Existence of long-time solutions to dynamic problems of viscoelasticity with rate-and-state friction

Elias Pipping*

March 13, 2017

We establish existence of long-time solutions to a dynamic problem of bilateral contact between a rigid surface and a viscoelastic body, subject to rate-and-state friction. The term *rate-and-state friction* is used here to refer to a set of functions and equations satisfying conditions which rule out the slip law but do cover the ageing law, and thus at least one of the rate-and-state friction laws commonly used in the geosciences.

MSC 2010: 35M31; 49J40, 74H20, 74H25, 35Q86.
Keywords: Variational inequality, existence, uniqueness, rate-and-state friction, viscoelasticity.

1 Introduction

We consider here the dynamic motion of a viscoelastic body $\Omega \subset \mathbb{R}^d$ in bilateral contact with a rigid foundation (on the boundary segment Γ_C), undergoing infinitesimal deformation and strain, subject to rate-and-state friction. To that end, we will derive a weak formulation of the following problem.

Problem 1. Find a displacement field $\boldsymbol{u}$ on Ω of the appropriate regularity that satisfies

$$\boldsymbol{\sigma} = \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}) + \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{u}) \qquad \text{in } \Omega \times I \qquad (1)$$

$$\nabla \cdot \boldsymbol{\sigma} + \boldsymbol{b} = \rho \ddot{\boldsymbol{u}} \qquad \text{in } \Omega \times I \qquad (2)$$

* *E-mail address*: elias.pipping@fu-berlin.de

with the boundary conditions

$$\dot{\boldsymbol{u}} = 0 \qquad \text{on } \Gamma_D \times I \tag{3}$$

$$\boldsymbol{\sigma n} = 0 \qquad \text{on } \Gamma_N \times I \tag{4}$$

$$\dot{\boldsymbol{u}} \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_C \times I \tag{5}$$

$$\left.\begin{aligned} -\boldsymbol{\sigma}_t &= \frac{\mu(|\dot{\boldsymbol{u}}|, \alpha)|\bar{\sigma}_n| + C}{|\dot{\boldsymbol{u}}|}\dot{\boldsymbol{u}} && \text{for } \dot{\boldsymbol{u}} \neq 0 \\ |\boldsymbol{\sigma}_t| &\leq \mu(0, \alpha) + C && \text{for } \dot{\boldsymbol{u}} = 0 \end{aligned}\right\} \qquad \text{on } \Gamma_C \times I \tag{6}$$

with prescribed $\boldsymbol{u}(0)$ and $\dot{\boldsymbol{u}}(0)$ as well as a scalar state field α on Γ_C that satisfies

$$\dot{\alpha} + A(\alpha) = f(|\dot{\boldsymbol{u}}|) \qquad \text{on } \Gamma_C \times I \tag{7}$$

with prescribed $\alpha(0)$.

Here, we write $\boldsymbol{u}$ for the displacement, $\boldsymbol{b}$ for the body force, $\boldsymbol{\sigma}$ for the stress tensor, and $\boldsymbol{\sigma}_t$ for its tangential component where the tangential direction is computed from the outer normal $\boldsymbol{n}$. Linear Kelvin–Voigt viscoelasticity is prescribed in (1), formulated in terms of the strain tensor $\boldsymbol{\varepsilon}$, a viscosity tensor $\boldsymbol{\mathcal{A}}$ and an elasticity tensor $\boldsymbol{\mathcal{B}}$. The friction law (6) on Γ_C is made up of the friction coefficient μ, the cohesion $C \geq 0$ and a prescribed, constant quantity denoted by $\bar{\sigma}_n$, meant to approximately equal the normal stress σ_n. Dirichlet and Neumann boundary conditions are, furthermore, imposed on the boundary segments Γ_D and Γ_N, respectively. The mass density is denoted by ρ.

2 Background

Rate-and-state friction plays an important role in the modelling of faults [3], which in turn play an important role in earthquake nucleation. It expresses frictional resistance in terms of the sliding velocity or *slip rate* $|\dot{\boldsymbol{u}}|$ and an abstract *state* variable α. Since the evolution of this state variable is again governed by the sliding velocity $|\dot{\boldsymbol{u}}|$, however, the dependence of the friction coefficient μ on the $|\dot{\boldsymbol{u}}|$ and α should rather be thought of as a means of depending on $|\dot{\boldsymbol{u}}|$ in two ways: once directly, in a monotone fashion, and once indirectly, through α, which reacts less immediately to changes in $|\dot{\boldsymbol{u}}|$, but generally in an antitone fashion.

Although laws that go by this name have been derived from experiments [2, 8], they could just as easily have been proposed as a regularisation of slip rate dependent friction (in which the coefficient of friction is a function of the sliding rate only but the dependence is generally antitone) due to the analytical and numerical difficulties that such ostensibly simpler stateless laws present [4].

The existence and uniqueness of solutions to (weak formulations of) dynamic problems of viscoelasticity and friction has been thoroughly studied. Rate-and-state friction falls outside the scope of these studies, however, because of the variable coupling between the rate and the state: Neither is typically known. The approach taken in this work is thus to consider the situation where α is known a-priori, to then compute $\dot{\boldsymbol{u}}$ under this assumption (such problems are covered by the current literature) and to then account for the actual lack of knowledge of α through a fixed-point iteration.

This work thus parallels earlier work from the author's dissertation in which the time-discrete setting was considered [6].

3 Examples

The following two rate-and-state friction laws are commonly used: the *ageing law* (also known as *slowness law*), which states

$$\mu = \mu_* + a \log \frac{r}{r_*} + b\alpha, \qquad \dot{\alpha} = \frac{r_* e^{-\alpha} - r}{L}, \tag{8}$$

and the *slip law*, which states

$$\mu = \mu_* + a \log \frac{r}{r_*} + b\alpha, \qquad \dot{\alpha} = -\frac{r}{L}\big(\log \frac{r}{r_*} + \alpha\big). \tag{9}$$

When presented in this form, both laws use the same expression for μ, so that their respective state variables α can be identified; consequently, the names of these laws are typically used to refer to the associated state evolution equations only.

The ageing law and the slip law as proposed by Dieterich and Ruina employ the term $\log(r/r_*)$, which becomes arbitrarily negative for sliding rates r close to zero; consequently, we have

$$\mu(r, \alpha) \to -\infty \qquad \text{whenever } r \to 0$$

for fixed α. They are thus unphysical for sufficiently small r, since they predict a negative coefficient of friction. If we introduce the quantity

$$r_\alpha = r_* \exp\left(-\frac{\mu_* + b\alpha}{a}\right),$$

this issue becomes even clearer, since now μ can be written as

$$\mu(r, \alpha) = a \log \frac{r}{r_\alpha}, \tag{10}$$

so that r_α denotes the rate at which the predicted coefficient of friction undergoes a sign change. In the literature, this undesirable behaviour of the

Dieterich–Ruina laws has been addressed by means of regularisation [7]. To be precise, the logarithm on the right-hand side of (10) is replaced by the nonnegative function $z \mapsto \operatorname{asinh}(z/2)$, yielding the *regularised law*

$$\mu_{\mathrm{r}}(r,\alpha) = a \operatorname{asinh}\left(\frac{r}{2r_\alpha}\right). \tag{11}$$

A different approach is to trust the original law as much as possible, and only modify it whenever it predicts a negative coefficient of friction. The requirement of monotonicity then leads to the *truncated law*

$$\mu_{\mathrm{t}}(r,\alpha) = a \log^+ \frac{r}{r_\alpha} \qquad \text{with} \qquad \log^+ z = \log \max(1,z) \tag{12}$$

Both adjustments clearly guarantee nonnegativity of the friction coefficient.

In what follows, rather than consider such laws directly, we choose to work in an abstract setting where friction is described through the friction coefficient $\mu\colon \mathbb{R}_0^+ \times \mathbb{R} \to \mathbb{R}_0^+$ and two functions $A\colon \mathbb{R} \to \mathbb{R}$, $f\colon \mathbb{R}_0^+ \to \mathbb{R}$ that govern the state evolution through the equation

$$\dot{\alpha} + A(\alpha) = f(r).$$

It is immediately clear that the slip law does not fall into this setting, unfortunately. The ageing law and potentially other laws of interest, however, do.

4 Abstract rate-and-state friction

In working with μ, A, and f, we find it necessary to make the following assumptions.

(A1) The function μ is nondecreasing and continuous in its first argument.

(A2) The function μ is uniformly Lipschitz in its second argument. In other words, we have

$$|\mu(r,\alpha) - \mu(r,\beta)| \le L_\mu |\alpha - \beta|$$

for any α, β, and $r \ge 0$.

(A3) The function μ can be bounded as follows:

$$0 \le \mu(r,\alpha) \le C_\mu(1 + r + |\alpha|)$$

for any α and $r \ge 0$.[1]

(A4) The function A is nondecreasing and continuous.

[1] Assumptions (A2) and (A3) are not independent. Indeed, if we assume the former, the latter reduces to requiring $\mu(r,0) \le C_\mu(1+r)$.

(A5) The function f is Lipschitz, so that we have

$$|f(r) - f(v)| \leq L_f |r - v|$$

for any r and v.

As mentioned earlier, the slip law clearly does not fit into this framework because of the requirement that $\dot{\alpha}$ can be written as a sum of two terms, one of which depends solely on α with the other depending solely on r.

The ageing law, in contrast, satisfies all of the assumptions made above.

Proposition 2. *Consider the ageing law* (8)*, either regularised as per* (11) *or truncated as per* (12)*. Then the resulting law satisfies assumptions (A1) to (A5).*

Proof. That μ_{r} and μ_{t} satisfy assumption (A1) is clear. To show that μ_{r} satisfies assumption (A2), it suffices to prove

$$|\mu_{\mathrm{r}}(r,\alpha) - \mu_{\mathrm{r}}(r,\beta)| = a\left|\operatorname{asinh}\left(\frac{r}{2r_\alpha}\right) - \operatorname{asinh}\left(\frac{r}{2r_\beta}\right)\right| \leq a\left|\log\frac{r_\beta}{r_\alpha}\right|$$

for any α, β, and $r \geq 0$, since the right-hand side equals $b \cdot |\alpha - \beta|$. For $r = 0$, this is immediate; for $r > 0$, it becomes clear once we prove the more general claim

$$|\operatorname{asinh}(x) - \operatorname{asinh}(y)| \leq |\log x - \log y|$$

for x, $y > 0$. Without loss of generality, assume $x \geq y$, so that we need to show

$$\operatorname{asinh}(x) - \operatorname{asinh}(y) \leq \log x - \log y.$$

From the logarithmic representation of the asinh function, we obtain that this is equivalent to

$$\log\frac{x + \sqrt{x^2+1}}{y + \sqrt{y^2+1}} \leq \log\frac{x}{y}$$

and thus

$$y\sqrt{x^2+1} \leq x\sqrt{y^2+1}$$

which is obviously true. For μ_{t}, we proceed analogously and prove

$$|\mu_{\mathrm{t}}(r,\alpha) - \mu_{\mathrm{t}}(r,\beta)| = a\left|\log^+\frac{r}{r_\alpha} - \log^+\frac{r}{r_\beta}\right| \leq a\left|\log\frac{r_\beta}{r_\alpha}\right|.$$

Again, this is trivially true if $r = 0$. For $r > 0$, we have

$$\begin{aligned}
\left|\log^+\frac{r}{r_\alpha} - \log^+\frac{r}{r_\beta}\right| &= \left|\log\max\left\{\frac{r}{r_\alpha}, 1\right\} - \log\max\left\{\frac{r}{r_\beta}, 1\right\}\right| \\
&= \left|\max\left\{\log\frac{r}{r_\alpha}, 0\right\} - \max\left\{\log\frac{r}{r_\beta}, 0\right\}\right| \\
&\leq \left|\log\left(\frac{r}{r_\alpha}\right) - \log\left(\frac{r}{r_\beta}\right)\right|
\end{aligned}$$

since $\max\{\cdot, 0\}$ is nonexpansive, so that the claim follows. To see that μ_{t} and μ_{r} satisfy assumption (A3), observe only

$$\mu_{\mathrm{t}}(r,\alpha) = a\log^+\frac{r}{r_\alpha} \le a\left(\log^+\frac{r}{r_*} + \left|\log\frac{r_\alpha}{r_*}\right|\right) \le a\frac{r}{r_*} + \mu_* + b|\alpha|.$$

and

$$\begin{aligned}\mu_{\mathrm{r}}(r,\alpha) &= a\,\mathrm{asinh}\,\frac{r}{2r_\alpha} = a\log\left(\frac{r}{2r_\alpha} + \sqrt{\left(\frac{r}{2r_\alpha}\right)^2 + 1}\right) \le a\log\left(\frac{r}{r_\alpha} + 1\right)\\ &\le a\log\left(2\max\left\{1, \frac{r}{r_\alpha}\right\}\right) = a\log 2 + \mu_{\mathrm{t}}(r,\alpha).\end{aligned}$$

Finally, each law clearly satisfies assumptions (A4) and (A5) with

$$A(\alpha) = -\frac{r_*}{L}e^{-\alpha}, \quad f(r) = r/L, \quad \text{and} \quad L_f = L. \qquad \square$$

5 Weak formulation

Here and in what follows, we will make the following typical assumptions on the domain Ω, the viscoelastic parameters, the body force, and the normal stress that we prescribe on the frictional boundary Γ_C.

(A6) The domain Ω is a bounded open subset of $\mathbb{R}^d$ with a Lipschitz boundary. In particular, the d-dimensional trace map γ is well-defined from $H^1(\Omega)^d$ to $L^2(\Gamma)^d$.

(A7) The viscosity tensor is symmetric as well as uniformly bounded from above and below through $0 < m_{\boldsymbol{\mathcal{A}}} \le M_{\boldsymbol{\mathcal{A}}}$, so that

$$m_{\boldsymbol{\mathcal{A}}}\|\boldsymbol{v}\|_V^2 \le \langle \mathfrak{A}\boldsymbol{v}, \boldsymbol{v}\rangle = \int_\Omega \langle \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle$$

and

$$\int_\Omega \langle \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle = \langle \mathfrak{A}\boldsymbol{v}, \boldsymbol{w}\rangle \le M_{\boldsymbol{\mathcal{A}}}\|\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V$$

hold for any $\boldsymbol{v}$, $\boldsymbol{w} \in V$.

(A8) The elasticity tensor is symmetric as well as uniformly bounded from above and below through $0 < m_{\boldsymbol{\mathcal{B}}} \le M_{\boldsymbol{\mathcal{B}}}$, so that

$$m_{\boldsymbol{\mathcal{B}}}\|\boldsymbol{v}\|_V^2 \le \langle \mathfrak{B}\boldsymbol{v}, \boldsymbol{v}\rangle = \int_\Omega \langle \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{v})\rangle$$

and

$$\int_\Omega \langle \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\boldsymbol{w})\rangle = \langle \mathfrak{B}\boldsymbol{v}, \boldsymbol{w}\rangle \le M_{\boldsymbol{\mathcal{B}}}\|\boldsymbol{v}\|_V\|\boldsymbol{w}\|_V$$

hold for any $\boldsymbol{v}$, $\boldsymbol{w} \in V$.

(A9) The body force $\boldsymbol{b}$ satisfies

$$\|\boldsymbol{b}\|_{L^2(0,T,V^*)} < \infty.$$

(A10) The prescribed normal stress $\bar{\sigma}_n$ satisfies

$$\|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} < \infty.$$

We will work with the spaces

$$V = \{\boldsymbol{v} \in H^1(\Omega)^d \colon \boldsymbol{v} = 0 \text{ on } \Gamma_D,\ \boldsymbol{v} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_C\} \quad \text{and} \quad H = L^2(\Omega)^d$$

which give rise to the Gelfand triple $V \subset H \subset V^*$, as well as the space

$$X = L^2(\Gamma_C).$$

In a standard fashion, by testing (2) with functions from V at fixed points in time, and putting (1) as well as (3) to (6) to use, we obtain the following weak rate problem.

Problem 3. For given $\alpha \in C(0,T,X)$, find $\boldsymbol{u} \in L^2(0,T,V)$ with $\dot{\boldsymbol{u}} \in L^2(0,T,V)$ and $\ddot{\boldsymbol{u}} \in L^2(0,T,V^*)$ such that[2]

$$\int_\Omega \rho\langle \ddot{\boldsymbol{u}}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle + \int_\Omega \langle \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\dot{\boldsymbol{u}}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\rangle + \int_\Omega \langle \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{u}(t)), \boldsymbol{\varepsilon}(\boldsymbol{v} - \dot{\boldsymbol{u}}(t))\rangle$$
$$+ \Phi_\alpha(t, \gamma\boldsymbol{v}) - \Phi_\alpha(t, \gamma\dot{\boldsymbol{u}}(t)) \geq \int_\Omega \langle \boldsymbol{b}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle \quad \forall \boldsymbol{v} \in V \quad (13)$$

for almost every $t \in [0,T]$ with prescribed $\boldsymbol{u}(0) = \boldsymbol{u}_0$, $\dot{\boldsymbol{u}}(0) = \dot{\boldsymbol{u}}_0$ and the friction nonlinearities given by

$$\Phi_\alpha(t, \boldsymbol{v}) = \int_{\Gamma_C} \varphi_\alpha(t, x, |\boldsymbol{v}(x)|)\,\mathrm{d}x \quad \text{and} \quad \varphi_\alpha(t,x,v) = \int_0^v \mu(r, \alpha(t,x))|\bar{\sigma}_n| + C\,\mathrm{d}r.$$

For the state field α, meanwhile, we stick to a strong formulation, requiring the following.

Problem 4. For given $\dot{\boldsymbol{u}} \in L^2(0,T,V)$, find $\alpha \in C(0,T,X)$ such that

$$\dot{\alpha}(t) + A(\alpha(t)) = f(|\gamma\dot{\boldsymbol{u}}(t)|) \text{ almost everywhere on } \Gamma_C$$

for almost every $t \in [0,T]$, with prescribed $\alpha(0) = \alpha_0$.

The reformulation of the coupled problem 1 we will work with from here on is thus the problem of finding a pair $(\dot{\boldsymbol{u}}, \alpha) \in L^2(0,T,V) \times C(0,T,X)$ such that $\dot{\boldsymbol{u}}$ solves problem 3 with state α and α solves problem 4 with rate $\dot{\boldsymbol{u}}$. To analyse this problem coupling, we first consider each problem separately

[2] The x-dependence of each integrand is not made explicit here.

6 Analysis of the rate problem

Remark 5. *In operator notation, we can also write* (13) *as the variational inequality*

$$\rho\langle \ddot{\boldsymbol{u}}(t) + \mathfrak{A}\dot{\boldsymbol{u}}(t) + \mathfrak{B}\boldsymbol{u}(t) - \boldsymbol{b}(t), \boldsymbol{v} - \dot{\boldsymbol{u}}(t)\rangle + \Phi_\alpha(t, \gamma\boldsymbol{v}) \geq \Phi_\alpha(t, \gamma\dot{\boldsymbol{u}}(t)) \quad \forall \boldsymbol{v} \in V \tag{14}$$

or the subdifferential inclusion

$$\boldsymbol{b}(t) \in \rho\ddot{\boldsymbol{u}}(t) + \mathfrak{A}\dot{\boldsymbol{u}}(t) + \mathfrak{B}\boldsymbol{u}(t) + \gamma^*\partial\Phi_\alpha(t, \cdot)(\gamma\dot{\boldsymbol{u}}(t)) \tag{15}$$

with $\mathfrak{A}$, $\mathfrak{B}\colon V \to V^*$ *given by*

$$\mathfrak{A}\boldsymbol{v} = \int_\Omega \langle \boldsymbol{\mathcal{A}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\cdot)\rangle \quad \text{and} \quad \mathfrak{B}\boldsymbol{v} = \int_\Omega \langle \boldsymbol{\mathcal{B}}\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\varepsilon}(\cdot)\rangle.$$

A result on second-order hemivariational inequalities now applies in particular to our variational setting.

Proposition 6. *Problem 3 has a unique solution for any* $\alpha \in C(0, T, X)$, $\boldsymbol{u}_0 \in V$, *and* $\dot{\boldsymbol{u}}_0 \in H$.

Proof. For existence of a solution see Migórski and Ochal [5, Corollary 12]. Uniqueness follows in particular from proposition 7 which we prove next.

A few comments are in order on why Theorem 8 and thus Corollary 12 from the previously cited work can be applied: Assumptions (A7) and (A8) make $\mathfrak{A}$ and $\mathfrak{B}$ strongly monotone and symmetric bounded linear operators. Assumption (A1), moreover, makes $\varphi_\alpha(t, x, \cdot)$ convex for almost every $(t, x) \in [0, T] \times \Gamma_C$, so that the Clarke subdifferential of $\varphi_\alpha(t, x, \cdot)$ is actually a regular subdifferential. Assumption (A3), finally, guarantees

$$|\partial\varphi_\alpha(t, x, \cdot)(v)| \leq C_\mu(1 + |v| + |\alpha(t, x)|)|\bar{\sigma}_n| + C. \tag{16}$$

While Theorem 8 in the aforementioned work, as stated, requires (16) to hold without a t- or x-dependent term, a look at the proof reveals that we are free to add any term from $L^2(0, T, X)$, and thus in particular $|\alpha|$. □

Proposition 7. *For two solutions* $\boldsymbol{u}$ *and* $\boldsymbol{w}$ *of problem 3 corresponding to* α *and* β*, respectively, with identical initial conditions and* $t \in [0, T]$*, we have*

$$\|\dot{\boldsymbol{w}} - \dot{\boldsymbol{u}}\|_{L^2(0,t,V)} \leq \sqrt{t}\frac{L_\mu\|\gamma\|}{m_{\boldsymbol{\mathcal{A}}}}\|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)}\|\beta - \alpha\|_{C(0,t,X)}.$$

In particular, the solution operator $R\colon \alpha \mapsto \dot{\boldsymbol{u}}$ *is single-valued and Lipschitz with the constant*

$$L_R = \sqrt{T}\frac{L_\mu\|\gamma\|}{m_{\boldsymbol{\mathcal{A}}}}\|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)}$$

from $C(0, T, X)$ *to* $L^2(0, T, V)$.

Proof. We test (14) for $\boldsymbol{u}$ with $\dot{\boldsymbol{w}}$ and for $\boldsymbol{w}$ with $\dot{\boldsymbol{u}}$ to obtain

$$\begin{aligned}
&\langle \rho(\ddot{\boldsymbol{w}}(s) - \ddot{\boldsymbol{u}}(s)) + \mathfrak{A}(\dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{u}}(s)) + \mathfrak{B}(\boldsymbol{w}(s) - \boldsymbol{u}(s)), \dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{u}}(s)\rangle \\
&\quad \leq \Phi_\alpha(s, \gamma\dot{\boldsymbol{w}}(s)) - \Phi_\alpha(s, \gamma\dot{\boldsymbol{u}}(s)) + \Phi_\beta(s, \gamma\dot{\boldsymbol{u}}(s)) - \Phi_\beta(s, \gamma\dot{\boldsymbol{w}}(s)) \\
&\quad = \int_{\Gamma_C} \int_{|\gamma\dot{\boldsymbol{u}}(s)|}^{|\gamma\dot{\boldsymbol{w}}(s)|} \big(\mu(r,\alpha) - \mu(r,\beta)\big)|\bar{\sigma}_n| \,\mathrm{d}r \\
&\quad \leq L_\mu \int_{\Gamma_C} |\gamma\dot{\boldsymbol{w}}(s) - \gamma\dot{\boldsymbol{u}}(s)||\beta(s) - \alpha(s)||\bar{\sigma}_n| \\
&\quad \leq L_\mu \|\gamma\| \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\dot{\boldsymbol{w}}(s) - \dot{\boldsymbol{u}}(s)\|_V \|\beta(s) - \alpha(s)\|_X
\end{aligned}$$

for almost every $s \in [0,T]$, where the second-to-last estimate makes use of assumption (A2). Integrating this inequality over the time interval $[0,t] \subset [0,T]$ and putting assumptions (A7) and (A8) to use yields

$$\begin{aligned}
&\frac{\rho}{2}\|\dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t)\|_H^2 + m_{\boldsymbol{\mathcal{A}}}\|\dot{\boldsymbol{w}} - \dot{\boldsymbol{u}}\|_{L^2(0,t,V)}^2 + \frac{m_{\boldsymbol{\mathcal{B}}}}{2}\|\dot{\boldsymbol{w}}(t) - \dot{\boldsymbol{u}}(t)\|_V^2 \\
&\qquad\qquad \leq L_\mu \|\gamma\| \|\bar{\sigma}_n\|_{L^\infty(\Gamma_C)} \|\dot{\boldsymbol{w}} - \dot{\boldsymbol{u}}\|_{L^2(0,t,V)} \|\beta - \alpha\|_{L^2(0,t,X)}.
\end{aligned}$$

The claim now follows from Hölder's inequality. □

7 Analysis of the state problem

In problem 4, we view A as an operator on the function space X and obtain a problem that has the structure of an evolution equation associated with a maximal monotone operator; in doing so, we do not put the superposition operator structure of A to use: To solve problem 4 is to solve a family of ordinary differential equations at once. In what follows, we apply the first and second line of thinking, in this order.

Proposition 8. *Problem 4 has a unique solution for any $\dot{\boldsymbol{u}} \in L^2(0,T,V)$ and $\alpha_0 \in X$.*

Proof. See for example Attouch and Damlamian [1, Theorem 1.3]. We remark that the requirement

$$\alpha_0 \in \overline{\mathrm{dom}(A)}$$

is automatically fulfilled since we have $L^\infty(\Gamma_C) \subset \mathrm{dom}(A)$ and $L^\infty(\Gamma_C)$ is dense in $L^1(\Gamma_c)$. □

The solution operator corresponding to proposition 8 additionally depends Lipschitz-continuously on the right-hand side.

Proposition 9. *For two solutions α and β of problem 4 corresponding to $\dot{\boldsymbol{u}}$ and $\dot{\boldsymbol{w}}$, respectively, with identical initial conditions and $t \in [0,T]$, we have*

$$\|\alpha(\cdot,x) - \beta(\cdot,x)\|_{C(0,t)} \leq L_f \|\gamma\dot{\boldsymbol{u}}(\cdot,x) - \gamma\dot{\boldsymbol{w}}(\cdot,x)\|_{L^1(0,t,\mathbb{R}^d)} \tag{17}$$

for almost every $x \in \Gamma_C$ and thus

$$\|\alpha - \beta\|_{C(0,T,X)} \leq \sqrt{T} L_f \|\gamma \dot{\boldsymbol{u}} - \gamma \dot{\boldsymbol{w}}\|_{L^2(0,T,X^d)}. \tag{18}$$

In particular, the solution operator $S\colon \dot{\boldsymbol{u}} \mapsto \alpha$ is Lipschitz with the constant

$$L_S = \sqrt{T}\|\gamma\| L_f$$

from $L^2(0,T,V)$ to $C(0,T,X)$.

Proof. For almost every $x \in \Gamma_C$ and $s \in [0,T]$, we have

$$\begin{aligned} \dot{\alpha}(s,x) + A(\alpha(s,x)) &= f(|\gamma \dot{\boldsymbol{u}}(s,x)|), \\ \dot{\beta}(s,x) + A(\beta(s,x)) &= f(|\gamma \dot{\boldsymbol{w}}(s,x)|) \end{aligned}$$

and thus a pair of evolution equations that have the same structure as problem 4 and are additionally one-dimensional. For each such pair we can derive

$$|\alpha(t,x) - \beta(t,x)| \leq \|f(|\gamma \dot{\boldsymbol{u}}(\cdot,x)|) - f(|\gamma \dot{\boldsymbol{w}}(\cdot,x)|)\|_{L^1(0,t,\mathbb{R}^n)}$$

for example from Attouch and Damlamian [1, Theorem 1.2(ii)]). Because of assumption (A5), this implies (17). To obtain (18), we apply Hölder's inequality, yielding

$$\begin{aligned} |\alpha(t,x) - \beta(t,x)| &\leq L_f \|\gamma \dot{\boldsymbol{u}}(\cdot,x) - \gamma \dot{\boldsymbol{w}}(\cdot,x)\|_{L^1(0,t,\mathbb{R}^d)} \\ &\leq \sqrt{t} L_f \|\gamma \dot{\boldsymbol{u}}(\cdot,x) - \gamma \dot{\boldsymbol{w}}(\cdot,x)\|_{L^2(0,t,\mathbb{R}^d)} \end{aligned}$$

for almost every $(t,x) \in [0,T] \times \Gamma_C$, so that by integrating over Γ_C we find

$$\|\alpha(t,\cdot) - \beta(t,\cdot)\|_X \leq \sqrt{t} L_f \|\gamma \dot{\boldsymbol{u}} - \gamma \dot{\boldsymbol{w}}\|_{L^2(0,t,X^d)}.$$

Since $t \in [0,T]$ was arbitrary, this proves (18). □

8 Analysis of the coupled problem

We first establish short-time existence and uniqueness of a solution.

Proposition 10. *For sufficiently small $T > 0$, problems 3 and 4 have a unique simultaneous solution $(\dot{\boldsymbol{u}}, \alpha) \in L^2(0,T,V) \times C(0,T,X)$ provided that $\boldsymbol{u}_0 \in V$, $\dot{\boldsymbol{u}}_0 \in H$, and $\alpha_0 \in X$.*

Proof. By propositions 7 and 9, the operator $R \circ S\colon L^2(0,T,V) \to L^2(0,T,V)$ is Lipschitz with the constant $L_{RS} = L_R L_S$, which satisfies $L_{RS} \to 0$ as $T \to 0$. In particular, the time T can be chosen such that we have $L_{RS} < 1$. The claim now follows from Banach's fixed point theorem. □

We note that T is not constrained in any way by the values of the initial data $\boldsymbol{u}_0$, $\dot{\boldsymbol{u}}_0$ or α_0. We can thus extend a solution provided by proposition 10 to the interval $[0, 2T]$ by applying the aforementioned proposition repeatedly: once with the actual initial data to obtain a solution on the time interval $[0, T]$ and once with the *final data* resulting from the first application, namely $\boldsymbol{u}(T)$, $\dot{\boldsymbol{u}}(T)$, and $\alpha(T)$, to obtain a solution on the interval $[T, 2T]$.

That this is indeed possible follows from the embeddings[3]

$$\boldsymbol{u} \in H^1(0, T, V) \subset C(0, T, V) \quad \text{and} \quad \dot{\boldsymbol{u}} \in H^1(0, T, V, V^*) \subset C(0, T, H)$$

which give us $\boldsymbol{u}(T) \in V$ and $\dot{\boldsymbol{u}}(T) \in H$ in addition to $\alpha(T) \in X$. Since the aforementioned continuation procedure can be repeated an arbitrary number of times, we can obtain solutions on $[0, nT]$ for arbitrary $n \in \mathbb{N}$ and thus intervals of arbitrary size.

Theorem 11. *For any $T > 0$, problems 3 and 4 have a unique simultaneous solution $(\dot{\boldsymbol{u}}, \alpha) \in L^2(0, T, V) \times C(0, T, X)$ provided that $\boldsymbol{u}_0 \in V$, $\dot{\boldsymbol{u}}_0 \in H$, and $\alpha_0 \in X$.*

Acknowledgements

This work was supported by the German Research Foundation through project B01 of the CRC 1114.

References

[1] H. Attouch and A. Damlamian. "On multivalued evolution equations in Hilbert spaces". In: *Isr. J. Math.* 12 (1972), pp. 373–390. DOI: 10.1007/BF02764629.

[2] J. H. Dieterich. "Modeling of Rock Friction: 1. Experimental Results and Constitutive Equations". In: *J. Geophys. Res.: Solid Earth* 84.B5 (1979), pp. 2161–2168. DOI: 10.1029/JB084iB05p02161.

[3] D. R. Faulkner et al. "A review of recent developments concerning the structure, mechanics and fluid flow properties of fault zones". In: *J. Struct. Geol.* 32.11 (2010), pp. 1557–1575. DOI: 10.1016/j.jsg.2010.06.009.

[4] I. R. Ionescu. "Viscosity solutions for dynamic problems with slip-rate dependent friction". In: *Q. Appl. Math.* 60.3 (2002), pp. 461–476. DOI: 10.1090/qam/1914436.

[3] By $H^1(0, T, V, V^*)$ we mean here the space of functions $\{\boldsymbol{v} \in L^2(0, T, V) : \dot{\boldsymbol{v}} \in L^2(0, T, V^*)\}$ equipped with the norm $\|\boldsymbol{v}\|_{H^1(0,T,V)} = \|\boldsymbol{v}\|_{L^2(0,T,V)} + \|\dot{\boldsymbol{v}}\|_{L^2(0,T,V^*)}$.

[5] S. Migórski and A. Ochal. "Hemivariational inequality for viscoelastic contact problem with slip-dependent friction". In: *Nonlinear Anal., Theory Methods Appl., Ser. A* 61.1–2 (2005), pp. 135–161. DOI: 10.1016/j.na.2004.11.018.

[6] E. Pipping. "Dynamic problems of rate-and-state friction in viscoelasticity". Dissertation. Freie Universität Berlin, 2014. URN: urn:nbn:de:kobv:188-fudissthesis000000098145-4.

[7] J. R. Rice, N. Lapusta, and K. Ranjith. "Rate and state dependent friction and the stability of sliding between elastically deformable solids". In: *J. Mech. Phys. Solids* 49.9 (2001), pp. 1865–1898. DOI: 10.1016/S0022-5096(01)00042-4.

[8] A. Ruina. "Slip Instability and State Variable Friction Laws". In: *J. Geophys. Res.: Solid Earth* 88.B12 (1983), pp. 10359–10370. DOI: 10.1029/JB088iB12p10359.