\mathbf{W}_r = \frac{1}{1+q_T}\partial_z(q_r V_r \mathbf{v}) \quad (101)$$

on the right hand side, see [3], [7], which is typically neglected in the literature. In an asymptotic analysis for deep convective clouds it was found not to play a role in the leading order dynamics on shorter time scales, but the term could get to be relevant for longer time scales or over large areas of precipitation, [7]. This is also in agreement with [3] and references therein. As we are interested here in analysing warm convective events and local processes, it is acceptable to assume

$$\mathbf{W}_r = 0\,, \quad (102)$$

below. Then the steady wind can be derived in a similar fashion as before and, following the earlier derivations for cloudy air by replacing q_T with (95) and q_l with (96), we obtain the same expressions for $B_{(r)}, \mathbf{G}_{(r)}, H_{(r)}, \Pi_{(r)}, \Lambda_{(r),i}$ as for $B_{(c)}, \mathbf{G}_{(c)}, H_{(c)}, \Pi_{(c)}, \Lambda_{(c),i}$. The precipitation terms, however, affect the entropy balance by constituting a source term

$$\frac{ds_{(r)}}{dt} = c_l \ln\frac{T}{T_0}\frac{dq_T}{dt} = c_l \ln\frac{T}{T_0}\frac{1}{\rho_d}\partial_z(\rho_d V_r q_r) =: Q_{s_{(r)}}\,, \quad (103)$$

and the generalised construction of the steady wind yields

$$\mathbf{v}_{st,(r)} = \frac{1}{\rho\Pi_{(r)}}\Big[\nabla s_{(r)} \times (\nabla B_{(r)} + \Lambda_{(r),1}\nabla q_v + \Lambda_{(r),2}\nabla q_l) + {} + \boldsymbol{\xi}_a\, Q_{s_{(r)}}\Big]\,. \quad (104)$$

Thus, for the $\mathrm{DSI}_{(r)}$ we have

$$\mathrm{DSI}_{(r)} = \frac{(\Pi_{(r)})^2}{\rho}\nabla\cdot\Big[\frac{1}{\Pi_{(r)}}\Big((\nabla B_{(r)} + \Lambda_{(r),1}\nabla q_T + {} + \Lambda_{(r),2}\nabla q_v) \times \nabla s_{(r)} - \boldsymbol{\xi}_a\, Q_{s_{(r)}}\Big)\Big]\,. \quad (105)$$

Relative to the DSI for dry air in (36), the $\mathrm{DSI}_{(r)}$ indicates deviations from a basic state to the balance of which both phase changes of water and the (vertical) transport of precipitation contribute substantially. In turn, comparisons of the $\mathrm{DSI}_{(r)}$ with the moist and cloudy air variants, $\mathrm{DSI}_{(v)}$ and $\mathrm{DSI}_{(c)}$, respectively, allows the user to distinguish regions in which different combinations of these processes balance each other.

IV. CONCLUSION

For the non-hydrostatic compressible governing equations without moist processes, non-zero values of the scalar dynamic state index, DSI, indicate non-stationary, diabatic and dissipative atmospheric processes. This work generalized this concept to moist atmospheric flows, ultimately including phase changes and precipitation. The point of departure for the present developments is the observation that the original dry air DSI has a representation in terms of the mass flow divergence of Schär's [13] "steady wind", $\mathbf{v}_{st}$,

$$\mathrm{DSI} \equiv \frac{1}{\rho}\frac{\partial(\theta, B, \Pi^\theta)}{\partial(x,y,z)} = -\frac{\Pi^2}{\rho}\nabla\cdot(\rho\mathbf{v}_{st}^\theta). \qquad (106)$$

It is difficult to see how the determinant of gradients of the constitutive Lagrangian conserved quantities (θ, B, Π^θ), i.e., of potential temperature, Bernoulli function, and potential vorticity, respectively, can be generalized to thermodynamically more complex situations. In contrast, generalization of the concept of the steady wind has turned out to be accessible, and has allowed us to achieve the stated goal.

Thus, in a hierarchical fashion we have introduced three generalizations to include moist processes in the DSI-concept. First, we included water vapor neglecting phase changes. Second, we considered water vapor together with phase changes to account for cloud formation, and, third, we have included the fall out of precipitation. For all models we first derived the associated steady wind representing the basic state, noticing that only the basic state for moist air without phase changes still characterizes adiabatic conditions. The second generalization of the basic state incorporates diabatic but also reversible processes. With these preliminaries, the DSI is given by expressions analogous to the last term in (106), which can be transferred to all models once appropriate analoga to the potential vorticity Π and the steady wind $\mathbf{v}_{st}$ are found. Stepwise extensions of the underlying flow models by different moist process descriptions leads to a hierarchy of steady wind representations, for

Dry air

$$\mathbf{v}_{st,(d)} = \frac{1}{\rho\Pi_{(d)}}\left[\nabla s_{(d)}\times\nabla B_{(d)}\right], \qquad (107)$$

Moist air

$$\mathbf{v}_{st,(v)} = \frac{1}{\rho\Pi_{(v)}}\left[\nabla s_{(v)}\times(\nabla B_{(v)}+\Lambda_{(v)}\nabla q_v)\right], \qquad (108)$$

Cloudy air

$$\mathbf{v}_{st,(c)} = \frac{1}{\rho\Pi_{(c)}}\left[\nabla s_{(c)}\times(\nabla B_{(c)}+\Lambda_{(c),1}\nabla q_v + \Lambda_{(c),2}\nabla q_l)\right], \qquad (109)$$

Fall out of precipitation

$$\mathbf{v}_{st,(r)} = \frac{1}{\rho\Pi_{(r)}}\left[\nabla s_{(r)}\times(\nabla B_{(r)}+\Lambda_{(r),1}\nabla q_v + \Lambda_{(r),2}\nabla q_l)+\boldsymbol{\xi}_a Q_{s_{(r)}}\right], \qquad (110)$$

where the density ρ and the pressure p denote the total density and total pressure for each of the different aerodynamic models. Comparing the dry air case, where the steady wind blows parallel to level sets of the Bernoulli function on isentropic surfaces, with the moist air case, an additional velocity component appears that is directed along the isolines of the mixing ratio of water vapor. If phase changes take place, liquid water needs to be accounted for, too, which generates an additional contribution to the steady wind. The latent heat then arises in the definition of the entropy and potential vorticity.

With these results, the new DSI for moist aerothermodynamics results from the respective generalizations of steady wind mass flux divergence term on the right of (106). For conservative systems the DSI can still equivalently be formulated based upon Jacobian-determinants,

$$\begin{aligned}\mathrm{DSI}_{(d)} &= \frac{1}{\rho_d}\nabla\Pi_{(d)}\cdot(\nabla s_{(d)}\times\nabla B_{(d)}) = \frac{1}{\rho}\frac{\partial(s,B,\Pi)}{\partial(x,y,z)}\\ &= \frac{\Pi_{(d)}^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi_{(d)}}\left(\nabla B_{(d)}\times\nabla s_{(d)}\right)\right]\end{aligned} \qquad (111)$$

$$\begin{aligned}\mathrm{DSI}_{(v)} &= \frac{\Pi_{(v)}^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi_{(v)}}\left((\nabla B_{(v)}+\Lambda_{(v)}\nabla q_v)\times\nabla s_{(v)}\right)\right]\\ &= \frac{1}{\rho}\left(\frac{\partial(s_{(v)},B_{(v)},\Pi_{(v)})}{\partial(x,y,z)} - \Lambda_{(v)}\frac{\partial(\Pi_{(v)},q_v,s_{(v)})}{\partial(x,y,z)} - \Pi_{(v)}\frac{\partial(\Lambda_{(v)},q_v,s_{(v)})}{\partial(x,y,z)}\right)\end{aligned} \qquad (112)$$

$$\mathrm{DSI}_{(c)} = \frac{\Pi_{(c)}^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi_{(c)}}\left(\nabla B_{(c)}+\Lambda_{(c),1}\nabla q_v + \Lambda_{(c),2}\nabla q_c\right)\times\nabla s_{(c)}\right] \qquad (113)$$

$$\mathrm{DSI}_{(r)} = \frac{\Pi_{(r)}^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi_{(r)}}\left(\left(\nabla B_{(r)}+\Lambda_{(r),1}\nabla q_v + \Lambda_{(r),2}\nabla(q_c+q_r)\right)\times\nabla s_{(r)} - \boldsymbol{\xi}_a Q_{s_{(r)}}\right)\right] \qquad (114)$$

Reformulating accordingly $\mathrm{DSI}_{(c)}$ as a sum of five Jacobi-determinants is straightforward. Only in the presence of precipitation, with the Lagrangian conservation of the relevant constituents of the DSI no longer guaranteed, can the DSI no longer be written solely in terms of Jacobi-determinants. For convencience we have summarized all formulae needed to directly evaluate the various DSI-variants in Appendix A.

For the derivations in this paper we have preferred using the entropy as the relevant advected scalar and in formulating a potential vorticity variable over any one of various possible potential temperatures. Our motivation is that the entropy has a unique physical meaning across all cases, which is generally not true for potential temperatures in complex multicomponent flows. This is not to say, however, that a particular choice of a potential temperature variable could not streamline some of the derivations or have advantages in terms of physical interpretability. For example, by adopting the purely pressure and density dependent potential temperature θ_ρ in [14] for the

TABLE II. Physical processes and characterizations, the advections of the entropy and the PV with respect to the steady wind vanish, if they are conserved.

	entropy	PV	Specific DSI signals
Dry air	$s_{(d)}$ conserved no advection	$\Pi_{(d)}$ conserved no advection	$\mathrm{DSI}_{(d)} \neq 0$ all diabatic (frictional, non-steady) processes
Moist air	$s_{(v)}$ conserved no advection	$\Pi_{(v)}$ conserved no advection	$\mathrm{DSI}_{(v)} - \mathrm{DSI}_{(d)} \neq 0$ transport of moist air variations of the humidity
Cloudy air	$s_{(c)}$ conserved no advection	$\Pi_{(c)}$ conserved no advection	$\mathrm{DSI}_{(c)} - \mathrm{DSI}_{(v)} \neq 0$ the generation and dissolving of clouds
Fallout of rain	$s_{(r)}$ not conserved advection	$\Pi_{(r)}$ not conserved advection	$\mathrm{DSI}_{(r)} - \mathrm{DSI}_{(c)} \neq 0$ variations of precipitation (e.g. in form of ice)

formulation, one can enforce the potential vorticity to remain a Lagrangian conserved quantity even in the presence of precipitation, and this may help interpretations or further in-depth analyses. The recent study by Baumgartner et al. [1], who investigates the potential temperature in terms of temperature dependent specific heat capacities $c_p(T)$, might also be of interest in this context, and it could be incorporated in the present framework as well.

The $\mathrm{DSI}_{(d)}$ for dry air, $\mathrm{DSI}_{(v)}$ for moist air, the $\mathrm{DSI}_{(c)}$ for cloudy air and the $\mathrm{DSI}_{(r)}$ for precipitating air indicate deviations of local flow conditions from inviscid and steady state motions. Differences of DSI variants that encode different steady balances can be utilized to identify and locate particular diabatic processes. While the classical $\mathrm{DSI}_{(d)}$ for dry air reflects general diabatic processes, the basic state of $\mathrm{DSI}_{(v)}$ additionally contains water vapor. Thus the difference $\mathrm{DSI}_{(d)} - \mathrm{DSI}_{(v)}$ indicates the transport of moist air. Considering cloudy air, *i.e.* adding the effects of liquid water and its phase changes to the equations of motion, and thus to the basic state, the $\mathrm{DSI}_{(c)}$ for cloudy air indicates general precipitation processes and other diabatic processes, such as radiation. The according difference $\mathrm{DSI}_{(v)} - \mathrm{DSI}_{(c)}$ therefore signals the generation and dissolution of clouds. Accounting, in addition, for the (vertical) transport of precipitation, the difference $\mathrm{DSI}_{(r)} - \mathrm{DSI}_{(c)}$ acts as an indicator for the occurrence and intensity of rain. The different DSI variants and their proper physical interpretations are summarized in table II.

An interesting further challenge will be the incorporation of the ice phase and its different conformations, such as snow, graupel, or hail.

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Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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SUMMARY OF THE DSI VARIANTS

In the following we summarise all DSI variants and highlight the additional terms entering when moving up the hierarchy of complexity involved in the according derivations. We shall emphasize that Δ in the following does not denote the Laplacian, but is used as a symbol for the deviation terms.

$\mathrm{DSI}_{(d)}$:

For the derivations based upon the dry air setting we have the classical definition of the DSI

$$\begin{aligned}\mathrm{DSI}_{(d)} &= \frac{1}{\rho_d}\nabla\Pi_{(d)}\cdot\left(\nabla s_{(d)}\times\nabla B_{(d)}\right)\\ &= \frac{\Pi_{(d)}^2}{\rho_d}\nabla\cdot\left[\frac{1}{\Pi_{(d)}}\left(\nabla B_{(d)}\times\nabla s_{(d)}\right)\right],\end{aligned}$$

where

$$\begin{aligned}s_{(d)} &= c_{pd}\ln\frac{T}{T_0}-R_d\ln\frac{p}{p_0}\\ H_{(d)} &= c_{pd}T\\ B_{(d)} &= \frac{1}{2}\mathbf{v}^2+H_{(d)}+\phi\\ \Pi_{(d)} &= \frac{\boldsymbol{\xi_a}\cdot\nabla s_{(d)}}{\rho_d}\end{aligned}$$

$\mathrm{DSI}_{(v)}$:

As a next step water vapor is included into the derivations, leading to:

$$\begin{aligned}\mathrm{DSI}_{(v)} &= \frac{\Pi_{(v)}^2}{\rho}\nabla\cdot\left[\frac{1}{\Pi_{(v)}}\left(\nabla B_{(v)}+\Lambda_{(v)}\nabla q_v\right)\times\nabla s_{(v)}\right]\\ &= \frac{(\Pi_{(d)}+\Delta\Pi_{(v)})^2}{\rho_d(1+q_v)}\nabla\cdot\left[\frac{1}{\Pi_{(d)}+\Delta\Pi_{(v)}}\left(\nabla(B_{(d)}+\Delta B_{(v)})\right.\right.\\ &\qquad\left.\left.+\Lambda_{(v)}\nabla q_v\right)\times\nabla(s_{(d)}+\Delta s_{(v)})\right]\end{aligned}$$

where

$$\begin{aligned}s_{(v)} &= s_{(d)}+\Delta s_{(v)},\\ \Delta s_{(v)} &= c_{pv}q_v\ln\frac{T}{T_0}-R_vq_v\ln\frac{e}{e_0}\\ H_{(v)} &= H_{(d)}+\Delta H_{(v)},\\ \Delta H_{(v)} &= -\frac{q_vH_{(d)}}{1+q_v}+\frac{c_{pv}q_vT}{1+q_v}\\ B_{(v)} &= B_{(d)}+\Delta B_{(v)},\\ \Delta B_{(v)} &= \Delta H_{(v)}\\ \Pi_{(v)} &= \Pi_{(d)}+\Delta\Pi_{(v)},\\ \Delta\Pi_{(v)} &= -\frac{q_v\Pi_{(d)}}{1+q_v}+\frac{\boldsymbol{\xi_a}\cdot\nabla\Delta s_{(v)}}{\rho_d(1+q_v)}\\ \Lambda_{(v)} &= \frac{1}{1+q_v}\Big(H_{(v)}+c_{pv}T\left(\ln\frac{T}{T_0}-1\right)-R_vT\ln\frac{e}{e_0}\Big)\end{aligned}$$

$\mathrm{DSI}_{(c)}$:

The next extension amounts to the inclusion of phase changes and liquid water in the form of cloud water:

$$\begin{aligned}\mathrm{DSI}_{(c)} &= \frac{\Pi_{(c)}^2}{\rho}\nabla\cdot\Big[\frac{1}{\Pi_{(c)}}\left(\nabla B_{(c)}+\Lambda_{(c),1}\nabla q_v+\Lambda_{(c),2}\nabla q_c\right)\times\nabla s_{(c)}\Big]\\ &= \frac{(\Pi_{(v)}+\Delta\Pi_{(c)})^2}{\rho_d(1+q_v+q_c)}\nabla\cdot\Big[\frac{1}{\Pi_{(v)}+\Delta\Pi_{(c)}}\Big(\big(\nabla(B_{(v)}+\Delta B_{(c)})+\\ &\quad+(\Lambda_{(v)}+\Delta\Lambda_{(c),1})\nabla q_v+\Lambda_{(c),2}\nabla q_c\big)\times\nabla(s_{(v)}+\Delta s_{(c)})\Big)\Big]\end{aligned}$$

where

$$
\begin{aligned}
s_{(c)} &= s_{(v)} + \Delta s_{(c)}\,, \\
\Delta s_{(c)} &= c_l q_c \ln \frac{T}{T_0}\,, \\
H_{(c)} &= H_{(v)} + \Delta H_{(c)}\,, \\
\Delta H_{(c)} &= -\frac{q_c H_{(v)}}{1+q_v+q_c} + \frac{c_l q_c T + L_0}{1+q_v+q_c}\,, \\
B_{(c)} &= B_{(v)} + \Delta B_{(c)}\,, \\
\Delta B_{(c)} &= \Delta H_{(c)}\,, \\
\Pi_{(c)} &= \Pi_{(v)} + \Delta\Pi_{(c)}\,, \\
\Delta\Pi_{(c)} &= -\frac{q_c \Pi_{(v)}}{1+q_v+q_c} + \frac{\boldsymbol{\xi_a}\cdot\nabla\Delta s_{(c)}}{\rho_d(1+q_v+q_c)}\,, \\
\Lambda_{(c),1} &= \Lambda_{(v)} + \Delta\Lambda_{(c),1}\,, \\
\Delta\Lambda_{(c),1} &= -\frac{q_c \Lambda_{(v)}}{1+q_v+q_c} + \frac{\Delta H_{(c)}}{1+q_v+q_c}\,, \\
\Lambda_{(c),2} &= \frac{1}{1+q_v+q_c}\Big(H_{(c)} + c_l T\big(\ln\frac{T}{T_0} - 1\big)\Big)
\end{aligned}
$$

<u>$\mathrm{DSI}_{(r)}$:</u>

Finally also precipitation is included into the derivation of the DSI:

$$
\begin{aligned}
\mathrm{DSI}_{(r)} &= \frac{\Pi_{(r)}^2}{\rho}\nabla\cdot\Bigg[\frac{1}{\Pi_{(r)}}\Big(\big(\nabla B_{(r)} + \Lambda_{(r),1}\nabla q_v \\
&\qquad + \Lambda_{(r),2}\nabla(q_c+q_r)\big)\times\nabla s_{(r)} - \xi_a Q_{s_{(r)}}\Big)\Bigg] \\
&= \frac{(\Pi_{(c)}+\Delta\Pi_{(r)})^2}{\rho_d(1+q_v+q_c+q_r)}\nabla\cdot\Bigg[\frac{1}{\Pi_{(c)}+\Delta\Pi_{(r)}}\Big(\big(\nabla(B_{(c)}+\Delta B_{(r)}) \\
&\quad + (\Lambda_{(c),1}+\Delta\Lambda_{(r),1})\nabla q_v + \ldots \\
&\quad + (\Lambda_{(c),2}+\Delta\Lambda_{(r),2})(\nabla q_c+\nabla q_r)\big)\times\nabla(s_{(c)}+\Delta s_{(r)}) \\
&\qquad - \xi_a Q_{s_{(r)}}\Big)\Bigg]
\end{aligned}
$$

where

$$
\begin{aligned}
s_{(r)} &= s_{(c)} + \Delta s_{(r)}\,, \\
\Delta s_{(r)} &= c_l q_r \ln\frac{T}{T_0} \\
H_{(r)} &= H_{(c)} + \Delta H_{(r)}\,, \\
\Delta H_{(r)} &= -\frac{q_r H_{(c)}}{1+q_v+q_c+q_r} + \frac{c_l q_r T}{1+q_v+q_c+q_r} \\
B_{(r)} &= B_{(c)} + \Delta B_{(r)}\,, \\
\Delta B_{(r)} &= \Delta H_{(r)} \\
\Pi_{(r)} &= \Pi_{(c)} + \Delta\Pi_{(r)}\,, \\
\Delta\Pi_{(r)} &= -\frac{q_r \Pi_{(c)}}{1+q_v+q_c+q_r} + \frac{\boldsymbol{\xi_a}\cdot\nabla\Delta s_{(r)}}{\rho_d(1+q_v+q_c+q_r)} \\
\Lambda_{(r),1} &= \Lambda_{(c),1} + \Delta\Lambda_{(r),1}\,, \\
\Delta\Lambda_{(r),1} &= -\frac{q_r \Lambda_{(c),1}}{1+q_v+q_c+q_r} + \frac{\Delta H_{(r)}}{1+q_v+q_c+q_r} \\
\Lambda_{(r),2} &= \Lambda_{(c),2} + \Delta\Lambda_{(r),2}\,, \\
\Delta\Lambda_{(r),2} &= -\frac{q_r \Lambda_{(c),2}}{1+q_v+q_c+q_r} + \frac{\Delta H_{(r)}}{1+q_v+q_c+q_r} \\
Q_{s_{(r)}} &= c_l \ln\frac{T}{T_0}\frac{1}{\rho_d}\partial_z(\rho_d V_r q_r)
\end{aligned}
$$