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GLOBAL WELL-POSEDNESS FOR PASSIVELY TRANSPORTED NONLINEAR MOISTURE DYNAMICS WITH PHASE CHANGES

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Abstract. We study a moisture model for warm clouds that has been used by Klein and Majda in [21] as a basis for multiscale asymptotic expansions for deep convective phenomena. These moisture balance equations correspond to a bulk microphysics closure in the spirit of Kessler [20] and Grabowski and Smolarkiewicz [15], in which water is present in the gaseous state as water vapor and in the liquid phase as cloud water and rain water. It thereby contains closures for the phase changes condensation and evaporation, as well as the processes of autoconversion of cloud water into rainwater and the collection of cloud water by the falling rain droplets. Phase changes are associated with enormous amounts of latent heat and therefore provide a strong coupling to the thermodynamic equation.

In this work we assume the velocity field to be given and prove rigorously the global existence and uniqueness of uniformly bounded solutions of the moisture model with viscosity, diffusion and heat conduction. To guarantee local well-posedness we first need to establish local existence results for linear parabolic equations, subject to the Robin boundary conditions on the cylindric type of domains under consideration. We then derive a priori estimates, for proving the maximum principle, using the Stampacchia method, as well as the iterative method by Alikakos [1] to obtain uniform boundedness. The evaporation term is of power law type, with an exponent in general less or equal to one and therefore making the proof of uniqueness more challenging. However, these difficulties can be circumvented by introducing new unknowns, which satisfy the required cancellation and monotonicity properties in the source terms.

1. Introduction

Latent heat conversions due to phase changes of water in the atmosphere do to a great extent influence the energy balance, which is made evident by the following statement of Emanuel in [34] on p. 5: *"If all the water vapor near the surface on a hot muggy day were condensed out, the air would warm by about* $35°C$*"*. Although the total amount of moisture even under saturated conditions is small compared to the dry air components, its effect on the dynamics can be enormous, getting visible e.g. in thunderstorms. To obtain a better understanding of such complex processes involving the interaction of deep convective phenomena with their environments, Klein and Majda [21] incorporated moisture processes into the asymptotic framework by performing very careful nondimensionalisations. Their studies involving multiple scales are thereby based on a warm cloud model, where water is

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present in gaseous and liquid form. Using this setting in further works they in particular also revealed interesting new results on the modulation of gravity waves by columnar clouds [31, 32]. The closure for the source terms of the moisture quantities in that work corresponds to a basic form of a bulk microphysics model in the spirit of Kessler [20] and Grabowski and Smolarkiewicz [15]. In the present work we establish global well-posedness of the moisture model coupled through the phase changes to the thermodynamic equation, where we assume the velocity field to be given. The coupling to the primitive equations will be studied in a forthcoming paper by taking over the ideas of Cao and Titi [7] for their recent breakthrough on the global solvability of the latter system. Moreover, cases of partial diffusions, arising from the asymptotical analysis in [21], will be analyzed in a future work too based on the results by Cao et al. [3, 4, 5, 6].

To the best of our knowledge the mathematical analysis of moisture models with phase changes has been investigated only in a few papers, see Coti-Zelati et al. [2, 8, 9, 10], Li and Titi [25], and Majda and Souganidis [27]. The models studied by Coti-Zelati et al. [2, 8, 9, 10] consist of one moisture quantity coupled to the temperature and contain only the process of condensation during upward motion, see e.g. [18]. Since the source term considered in [2, 8, 9, 10] is modeled via a Heavy side function as a switching term between saturated and undersaturated regions, an approach based on differential inclusions and variational techniques was used in these papers. The moisture model investigated in [25, 27] is a coupled nonlinear system of the moisture to the barotropic and the first baroclinic models of the velocity, where the source term in the moisture is the precipitation. This model proposed by Frierson et al. [14] for the tropical atmosphere involves a small convective adjustment time scale parameter in the precipitation term. The rigorous analysis was carried out in [25] for both the finite-time relaxation system and the instantaneous relaxation limiting system, including the global well-posedness and the strong convergence of the relaxation limit. Some moisture models without phase changes coupled to the primitive equations were also considered by Guo and Huang [16, 17].

The model we are analyzing in this paper is physically more refined and consists of three moisture quantities for water vapor, cloud water and rain water and contains besides all the phase changes due to condensation and evaporation also the autoconversion of cloud water to rain water after a certain threshold is reached, as well as a closure for the collection of cloud water by the falling rain droplets.

In the remainder of the introduction we first introduce the moisture model in the setting of Klein and Majda [21] with additional diffusion, viscosity and heat conduction in cartesian coordinates. We then reformulate the system in pressure coordinates, which have the advantage that under the assumption of hydrostatic balance the continuity equation takes the form of the incompressibility condition. Although we assume the velocity field to be given this property is very useful when deriving a priori estimates, since it allows for cancellations of integral terms involving the advection terms. In the pressure coordinates the vertical diffusion term becomes nonlinear and we make here the standard approximation by linearizing around a given background temperature profile, see also [2, 8, 26, 30]. In section 2 we then formulate the full problem with boundary and side conditions and state the main result on the global existence and uniqueness of solutions.

Section 3 contains the proof of local existence and some appropriate a priori estimates. The local existence is proved based on the results established in section A for linear parabolic equations, subject to the Robin boundary conditions on the cylindric type of domains. The a priori estimates are derived by using the Stampaccia type arguments and the iterative methods of Alikakos [1].

Section 4 contains the proof of the well-posedness, i.e., the uniqueness and the continuous dependence of the solutions on the initial data. The proof is based on the typical L^2-type estimates for the difference between two solutions. Here the main difficulty is caused by the evaporation source term which is of power law type in q_r with an exponent, generally, less or equal to 1, see (1.18), below. This possible lack of Lipschitz continuity is overcome by introducing new unknowns, for which the source terms satisfy certain cancellation and monotonicity properties in the estimates, such that the uniqueness of the solution can be concluded also for the original unknowns.

The Appendix is a technical section, which gives the necessary results for some linear parabolic equations subject to the Robin boundary conditions on the cylindric domains. These parabolic equations are motivated by the problem considered in the present paper. Due to the cylindric domains not being smooth, the corresponding parabolic theory required for our analysis cannot be found explicitly in the existing literature. Therefore, for the sake of completeness, the derivations are carried out here in detail.

1.1. Moisture model in cartesian coordinates.

In the following $\rho, T, p, \mathbf{u} = (u, v), w$ are the density, temperature, pressure, as well as horizontal and vertical velocity components respectively. The thermodynamic equation is given by

$$\frac{DT}{Dt} - \frac{RT}{c_p p}\frac{Dp}{Dt} = S_T + \mathcal{D}^T T\,, \tag{1.1}$$

where $\mathcal{D}$ denotes here, and in what follows, the turbulent or molecular diffusion/viscosity operators, for which we will assume the simple closure in form of the full Laplacian. The term S_T accounts for the diabatic source and sink terms, such as latent heating, radiation effects, etc., (see e.g. also [8, 9, 21]), but we will in the following only focus on the effect of latent heat in association with phase changes. The total derivative contains the advection with respect to the given velocity field (u, v, w) is denoted by:

$$\frac{D}{Dt} = \partial_t + u\partial_x + v\partial_y + w\partial_z\,. \tag{1.2}$$

The ideal gas law with the gas constant R reads

$$p = R\rho T\,. \tag{1.3}$$

Remark 1.1. *Due to the difference of the gas constant for dry air and water vapor, for moist air more precisely the virtual temperature, or the density temperature respectively if also condensed water is present, should be used in* (1.3). *Moreover, the heat capacities in the presence of moisture vary as well. Notably, since these corrections are small, they are* *neglected in the following. For more details on the thermodynamics of moist air we refer, e.g., to* [13].

To describe the state of the atmosphere a common thermodynamic quantity used is the potential temperature

$$\theta = T\left(\frac{p_0}{p}\right)^{\frac{\gamma-1}{\gamma}}, \tag{1.4}$$

which is conserved during isentropic motions. Here and in the following the isentropic exponent γ denotes the ratio of heat capacities $\gamma = c_p/c_V$ satisfying

$$\frac{\gamma-1}{\gamma} = \frac{R}{c_p}.$$

Then θ satisfies

$$\frac{D\theta}{Dt} = \frac{\theta}{T}(S_T + \mathcal{D}^T T). \tag{1.5}$$

In the case of moisture being present typically the water vapor mixing ratio, defined as the ratio of the density of ρ_v over the density of dry air ρ_d,

$$q_v = \frac{\rho_v}{\rho_d}, \tag{1.6}$$

is used for a measure of quantification. If saturation effects occur, then water is also present in liquid form as cloud water and rain water which are represented by the additional moisture quantities

$$q_c = \frac{\rho_c}{\rho_d}, \qquad q_r = \frac{\rho_r}{\rho_d}. \tag{1.7}$$

We focus here on warm clouds, where water is present only in gaseous and liquid form, i.e., no ice or snow phases occur.

For these mixing ratios for water vapor, cloud water and rain water we have the following moisture balances

$$\frac{Dq_v}{Dt} = S_{ev} - S_{cd} + \mathcal{D}^{q_v} q_v, \tag{1.8}$$

$$\frac{Dq_c}{Dt} = S_{cd} - S_{ac} - S_{cr} + \mathcal{D}^{q_c} q_c, \tag{1.9}$$

$$\frac{Dq_r}{Dt} - \frac{V}{g\rho}\partial_z(\rho q_r) = S_{ac} + S_{cr} - S_{ev} + \mathcal{D}^{q_r} q_r, \tag{1.10}$$

where $S_{ev}, S_{cd}, S_{ac}, S_{cr}$ are the rates of evaporation of rain water, the condensation of water vapor to cloud water and the inverse evaporation process, the auto-conversion of cloud water into rainwater by accumulation of microscopic droplets, and the collection of cloud water by falling rain. Moreover V denotes the terminal velocity of falling rain and is assumed to be constant.

The key quantity to appear in the following explicit expressions for the source terms is the saturation mixing ratio

$$q_{vs} = \frac{\rho_{vs}}{\rho_d}, \tag{1.11}$$

which gives the threshold for saturation as follows

- undersaturated region: $q_v < q_{vs}$,
- saturated region: $q_v = q_{vs}$,
- oversaturated region: $q_v > q_{vs}$.

Denoting $E = R/R_v$ as the ratio of the individual gas constants for dry air and water vapor, the saturation vapor mixing ratio satisfies

$$q_{vs}(p,T) = \frac{Ee_s(T)}{p - e_s(T)}, \tag{1.12}$$

with the saturation vapor pressure e_s as a function of T being defined by the Clausius-Clapeyron equation:

$$e_s(T) = e_s(T_0)\exp\left(\frac{L}{R_v}\left(\frac{1}{T_0} - \frac{1}{T}\right)\right), \tag{1.13}$$

where the latent heat per unit mass of water vapor L is here and in the following assumed to be constant, which in general varies slightly with temperature. From this formula it is obvious that e_s increases in T, thereby quantifying the fact that the warmer the air is, the more moisture it can carry. Only positive temperature values are meaningful. Typically, the reference temperature $T_0 = 273.15K$ is used.

The Clausius-Clapeyron equation is only meaningful for temperature ranges found in the troposphere, thus in particular as e_s vanishes with decreasing values in T, we shall pose in the following the natural assumption

$$e_s(T) = 0 \quad \text{and} \quad q_{vs}(p,T) = 0, \quad \text{for } T \leq \underline{T}, \tag{1.14}$$

for some constant $\underline{T} \geq 0\,K$, which will also be helpful for proving nonnegativity of the solutions. Moreover the saturation mixing ratio q_{vs} is assumed to be nonnegative and bounded, that is

$$0 \leq q_{vs}(p,T) \leq q_{vs}^*, \tag{1.15}$$

for a positive constant q_{vs}^*. For deriving uniqueness of the solutions we need additionally the Lipschitz continuity of q_{vs}, i.e., we assume

$$|q_{vs}(p,T_1) - q_{vs}(p,T_2)| \leq C|T_1 - T_2|, \tag{1.16}$$

where C is a positive constant.

1.2. **Explicit expressions for the source terms.** Condensation sets free enormous amounts of latent heat, which causes a warming of the surrounding environment, and therefore is a source in the thermodynamic equation. Evaporation on the other hand enters as a sink displaying its cooling effect. The temperature source term accounting for the effects due to latent heat is therefore given by

$$S_T = \frac{L}{c_p}(S_{cd} - S_{ev}), \tag{1.17}$$

see e.g. [13, 21]. All other non-moisture related diabatic sources for potential temperature like e.g. radiation effects are not being considered here. For the source terms of the mixing ratios we take over the setting of Klein and Majda [21] corresponding to a bulk microphysics closure in the spirit of Kessler [22] and Grabowski and Smolarkiewicz [15]:

$$S_{ev} = C_{ev}T(q_r^+)^\beta (q_{vs} - q_v)^+ , \qquad \beta \in (0,1], \tag{1.18}$$

$$S_{cr} = C_{cr} q_c q_r \tag{1.19}$$

$$S_{ac} = C_{ac}(q_c - q_{ac}^*)^+, \tag{1.20}$$

where C_{ev}, C_{cr}, C_{ac} are dimensionless rate constants. Moreover $(g)^+ = \max\{0, g\}$ and the constant $q_{ac}^* \geq 0$ denotes the threshold for cloud water mixing ratio beyond which autoconversion of cloud water into precipitation becomes active. The exponent β in the evaporation term S_{ev} in the literature typically appears to be chosen as $\beta \approx 0.5$, see e.g. [21, 15] and references therein.

As we will see below, exponents $\beta \in (0,1)$ cause difficulties in the analysis, in particular for the uniqueness of the solutions. In a forthcoming paper we will also demonstrate how in the case $\beta = 1$, which is used in the modelling of precipitating clouds e.g. in [11, 19, 28], existence and uniqueness of solutions can be proven without diffusivity imposing higher regularity on the initial data.

We shall use the closure of the condensation term in a similar fashion to [21]

$$S_{cd} = C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ , \tag{1.21}$$

which is in the literature often defined implicitely via the equation of water vapor at saturation, see e.g. [15].

1.3. **The dynamics in pressure coordinates.** Since the density of air varies strongly throughout the troposphere, the incompressibility assumption is only justified when describing shallow phenomena, and thus in general the full compressible governing equations need to be considered. However, under the assumption of hydrostatic balance

$$\frac{\partial p}{\partial z} = -g\rho \, ,$$

which in particular guarantees the pressure to decrease monotonically in height, the pressure can be used as the vertical coordinate. This has the main advantage that the continuity equation takes the form of the incompressibility condition

$$\partial_x u + \partial_y v + \partial_p \omega = 0 \qquad \text{where} \quad \omega = \frac{dp}{dt} \, , \tag{1.22}$$

see Lions et al. [26] and Petcu et al. [30].

We therefore reformulate the dynamics in pressure coordinates

$$(x, y, p)$$

using hereafter the notation

$$\mathbf{v}_h = (u, v) \, , \qquad \nabla_h = (\partial_x, \partial_y) \, , \qquad \Delta_h = \partial_x^2 + \partial_y^2 \, .$$

Due to the different units of the horizontal and the vertical derivatives in pressure coordinates, it is inappropriate to combine them into a single gradient operator. The same holds for the velocity components having different units in vertical and horizontal directions. Nevertheless the total derivative in pressure coordinates reads

$$\frac{D}{Dt} = \partial_t + \mathbf{v}_h \cdot \nabla_h + \omega \partial_p \,. \tag{1.23}$$

For the closure of the turbulent and molecular transport terms we use

$$\mathcal{D}^* = \mu_* \Delta_h + \nu_* \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p \right) \tag{1.24}$$

where $\bar{T} = \bar{T}(p)$ corresponds to some background distribution being uniformly bounded from above and from below away from 0. The operator $\mathcal{D}^*$ thereby provides a close approximation to the full Laplacian in cartesian coordinates, see also [26, 30]. This linearization around the reference profile $\bar{T}$ is also applied to the vertical transport term of q_r after replacing the density using the ideal gas law, such that the moisture equations in pressure coordinates (x, y, p) with corresponding velocities (u, v, ω) become

$$\frac{Dq_v}{Dt} = S_{ev} - S_{cd} + \mathcal{D}^{q_v} q_v \,, \tag{1.25}$$

$$\frac{Dq_c}{Dt} = S_{cd} - S_{ac} - S_{cr} + \mathcal{D}^{q_c} q_c \,, \tag{1.26}$$

$$\frac{Dq_r}{Dt} + V \partial_p \left(\frac{p}{\bar{T}} q_r \right) = S_{ac} + S_{cr} - S_{ev} + \mathcal{D}^{q_r} q_r \,, \tag{1.27}$$

where according to (1.23)

$$\frac{Dq_j}{Dt} = \partial_t q_j + u \partial_x q_j + v \partial_y q_j + \omega \partial_p q_j \tag{1.28}$$

and according to (1.24)

$$\mathcal{D}^{q_j} q_j = \mu_{q_j} (\partial_x^2 q_j + \partial_y^2 q_j) + \nu_{q_j} \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p q_j \right) \,. \tag{1.29}$$

The closure of the source terms thereby remains unchanged. The temperature equation in pressure coordinates reads

$$\frac{DT}{Dt} - \frac{RT}{c_p p} \omega = \frac{L}{c_p} (S_{cd} - S_{ev}) + \mathcal{D}^T T \,, \tag{1.30}$$

which can again be reformulated in terms of the potential temperature as follows

$$\frac{D\theta}{Dt} = \frac{\theta}{T} \left(S_T + \mathcal{D}^T T \right) = \left(\frac{p_0}{p} \right)^{\frac{R}{c_p}} \frac{L}{c_p} (S_{cd} - S_{ev}) + \widetilde{\mathcal{D}}^\theta \theta \,, \tag{1.31}$$

where the diffusion in vertical direction however deviates due to the additional pressure function arising when replacing T in terms of θ according to (1.4)

$$\widetilde{\mathcal{D}}^\theta \theta = \mu_T \Delta_h \theta + \nu_T \left(\frac{p_0}{p} \right)^{R/c_p} \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p \left(\left(\frac{p}{p_0} \right)^{R/c_p} \theta \right) \right) \,. \tag{1.32}$$

2. Formulation of the problem and main result

As in Coti Zelati et al. [8] we let $\mathcal{M}$ be a cylinder of the form

$$\mathcal{M} = \{(x, y, p) : (x, y) \in \mathcal{M}', p \in (p_1, p_0)\}, \tag{2.1}$$

where $\mathcal{M}'$ is a smooth bounded domain in $\mathbb{R}^2$ and $p_0 > p_1 > 0$. The boundary is given by

$$\Gamma_0 = \{(x, y, p) \in \overline{\mathcal{M}} : p = p_0\}, \tag{2.2}$$

$$\Gamma_1 = \{(x, y, p) \in \overline{\mathcal{M}} : p = p_1\}, \tag{2.3}$$

$$\Gamma_\ell = \{(x, y, p) \in \overline{\mathcal{M}} : (x, y) \in \partial\mathcal{M}', p_0 \geq p \geq p_1\}. \tag{2.4}$$

The boundary conditions read

$$\Gamma_0 : \ \partial_p T = \alpha_{0T}(T_{b0} - T), \qquad \partial_p q_j = \alpha_{0j}(q_{b0j} - q_j), \quad j \in \{v, c, r\}, \tag{2.5}$$

$$\Gamma_1 : \ \partial_p T = 0, \qquad \partial_p q_j = 0, \quad j \in \{v, c, r\}, \tag{2.6}$$

$$\Gamma_\ell : \ \partial_n T = \alpha_{\ell T}(T_{b\ell} - T), \qquad \partial_n q_j = \alpha_{\ell j}(q_{b\ell j} - q_j), \quad j \in \{v, c, r\}, \tag{2.7}$$

where the multipliers $\alpha_{0T}, \alpha_{0j}, \alpha_{\ell T}, \alpha_{\ell j}$ are nonnegative quantities, which are usually constants, but for the sake of generality are allowed to vary on the corresponding boundaries and also with time. The functions $T_{b0}, q_{b0j}, T_{b\ell}, q_{b\ell j}$ are typical temperature and moisture profiles, which are again defined on the corresponding boundaries and may vary additionally in time. Note that the special case of $\alpha_{\ell T} = \alpha_{\ell v} = \alpha_{\ell c} = \alpha_{\ell r} = 0$ in (2.5)–(2.7) reduces to a similar setting used for the analysis of the moisture model in [8]. We assume all given functions $\alpha_{0j}, \alpha_{\ell j}, \alpha_{0T}, \alpha_{\ell T}$ as well as $T_{b0}, T_{b\ell}, q_{b0j}, q_{b\ell j}$ to be nonnegative, sufficiently smooth and uniformly bounded, say be in C^1. We assume the velocity field to be given and to satisfy

$$u, v, \omega \in L^\infty(0, \mathcal{T}; L^2(\mathcal{M})) \cap L^2(0, \mathcal{T}; H^1(\mathcal{M})) \cap L^r(0, \mathcal{T}; L^\sigma(\mathcal{M})), \tag{2.8}$$

for any finite time $\mathcal{T}$, and for some $r \in [2, \infty]$ and $\sigma \in [3, \infty]$ with

$$\frac{2}{r} + \frac{3}{\sigma} < 1, \tag{2.9}$$

satsifying the well-known Prodi-Serrin regularity condition [33], [29]. Moreover we assume mass conservation, which reduces in pressure coordinates as mentioned above to

$$\nabla_h \cdot \mathbf{v}_h + \partial_p \omega = 0 \qquad \text{in } \ \mathcal{M}, \tag{2.10}$$

as well as the no-flux conditions on the boundary of the domain

$$\mathbf{v}_h \cdot \mathbf{n}_h + \omega\, n_p = 0 \qquad \text{on } \ \partial\mathcal{M}. \tag{2.11}$$

Remark 2.1. *Typically the vertical velocity component ω is determined via the divergence constraint in* (2.10)*, which results in different regularity properties for ω in comparison to the vertical velocity comonents u, v. For the primitive equations it was proven that the latter satisfy additionally $u, v \in L^2(0, \mathcal{T}; H^2(\mathcal{M}))$, see* [7].

In the following we use the abbreviation

$$\|f\| = \|f\|_{L^2(\mathcal{M})}\,, \qquad \|f\|_{L^p} = \|f\|_{L^p(\mathcal{M})}\,. \tag{2.12}$$

According to the weight in the vertical diffusion terms, we also introduce the weighted norms

$$\|f\|_w = \left\| \left(\frac{gp}{R\bar{T}}\right) f \right\|, \qquad \|f\|^2_{H^1_w} = \|f\|^2 + \|\nabla_h f\|^2 + \|\partial_p f\|^2_w\,, \tag{2.13}$$

where we emphasize, that since the weight $\frac{gR}{p\bar{T}}$ is uniformly bounded from above and below by positive constants, the $H^1_w(\mathcal{M})$-norm is equivalent to the $H^1(\mathcal{M})$-norm.

We summarize the results obtained below in the following theorem:

Theorem 2.1. *Let the initial data* $(T_0, q_{v0}, q_{c0},\, q_{r0}) \in (L^\infty(\mathcal{M}))^4 \cap (H^1(\mathcal{M}))^4$ *be nonnegative and let the given velocity field* $(\mathbf{v}_h, \omega)$ *satisfy the above assumptions* (2.8)–(2.11). *Then, for any (arbitrarily large)* $\mathcal{T} \in (0, \infty)$, *there exists a unique nonnegative solution* (T, q_v, q_c, q_r), *with initial value* $(T_0, q_{v0}, q_{c0}, q_{r0})$, *of the boundary value problem* (1.25)–(1.30) *subject to* (2.5)–(2.7), *on* $\mathcal{M} \times (0, \mathcal{T})$, *satisfying*

$$(T, q_v, q_c, q_r) \in L^\infty((0, \mathcal{T}) \times \mathcal{M}) \cap L^2(0, \mathcal{T}; H^2(\mathcal{M})), \tag{2.14}$$

$$(T, q_v, q_c, q_r) \in C([0, \mathcal{T}]; H^1(\mathcal{M})), \quad (\partial_t T, \partial_t q_v, \partial_t q_c, \partial_t q_r) \in L^2(0, \mathcal{T}; L^2(\mathcal{M})). \tag{2.15}$$

3. Local well-posedness and a priori estimates

In this section we prove the local existence and some a priori estimates of strong solutions to system (1.25)–(1.30), or equivalently system (1.25)–(1.27) with (1.31), subject to the boundary conditions (2.5)–(2.7).

Local existence of strong solutions is stated in the following proposition.

Proposition 3.1. *Let the initial data* $(T_0, q_{v0}, q_{c0},\, q_{r0}) \in (H^1(\mathcal{M}))^4$ *be nonnegative and let the given velocity field* $(\mathbf{v}_h, \omega)$ *satisfy the above assumptions* (2.8)–(2.11). *Then there exists a unique local solution* (q_v, q_c, q_r, θ), *which depends continuously on the initial data, in some short time interval* $(0, \mathcal{T}_0)$, *to system* (1.25)–(1.27) *with (1.31), subject to* (2.5)–(2.7), *with initial data* $(q_{v0}, q_{c0}, q_{r0}, \theta_0)$, *satisfying*

$$(q_v, q_c, q_r, \theta) \in C([0, \mathcal{T}_0]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}_0; H^2(\mathcal{M})), \tag{3.1}$$

$$(\partial_t q_v, \partial_t q_c, \partial_t q_r, \partial_t \theta) \in L^2(0, \mathcal{T}_0; L^2(\mathcal{M})). \tag{3.2}$$

Proof. The well-posedness, i.e., the uniqueness and continuous dependence on initial data, will be postponed to section 4, below. Therefore, we only prove the existence part here. We are going to prove the local existence by iteration procedure, that is we construct a sequence of vector fields $\{(q_v^n, q_c^n, q_r^n, \theta^n)\}_{n=0}^\infty$, and show that this sequence converges to a vector field (q_v, q_c, q_r, θ), which is the desired strong solution. Let $\mathcal{T}_0 \le 1$ be a positive number to be determined later. For simplicity of notations, we set $\mathbf{U} = (q_v, q_c, q_r, \theta)$, and denote $X_{\mathcal{T}_0} := C([0, \mathcal{T}_0]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}_0; H^2(\mathcal{M}))$.

Set $\mathbf{U}^0 = 0$, and define $\mathbf{U}^{n+1} = (q_v^{n+1}, q_c^{n+1}, q_r^{n+1}, \theta^{n+1})$, $n = 0, 1, \dots$, to be the unique strong solution to the linear parabolic system

$$\partial_t q_v^{n+1} - \mathcal{D}^{q_v} q_v^{n+1} = S_{q_v}^n - \mathbf{v}_h \cdot \nabla_h q_v^n - \omega \partial_p q_v^n, \tag{3.3}$$

$$\partial_t q_c^{n+1} - \mathcal{D}^{q_c} q_c^{n+1} = S_{q_c}^n - \mathbf{v}_h \cdot \nabla_h q_c^n - \omega \partial_p q_c^n, \tag{3.4}$$

$$\partial_t q_r^{n+1} - \mathcal{D}^{q_r} q_r^{n+1} = S_{q_r}^n - \mathbf{v}_h \cdot \nabla_h q_r^n - \omega \partial_p q_r^n - V \partial_p \left(\frac{p}{\overline{T}} q_r^n \right), \tag{3.5}$$

$$\partial_t \theta^{n+1} - \tilde{\mathcal{D}}^\theta \theta^{n+1} = S_\theta^n - \mathbf{v}_h \cdot \nabla_h \theta^n - \omega \partial_p \theta^n, \tag{3.6}$$

subject to the boundary conditions (2.5)–(2.7) and the initial condition

$$(q_v^{n+1}, q_c^{n+1}, q_r^{n+1}, \theta^{n+1})|_{t=0} = (q_{v0}, q_{c0}, q_{r0}, \theta_0). \tag{3.7}$$

Here, $S_{q_v}^n, S_{q_c}^n, S_{q_r}^n$ and S_θ^n are the source terms, respectively, expressed by $S_{ev}^n - S_{cd}^n$, $S_{cd}^n - S_{ac}^n - S_{cr}^n$, $S_{ac}^n + S_{cr}^n - S_{ev}^n$ and $(\frac{p_0}{p})^{\frac{R}{c_p}} \frac{L}{c_p}(S_{cd}^n - S_{ev}^n)$, with $S_{ev}^n, S_{cr}^n, S_{ac}^n$ and S_{cd}^n given by (1.18)–(1.21), by replacing (q_v, q_c, q_r, θ) with $(q_v^n, q_c^n, q_r^n, \theta^n)$ there.

For simplicity of notations, we denote the vector field of source term $\mathbf{S}_U^n$ as

$$\mathbf{S}_U^n = (S_{q_v}^n, S_{q_c}^n, S_{q_r}^n, S_\theta^n). \tag{3.8}$$

Recalling the expressions of S_{ev}, S_{cr}, S_{ac} and S_{cd} in (1.18)–(1.21) and the regularity assumptions on $(\mathbf{v}_h, \omega)$ in (2.8), one can check (see the calculations (3.10)–(3.11), below) by the Hölder and Sobolev embedding inequalities that $\mathbf{S}_U^n, (\mathbf{v}_h \cdot \nabla_h \mathbf{U}^n + \omega \partial_p \mathbf{U}^n) \in L^2(Q_{\mathcal{T}_0})$, for any $\mathbf{U}^n \in X_{\mathcal{T}_0}$, where $Q_{\mathcal{T}_0} = \mathcal{M} \times (0, \mathcal{T}_0)$. Thus, by Corollary A.1 in the appendix, below, $(q_v^{n+1}, q_c^{n+1}, q_r^{n+1}, \theta^{n+1})$ is well-defined and satisfies

$$\begin{aligned} \|\mathbf{U}^{n+1}\|_{X_{\mathcal{T}_0}} + \|\partial_t \mathbf{U}^{n+1}\|_{L^2(Q_{\mathcal{T}_0})} \leq{}& C\bigg(\|\mathbf{S}_U^n\|_{L^2(Q_{\mathcal{T}_0})} + \|\mathbf{v}_h \cdot \nabla_h \mathbf{U}^n\|_{L^2(Q_{\mathcal{T}_0})} \\ & + \|\omega \partial_p \mathbf{U}^n\|_{L^2(Q_{\mathcal{T}_0})} + \left\| \partial_p \left(\frac{p}{\overline{T}} q_r^n \right) \right\|_{L^2(Q_{\mathcal{T}_0})} + 1 \bigg), \end{aligned} \tag{3.9}$$

for a positive constant C, which is independent of n and $\mathcal{T}_0$, while $\mathcal{T}_0 \in (0, 1]$.

Noticing that $|\mathbf{S}_U^n| \leq C(|\mathbf{U}^n| + |\mathbf{U}^n|^3)$, it follows from the Young and Sobolev embedding inequalities that

$$\begin{aligned} \|\mathbf{S}_U^n\|_{L^2(Q_{\mathcal{T}_0})} \leq{}& C(\|\mathbf{U}^n\|_{L^2(Q_{\mathcal{T}_0})} + \|\mathbf{U}^n\|_{L^6(Q_{\mathcal{T}_0})}^3) \\ \leq{}& C(\|\mathbf{U}^n\|_{L^2(0,\mathcal{T}_0;L^2(\mathcal{M}))} + \|\mathbf{U}^n\|_{L^6(0,\mathcal{T}_0;H^1(\mathcal{M}))}^3) \\ \leq{}& C\sqrt{\mathcal{T}_0}(\|\mathbf{U}^n\|_{L^\infty(0,\mathcal{T}_0;L^2(\mathcal{M}))} + \|\mathbf{U}^n\|_{L^\infty(0,\mathcal{T}_0;H^1(\mathcal{M}))}^3) \\ \leq{}& C\sqrt{\mathcal{T}_0}(1 + \|\mathbf{U}^n\|_{X_{\mathcal{T}_0}}^3), \end{aligned} \tag{3.10}$$

for a constant C, which is independent of n and $\mathcal{T}_0$, while $\mathcal{T}_0 \in (0, 1]$. Recalling the regularity assumption in (2.8), it follows from the Hölder and Sobolev inequalities that

$$\begin{aligned} &\|\mathbf{v}_h \cdot \nabla_h \mathbf{U}^n\|_{L^2(Q_{\mathcal{T}_0})} \leq \|\mathbf{v}_h\|_{L^{\frac{2\sigma}{\sigma-3}}(0,\mathcal{T}_0;L^\sigma(\mathcal{M}))} \|\nabla_h \mathbf{U}^n\|_{L^{\frac{2\sigma}{3}}(0,\mathcal{T}_0;L^{\frac{2\sigma}{\sigma-2}}(\mathcal{M}))} \\ \leq{}& C\mathcal{T}_0^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}} \|\mathbf{v}_h\|_{L^r(0,\mathcal{T}_0;L^\sigma(\mathcal{M}))} \|\nabla_h \mathbf{U}^n\|_{L^\infty(0,\mathcal{T}_0;L^2(\mathcal{M}))}^{1-\frac{3}{\sigma}} \|\nabla_h \mathbf{U}^n\|_{L^2(0,\mathcal{T}_0;L^6(\mathcal{M}))}^{\frac{3}{\sigma}} \\ \leq{}& C\mathcal{T}_0^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}} \|\mathbf{v}_h\|_{L^r(0,\mathcal{T}_0;L^\sigma(\mathcal{M}))} \|\mathbf{U}^n\|_{L^\infty(0,\mathcal{T}_0;H^1(\mathcal{M}))}^{1-\frac{3}{\sigma}} \|\mathbf{U}^n\|_{L^2(0,\mathcal{T}_0;H^2(\mathcal{M}))}^{\frac{3}{\sigma}} \end{aligned}$$

$$\leq C\mathcal{T}_0^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}}\|\mathbf{v}_h\|_{L^r(0,\mathcal{T};L^\sigma(\mathcal{M}))}\|\mathbf{U}^n\|_{X_{\mathcal{T}_0}} \leq C\mathcal{T}_0^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}}\|\mathbf{U}^n\|_{X_{\mathcal{T}_0}}, \tag{3.11}$$

for a positive constant C, which is independent of n and $\mathcal{T}_0$, while $\mathcal{T}_0 \in (0,1]$; with an analogous estimate that can be established for $\|\omega\partial_p\mathbf{U}^n\|_{L^2(Q_{\mathcal{T}_0})}$. Moreover,

$$\left\|\partial_p\left(\frac{p}{\overline{T}}q_r^n\right)\right\|_{L^2(Q_{\mathcal{T}_0})} \leq C\|q_r^n\|_{L^2(0,\mathcal{T}_0;H^1(\mathcal{M}))} \leq C\sqrt{\mathcal{T}}_0\|q_r^n\|_{X_{\mathcal{T}_0}}, \tag{3.12}$$

for a positive constant C, which is independent of n and $\mathcal{T}_0$, while $\mathcal{T}_0 \in (0,1]$.

Therefore, we have

$$\begin{aligned}\|\mathbf{U}^{n+1}\|_{X_{\mathcal{T}_0}} + \|\partial_t\mathbf{U}^{n+1}\|_{L^2(Q_{\mathcal{T}_0})} &\leq C[\mathcal{T}_0^\delta(\|\mathbf{U}^n\|_{X_{\mathcal{T}_0}} + \|\mathbf{U}^n\|_{X_{\mathcal{T}_0}}^3) + 1]\\ &\leq C_1(\mathcal{T}_0^\delta\|\mathbf{U}^n\|_{X_{\mathcal{T}_0}}^3 + 1),\end{aligned} \tag{3.13}$$

for a positive constant C_1, which is independent of n and $\mathcal{T}_0$, while $\mathcal{T}_0 \in (0,1]$; and $\delta = \min\{\frac{1}{2}, \frac{\sigma-3}{2\sigma} - \frac{1}{r}\} = \frac{\sigma-3}{2\sigma} - \frac{1}{r} > 0$. Set $M = 2C_1$ and $\mathcal{T}_0 = \min\left\{1, (2C_1)^{-\frac{3}{\delta}}\right\}$. Then, thanks to (3.13), and recalling that $\mathbf{U}^0 = 0$, one can easily show by induction that

$$\|\mathbf{U}^{n+1}\|_{X_{\mathcal{T}_0}} + \|\partial_t\mathbf{U}^{n+1}\|_{L^2(Q_{\mathcal{T}_0})} \leq M, \quad n = 0,1,2,\ldots. \tag{3.14}$$

Next, we show that $\{U^n\}_{n=1}^\infty$ is a Cauchy sequence in the space $C([0,\mathcal{T}_0^*];L^2(\mathcal{M}))$, for a positive number $\mathcal{T}_0^* \in (0,\mathcal{T}_0)$. As we have pointed out in the Introduction, if the exponent in the evaporation term S_{ev}, $\beta \in (0,1)$, we need to introduce the following new unknowns

$$Q^n = q_v^n + q_r^n, \quad H^n = T^n - \frac{L}{c_p}(q_c^n + q_r^n), \quad n = 0,1,2,\cdots.$$

To show that $\{U^n\}_{n=1}^\infty$ is a Cauchy sequence in the space $L^\infty(0,\mathcal{T}_0^*;L^2(\mathcal{M}))$, instead of carrying out the L^2-estimate for the difference $U^{n+1} - U^n$, we will perform the corresponding estimate for $(Q^{n+1} - Q^n, q_c^{n+1} - q_c^n, q_r^{n+1} - q_r^n, H^{n+1} - H^n)$.

To simplify the notation, we set

$$\begin{aligned}\alpha_n(t) &= 1 + \|q_c^{n-1}\|_{L^\infty}^2 + \|q_c^n\|_{L^\infty}^2 + \|q_r^{n-1}\|_{L^\infty}^2 + \|q_r^n\|_{L^\infty}^2 + \|q_v^n\|_{L^\infty}^2 + \|\theta^n\|_{L^\infty}^2,\\ \phi_n(t) &= \|Q^n - Q^{n-1}\|^2 + \|q_r^n - q_r^{n-1}\|^2 + \|q_c^n - q_c^{n-1}\|^2 + \|H^n - H^{n-1}\|^2,\end{aligned}$$

for $n = 1,2,\cdots$. Then, thanks to (3.14), and using the Gagliardo-Nirenberg inequality, $\|f\|_{L^\infty(\mathbb{R}^3)} \leq C\|f\|_{L^2(\mathbb{R}^3)}^{\frac{1}{4}}\|f\|_{H^2(\mathbb{R}^3)}^{\frac{3}{4}}$, and the Hölder inequality, one deduces

$$\begin{aligned}\int_0^{\mathcal{T}_0^*}\alpha_n(t)dt &\leq \mathcal{T}_0^* + C\sum_{f\in\{q_c^{n-1},q_c^n,q_r^{n-1},q_r^n,q_v^n,\theta^n\}}\int_0^{\mathcal{T}_0^*}\|f\|_{L^2}^{\frac{1}{2}}\|f\|_{H^2}^{\frac{3}{2}}dt\\ &\leq \mathcal{T}_0^* + C(\mathcal{T}_0^*)^{\frac{1}{4}}M^{\frac{1}{2}}\sum_{f\in\{q_c^{n-1},q_c^n,q_r^{n-1},q_r^n,q_v^n,\theta^n\}}\left(\int_0^{\mathcal{T}_0^*}\|f\|_{H^2}^2dt\right)^{\frac{3}{4}}\\ &\leq \mathcal{T}_0^* + CM^2(\mathcal{T}_0^*)^{\frac{1}{4}}.\end{aligned} \tag{3.15}$$

We now perform the L^2-estimate for $(Q^{n+1}-Q^n, q_c^{n+1}-q_c^n, q_r^{n+1}-q_r^n, H^{n+1}-H^n)$. Since the proof is similar to that of showing the uniqueness in Proposition 4.1, below, we only sketch the proof here. First, for the estimate of $Q^{n+1} - Q^n$, following the same arguments for deriving (4.13), one ends up with an energy inequality, which is the same as (4.13), but replacing $Q_1 - Q_2$ and $q_{r1} - q_{r2}$ by $Q^{n+1} - Q^n$ and $q_r^{n+1} - q_r^n$, respectively, on the left and the first line on the right of (4.13), and replacing $q_{r1}, q_{r2}, Q_1, Q_2, q_{c1}, q_{c2}, q_{v1}, q_{v2}, T_1, T_2$ by $q_r^n, q_r^{n-1}, Q^n, Q^{n-1}$, q_c^n, q_c^{n-1}, $q_v^n, q_v^{n-1}, T^n, T^{n-1}$, respectively, in the other terms on the right of (4.13). Then, by using (4.14), below, as well as the Young inequality, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|Q^{n+1} - Q^n\|^2 + \frac{\mu_{q_c}}{4}\|\nabla_h(Q^{n+1} - Q^n)\|^2 + \frac{\nu_{q_v}}{4}\|\partial_p(Q^{n+1} - Q^n)\|_w^2 \\ \leq\ & C_Q(\mu_{q_r}\|q_r^{n+1} - q_r^n\|^2 + \nu_{q_r}\|\partial_p(q_r^{n+1} - q_r^n)\|_w^2) + C[\phi_{n+1}(t) + \alpha_n(t)\phi_n(t)]. \end{aligned} \tag{3.16}$$

Next, for the estimate of $q_r^{n+1} - q_r^n$, similar as before, following the same arguments for deriving (4.17), one obtains a similar energy inequality as (4.17), from which, by using the monotonicity in the evaporation term $[(q_r^n)^\beta - (q_r^{n-1})^\beta](q_r^n - q_r^{n-1}) \geq 0$, as well as the Young inequality, one has the following inequality

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|q_r^{n+1} - q_r^n\|^2 + \mu_{q_r}\|\nabla_h(q_r^{n+1} - q_r^n)\|^2 + \frac{\mu_{q_r}}{2}\|\partial_p(q_r^{n+1} - q_r^n)\|_w^2 \\ &\leq C[\phi_{n+1}(t) + \alpha_n(t)\phi_n(t)]. \end{aligned} \tag{3.17}$$

And finally, similar arguments as for (4.19) and (4.21) yield the estimates for $q_c^{n+1} - q_c^n$ and $H^{n+1} - H^n$ as follows

$$\frac{1}{2}\frac{d}{dt}\|q_c^{n+1} - q_c^n\|^2 + \mu_{q_c}\|\nabla_h(q_c^{n+1} - q_c^n)\|^2 + \nu_{q_c}\|\partial_p(q_c^{n+1} - q_c^n\|_w^2 \leq C\alpha_n(t)\phi_n(t), \tag{3.18}$$

and

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|H^{n+1} - H^n\|^2 + \frac{\mu_T}{2}\|\nabla_h(H^{n+1} - H^n)\|^2 + \frac{\nu_T}{2}\|\partial_p(H^{n+1} - H^n)\|_w^2 \\ \leq\ & C_H\Big(\mu_{q_c}\|\nabla_h(q_c^{n+1} - q_c^n)\|^2 + \nu_{q_c}\|\partial_p(q_c^{n+1} - q_c^n)\|_w^2 + \mu_{q_r}\|\nabla_h(q_r^{n+1} - q_r^n)\|^2 \\ & + \nu_{q_r}\|\partial_p(q_r^{n+1} - q_r^n)\|_w^2\Big) + C[\phi_{n+1}(t) + \alpha_n(t)\phi_n(t)]. \end{aligned} \tag{3.19}$$

Set

$$J_n(t) = \frac{1}{2C_Q}\|Q^n - Q^{n-1}\|^2 + \|q_r^n - q_r^{n-1}\|^2 + \|q_c^n - q_c^{n-1}\|^2 + \frac{1}{2C_H}\|H^n - H^{n-1}\|^2.$$

Noticing that $\phi_n(t) \leq CJ_n(t)$, it follows from (3.16)–(3.19) that

$$\frac{d}{dt}J_{n+1}(t) \leq C[J_{n+1}(t) + \alpha_n(t)J_n(t)].$$

Since $J_{n+1}(0) = 0$ it follows from the above, by virtue the Gronwall inequality, that

$$J_{n+1}(t) \leq e^{Ct}\int_0^t \alpha_n(s)J_n(s)ds, \quad \text{for all} \quad t \in [0, \mathcal{T}_0^*] \subset [0, \mathcal{T}_0].$$

Thanks to the above, recalling (3.15), and choosing $\mathcal{T}_0^*$ small enough, we have

$$\sup_{0\leq t\leq\mathcal{T}_0^*} J_{n+1}(t) \leq e^{C\mathcal{T}_0^*}\left(\int_0^{\mathcal{T}_0^*}\alpha_n(t)dt\right)\sup_{0\leq t\leq\mathcal{T}_0^*} J_n(t) \leq \frac{1}{2}\sup_{0\leq t\leq\mathcal{T}_0^*} J_n(t),$$

and thus

$$\sup_{0\leq t\leq\mathcal{T}_0^*} J_{n+1}(t) \leq \frac{C}{2^n}.$$

Thanks to the above estimate, it is clear that $\{(Q^n, q_c^n, q_r^n, H^n)\}_{n=1}^\infty$ is a Cauchy sequence in $L^\infty(0, \mathcal{T}_0^*; L^2(\mathcal{M}))$, and consequently $\{U^n\}_{n=1}^\infty$ is a Cauchy sequence in the same space.

By virtue of this fact, and recalling the a priori estimate (3.14), by the Aubin-Lions lemma, there is a vector field $\mathbf{U} \in X_{\mathcal{T}_0^*}$, with $\partial_t \mathbf{U} \in L^2(Q_{\mathcal{T}_0^*})$, such that

$$\mathbf{U}^n \rightharpoonup \mathbf{U}, \quad \text{in } L^2(0, \mathcal{T}_0^*; H^2(\mathcal{M})), \tag{3.20}$$

$$\partial_t \mathbf{U}^n \rightharpoonup \partial_t \mathbf{U}, \quad \text{in } L^2(Q_{\mathcal{T}_0^*}), \tag{3.21}$$

$$\mathbf{U}^n \to \mathbf{U}, \quad \text{in } L^2(0, \mathcal{T}_0^*; H^1(\mathcal{M})) \cap C([0, \mathcal{T}_0^*]; L^2(\mathcal{M})). \tag{3.22}$$

We point out that the above convergence holds for the whole sequence $\{U^n\}_{n=1}^\infty$, rather than only for a subsequence.

Using the above convergences, one can take the limit, as $n \to \infty$, in (3.3)–(3.6) to show that $\mathbf{U} = (q_v, q_c, q_r, \theta)$ is a strong solution of the boundary value problem (1.25)–(1.27) with (1.31), subject to (2.5)–(2.7), with initial data $(q_{v0}, q_{c0}, q_{r0}, \theta_0)$. $\square$

In the following proposition we derive nonnegativity and uniform boundedness for the moisture quantities and the temperature. Here the sequence of the derivation of the individual bounds needs to be in the right order to close the estimates consecutively.

Proposition 3.2. *Let $\mathcal{T} \in (0, \infty)$ and (T, q_v, q_c, q_r) be a solution to (1.25)–(1.30) in $\mathcal{M} \times (0, \mathcal{T})$ subject to the boundary conditions (2.5)–(2.7) with non-negative intial data $(T_0, q_{v0}, q_{c0}, q_{r0}) \in (L^\infty(\mathcal{M}))^4 \cap (H^1(\mathcal{M}))^4$, satisfying the regularities stated in Proposition 3.1 by replacing $\mathcal{T}_0$ with $\mathcal{T}$. Then for every $t \in [0, \mathcal{T}]$ the solution $(T, q_v, q_c, q_r)(t)$ satisfies*

$$0 \le q_v \le q_v^*, \quad 0 \le q_c \le q_c^*, \quad 0 \le q_r \le q_r^*, \quad 0 \le T \le T^*, \tag{3.23}$$

where

$$q_v^* = \max\left\{\|q_{v0}\|_{L^\infty(\mathcal{M})}, \|q_{b0v}\|_{L^\infty((0,\mathcal{T})\times\mathcal{M}')}, \|q_{b\ell v}\|_{L^\infty((0,\mathcal{T})\times\Gamma_\ell)}, q_{vs}^*\right\} \tag{3.24}$$

with q_{vs}^* being the constant in (1.15), and q_c^*, q_r^*, T^* are constants depending on the following quantities:

$$q_c^* = C_{q_c}\big(\mathcal{T}, \|q_{c0}\|_{L^\infty(\mathcal{M})}, \|q_{b0c}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell c}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_v^*, q_{vs}^*\big), \tag{3.25}$$

$$q_r^* = C_{q_r}\big(\mathcal{T}, \|q_{r0}\|_{L^\infty(\mathcal{M})}, \|q_{b0r}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|q_{b\ell r}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_c^*\big), \tag{3.26}$$

$$T^* = C_T\big(\mathcal{T}, \|T_0\|_{L^\infty(\mathcal{M})}, \|T_{b0}\|_{L^\infty((0,T)\times\mathcal{M}')}, \|T_{b\ell}\|_{L^\infty((0,T)\times\Gamma_\ell)}, q_v^*, q_c^*, q_{vs}^*\big). \tag{3.27}$$

Proof. (i) *Nonnegativity of q_v, q_c, q_r and θ.* For deriving this first part of the comparison principles we employ the Stampacchia method and therefore test the equations of the mixing ratios with their negative parts, where in the following we use

$$f = f^+ - f^- \tag{3.28}$$

for splitting a function f into its positive and negative parts, with $f^+ = \max\{f, 0\}$ and $f^- = \max\{-f, 0\}$.

For the aim of the later uses, we first carry out some calculations on the integrals over the domain $\mathcal{M}$ of the products of the diffusion and convection terms with q_j^-, $j \in \{v, c, r\}$.

Integration by parts and using the boundary conditions (2.5)–(2.7), one deduces

$$\begin{aligned}
\int_{\mathcal{M}} q_j^- \mathcal{D}^{q_j} q_j d\mathcal{M} &= \int_{\mathcal{M}} \left[\mu_{q_j}\Delta_h q_j + \nu_{q_j}\partial_p\left(\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p q_j\right)\right] q_j^- d\mathcal{M} \\
&= \mu_{q_j}\int_{\Gamma_\ell}(\partial_n q_j)q_j^- d\Gamma_\ell + \nu_{q_j}\int_{\mathcal{M}'}\left(\frac{gp}{R\bar{T}}\right)^2(\partial_p q_j)q_j^- d\mathcal{M}'\Big|_{p_1}^{p_0} \\
&\quad -\int_{\mathcal{M}}\left[\mu_{q_j}\nabla_h q_j\cdot\nabla_h q_j^- + \nu_{q_j}\left(\frac{gp}{R\bar{T}}\right)^2\partial_p q_j\partial_p q_j^-\right]d\mathcal{M} \\
&= \mu_{q_j}\|\nabla_h q_j^-\|^2 + \nu_{q_j}\|\partial_p q_j^-\|_w^2 + \mu_{q_j}\int_{\Gamma_\ell}\alpha_{\ell j}(q_{\ell j}-q_j)q_j^- d\Gamma_\ell \\
&\quad +\nu_{q_j}\int_{\mathcal{M}'}\left(\frac{gp_0}{R\bar{T}}\right)^2\alpha_{b0j}(q_{b0j}-q_j)q_j^- d\mathcal{M}'. \qquad (3.29)
\end{aligned}$$

Since the functions $q_{\ell j}$ and q_{0j} are both nonnegative and $q_jq_j^- = -(q_j^-)^2$ a.e., the last two boundary integrals are nonnegative, and we obtain

$$\int_{\mathcal{M}} q_j^- \mathcal{D}^{q_j} q_j d\mathcal{M} \geq \mu_{q_j}\|\nabla_h q_j^-\|^2 + \nu_{q_j}\|\partial_p q_j^-\|_w^2. \qquad (3.30)$$

The integral containing the advection term vanishes due to (2.10) and (2.11), since

$$\begin{aligned}
&\int_{\mathcal{M}}(\mathbf{v}_h\cdot\nabla_h q_j + \omega\partial_p q_j)q_j^- d\mathcal{M} = -\frac{1}{2}\int_{\mathcal{M}}(\mathbf{v}_h\cdot\nabla_h + \omega\partial_p)(q_j^-)^2 d\mathcal{M} \\
= & -\frac{1}{2}\int_{\partial\mathcal{M}}(\mathbf{v}_h\cdot\mathbf{n}_h + \omega n_p)(q_j^-)^2 d(\partial\mathcal{M}) + \frac{1}{2}\int_{\mathcal{M}}(q_j^-)^2(\nabla_h\cdot\mathbf{v}_h + \partial_p\omega)d\mathcal{M} = 0\,. \qquad (3.31)
\end{aligned}$$

We now proceed with the derivation of the nonnegativity of q_v, q_c, q_r and T, where we start with the cloud water mixing ratio q_c. Multiplying equation (1.26) by $-q_c^-$, recalling (3.30)–(3.31), noticing that $q_c^-(q_c - q_{ac}^*)^+ = 0$ since $q_{ac}^* \geq 0$ and applying the Sobolev embedding inequality, one deduces

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}}(q_c^-)^2 d\mathcal{M} \leq -\int_{\mathcal{M}} q_c^-(S_{cd} - S_{ac} - S_{cr})d\mathcal{M} \\
= & -\int_{\mathcal{M}} q_c^-\big(C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ - C_{ac}(q_c - q_{ac}^*)^+ - C_{cr}q_cq_r\big)d\mathcal{M} \\
\leq & \int_{\mathcal{M}}(C_{cr}q_cq_r - C_{cd}(q_v - q_{vs})q_c)q_c^- d\mathcal{M} = \int_{\mathcal{M}}(C_{cd}(q_v - q_{vs}) - C_{cr}q_r)(q_c^-)^2 d\mathcal{M} \\
\leq & C(1 + \|(q_v, q_r)\|_{L^\infty(\mathcal{M})})\|q_c^-\|^2 \leq C(1 + \|(q_v, q_r)\|_{H^2(\mathcal{M})})\|q_c^-\|^2, \qquad (3.32)
\end{aligned}$$

from which due to the Gronwall inequality we have

$$\|q_c^-\|^2(t) \leq e^{C\int_0^t(1+\|(q_v,q_r)\|_{H^2(\mathcal{M})})dt}\|q_{c0}^-\|^2 = 0\,, \qquad (3.33)$$

implying $q_c^- \equiv 0$, and thus the nonnegativity of q_c.

We next prove the nonnegativity of q_v. Multiplying equation (1.25) by $-q_v^-$, integrating the resultant over $\mathcal{M}$, using (3.30)–(3.31) and noticing that $(q_v - q_{vs})^+q_v^- = 0$ due to $q_{vs} \geq 0$, we get

$$\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}}(q_v^-)^2 d\mathcal{M} \leq -\int_{\mathcal{M}} q_v^-(S_{ev} - S_{cd})d\mathcal{M}$$

$$
\begin{aligned}
&= \int_{\mathcal{M}} \big(C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ - C_{ev}T(q_r^+)^\beta (q_{vs} - q_v)^+\big) q_v^- d\mathcal{M} \\
&= \int_{\mathcal{M}} \big(C_{cd}(q_v - q_{vs})q_c - C_{ev}T(q_r^+)^\beta (q_{vs} - q_v)^+\big) q_v^- d\mathcal{M} \\
&\le \int_{\mathcal{M}} \big(C_{cd}(q_v - q_{vs})q_c + C_{ev}T^-(q_r^+)^\beta (q_{vs} - q_v)^+\big) q_v^- d\mathcal{M}. \qquad (3.34)
\end{aligned}
$$

Recalling $q_{vs} \ge 0$, $q_c \ge 0$ and $q_{vs}(p,T) = 0$ for $T \le 0$ from (1.14), one can deduce $q_{vs}q_c q_v^- \ge 0$ and $T^-(q_r^+)^\beta (q_{vs} - q_v)^+ q_v^- = T^-(q_r^+)^\beta(-q_v)^+ q_v^- = T^-(q_r^+)^\beta (q_v^-)^2$. Using moreover the Sobolev and Young inequalities, one obtains

$$
\begin{aligned}
\frac{1}{2}\frac{d}{dt}\|q_v^-\|^2 &\le \int_{\mathcal{M}} \big(-C_{cd}q_c + C_{ev}T^-(q_r^+)^\beta\big)(q_v^-)^2 d\mathcal{M} \\
&\le C(\|q_c\|_{L^\infty(\mathcal{M})} + \|T\|_{L^\infty(\mathcal{M})}\|q_r\|^\beta_{L^\infty(\mathcal{M})})\|q_v^-\|^2 \\
&\le C(1 + \|(q_c, q_r, \theta)\|^2_{H^2(\mathcal{M})})\|q_v^-\|^2. \qquad (3.35)
\end{aligned}
$$

In (3.35) we made use of the fact that $\beta \in (0,1]$. Applying the Gronwall inequality to the resulting estimate, one obtains

$$
\|q_v^-\|^2(t) \le e^{C\int_0^t (1+\|(q_c,q_r,\theta)\|^2_{H^2(\mathcal{M})})dt}\|q_{v0}^-\|^2 = 0. \qquad (3.36)
$$

Therefore $q_v^- \equiv 0$, implying again $q_v \ge 0$.

We now turn to the rain water mixing ratio q_r. Before proceeding in proving the nonnegativity, we show how to deal with the integral involving the terminal velocity by applying the Young inequality as follows

$$
\begin{aligned}
V\int_{\mathcal{M}} q_r^- \partial_p\left(\frac{p}{\overline{T}}q_r\right) d\mathcal{M} &= -V\int_{\mathcal{M}} \left[(q_r^-)^2\partial_p\left(\frac{p}{\overline{T}}\right) + \left(\frac{p}{\overline{T}}\right) q_r^- \partial_p q_r^-\right] d\mathcal{M} \\
&\le C\|q_r^-\|^2 + \frac{\nu_{q_r}}{2}\|\partial_p q_r^-\|_w^2. \qquad (3.37)
\end{aligned}
$$

Testing now (1.27) with q_r^- and employing (3.30)–(3.37), we obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathcal{M}} (q_r^-)^2 d\mathcal{M} \le C\|q_r^-\|^2 - \int_{\mathcal{M}} q_r^-(S_{ac} + S_{cr} - S_{ev}) d\mathcal{M} \\
&= C\|q_r^-\|^2 - \int_{\mathcal{M}} q_r^- \Big(C_{ac}(q_c - q_{ac}^*)^+ + C_{cr}q_c q_r - C_{ev}T(q_r^+)^\beta (q_{vs} - q_v)^+\Big) d\mathcal{M} \\
&\le C\|q_r^-\|^2 + C_{cr}\int_{\mathcal{M}} q_c (q_r^-)^2 d\mathcal{M} \le C(1 + \|q_c\|_{L^\infty(\mathcal{M})})\|q_r^-\|^2, \qquad (3.38)
\end{aligned}
$$

where we have used $q_r^-(q_r^+)^\beta = 0$. Applying the Gronwall inequality to the above inequality and using the Sobolev embedding theorem, we deduce

$$
\begin{aligned}
\|q_r^-\|^2(t) &\le e^{C\int_0^t (1+\|q_c\|_{L^\infty(\mathcal{M})})dt}\|q_{r0}^-\|^2 \\
&\le e^{C\int_0^t (1+\|q_c\|_{H^2(\mathcal{M})})dt}\|q_{r0}^-\|^2 = 0. \qquad (3.39)
\end{aligned}
$$

Thus $q_r^- \equiv 0$ and we obtain the desired nonnegativity $q_r \ge 0$.

Finally, we prove the nonnegativity of θ and therefore first deal with the integrals involving the diffusion terms. Integration by parts yields

$$\int_{\mathcal{M}} \theta^- \widetilde{\mathcal{D}}^\theta \theta d\mathcal{M} = \int_{\mathcal{M}} \left[\mu_T \Delta_h \theta + \nu_T \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \partial_p \left(\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right)\right)\right] \theta^- d\mathcal{M}$$

$$= - \int_{\mathcal{M}} \left[\mu_T \nabla_h \theta \cdot \nabla_h \theta^- + \nu_T \left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \theta^- \right) \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right)\right] d\mathcal{M}$$

$$+ \mu_T \int_{\Gamma_\ell} \theta^- \partial_n \theta d\Gamma_\ell + \nu_T \int_{\mathcal{M}'} \left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right) \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \theta^- d\mathcal{M}' \Bigg|_{p_1}^{p_0} . \quad (3.40)$$

We recall that $\theta = T(\frac{p_0}{p})^{R/c_p}$, as well as $T_{b\ell} \geq 0$ and $T_{b0} \geq 0$. By the boundary conditions (2.5)–(2.7) we then have

$$\text{on } \Gamma_\ell : \quad \theta^- \partial_n \theta = \left(\frac{p_0}{p}\right)^{\frac{2R}{c_p}} T^- \partial_n T = \left(\frac{p_0}{p}\right)^{\frac{2R}{c_p}} T^- \alpha_{\ell T}(T_{b\ell} - T) \geq 0\,, \quad (3.41)$$

$$\text{on } \Gamma_0 : \quad \theta^- \partial_p \left(\left(\frac{p}{p_0}\right)^{R/c_p} \theta \right) = \theta^- \partial_p T = T^- \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \alpha_{0T}(T_{b0} - T) \geq 0\,, \quad (3.42)$$

$$\text{on } \Gamma_1 : \quad \theta^- \partial_p \left(\left(\frac{p}{p_0}\right)^{R/c_p} \theta \right) = \theta^- \partial_p T = 0. \quad (3.43)$$

Straightforward computation of the integral containing the $p-$derivatives gives

$$\nu_T \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p_0}{p}\right)^{R/c_p} \theta^- \right) \partial_p \left(\left(\frac{p}{p_0}\right)^{R/c_p} \theta \right) d\mathcal{M}$$

$$= \nu_T \int_{\mathcal{M}} \left(- \left(\frac{gp}{R\bar{T}}\right)^2 (\partial_p \theta^-)^2 + \left(\frac{g}{c_p \bar{T}}\right)^2 (\theta^-)^2 \right) d\mathcal{M}\,. \quad (3.44)$$

Thus, we have

$$\int_{\mathcal{M}} \theta^- \widetilde{\mathcal{D}}^\theta \theta d\mathcal{M} \geq \mu_T \left\|\nabla_h \theta^-\right\|^2 + \nu_T \left\|\partial_p \theta^-\right\|_w^2 - \nu_T \int_{\mathcal{M}} \left(\frac{g}{c_p \bar{T}}\right)^2 (\theta^-)^2 d\mathcal{M}\,. \quad (3.45)$$

By the aid of the above, multiplying equation (1.31) by θ^-, it follows from integration by parts that

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|\theta^-\|^2 \quad \leq \quad & -\frac{L}{c_p} \int_{\mathcal{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \theta^- \big(C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ q_c\big) d\mathcal{M} \\ & + \frac{L}{c_p} \int_{\mathcal{M}} \theta^- \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} C_{ev} T (q_r^+)^\beta (q_{vs} - q_v)^+ d\mathcal{M} \\ & + \nu_T \int_{\mathcal{M}} \left(\frac{g}{c_p \bar{T}}\right)^2 (\theta^-)^2 d\mathcal{M} \leq C \|\theta^-\|^2, \end{aligned} \quad (3.46)$$

where we used the assumption (1.14) that $q_{vs} = 0$ for $T \leq 0$ (or $\theta \leq 0$ respectively), and the nonnegativity of q_v, q_c and q_r. Since $\theta_0^- \equiv 0$, the Gronwall inequality implies again $\theta^- \equiv 0$ for all $t > 0$, and thus $\theta \geq 0$.

(ii) Boundedness of q_v. We will test the equation (1.25) with $(q_v - q_v^*)^+$. For the diffusion operator we thereby proceed similar to above:

$$
\begin{aligned}
&\int_{\mathcal{M}} (q_v - q_v^*)^+ \, \mathcal{D}^{q_v} q_v \, d\mathcal{M} \\
= \;& \int_{\mathcal{M}} \left[\mu_{q_v} \Delta_h q_v + \nu_{q_v} \partial_p \left(\left(\frac{gp}{R\bar{T}} \right)^2 \partial_p q_v \right) \right] (q_v - q_v^*)^+ d\mathcal{M} \\
= \;& \mu_{q_v} \int_{\Gamma_\ell} (q_v - q_v^*)^+ \partial_n q_v d\Gamma_\ell + \nu_{q_v} \int_{\mathcal{M}'} \left(\frac{gp}{R\bar{T}} \right)^2 (q_v - q_v^*)^+ \partial_p q_v dxdy \bigg|_{p_1}^{p_0} \\
& - \int_{\mathcal{M}} \left[\mu_{q_v} \nabla_h q_v \cdot \nabla_h (q_v - q_v^*)^+ + \nu_{q_v} \left(\frac{gp}{R\bar{T}} \right)^2 \partial_p q_v \partial_p (q_v - q_v^*)^+ \right] d\mathcal{M} \\
= \;& -\mu_{q_v} \|\nabla_h (q_v - q_v^*)^+\|^2 - \nu_{q_v} \|\partial_p (q_v - q_v^*)^+\|_w^2 \\
& + \mu_{q_v} \int_{\Gamma_\ell} \alpha_{\ell v} (q_{b\ell v} - q_v)(q_v - q_v^*)^+ d\Gamma_\ell \\
& + \nu_{q_v} \int_{\mathcal{M}'} \left(\frac{gp_0}{R\bar{T}} \right)^2 \alpha_{b0v} (q_{b0v} - q_v)(q_v - q_v^*)^+ d\mathcal{M}' \bigg|_{p=p_0} . \qquad (3.47)
\end{aligned}
$$

By the definition of q_v^* we have

$$
(q_{b\ell v} - q_v)(q_v - q_v^*)^+ \leq 0 \quad \text{on } \Gamma_\ell \, , \qquad (q_{b0v} - q_v)(q_v - q_v^*)^+ \leq 0 \quad \text{on } \Gamma_0, \qquad (3.48)
$$

implying

$$
\int_{\mathcal{M}} (q_v - q_v^*)^+ \mathcal{D}^{q_v} q_v d\mathcal{M} \leq -\mu_{q_v} \|\nabla_h (q_v - q_v^*)^+\|^2 - \nu_{q_v} \|\partial_p (q_v - q_v^*)^+\|_w^2, \qquad (3.49)
$$

and leading further to the inequality

$$
\begin{aligned}
&\frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} ((q_v - q_v^*)^+)^2 d\mathcal{M} \leq \int_{\mathcal{M}} (q_v - q_v^*)^+ (S_{ev} - S_{cd}) d\mathcal{M} \qquad (3.50) \\
= \;& \int_{\mathcal{M}} (q_v - q_v^*)^+ \left(C_{ev} T (q_r^+)^\beta (q_{vs} - q_v)^+ - C_{cd} (q_v - q_{vs}) q_c - C_{cn} (q_v - q_{vs})^+ \right) d\mathcal{M}.
\end{aligned}
$$

Thanks to the above inequality, the definition of q_v^* in (3.24) (implying in particular $q_v^* \geq q_{vs}$) and the nonnegativity of T and q_c, one has

$$
\frac{1}{2} \frac{d}{dt} \int_{\mathcal{M}} ((q_v - q_v^*)^+)^2 d\mathcal{M} \leq 0. \qquad (3.51)
$$

Therefore, $\|(q_v - q_v^*)^+\|^2(t) \leq \|(q_{v0} - q_v^*)^+\|^2 = 0$, such that $q_v \leq q_v^*$.

(iii) *Boundedness of q_c, q_r.* We start with the derivation of the uniform boundedness of q_j, with $j \in \{c, r\}$. For any $m \geq 2$, we denote the cutoff function q_{j,k_j} as

$$
q_{j,k_j} = (q_j - k_j)^+ \qquad (3.52)
$$

with

$$
k_j = \sup_{t \in [0,\mathcal{T}]} (\|q_{b\ell j}\|_{L^\infty(\Gamma_\ell)} + \|q_{b0j}\|_{L^\infty(\partial\mathcal{M}')} + \|q_{j0}\|_{L^\infty(\mathcal{M})}), \qquad (3.53)
$$

for $j \in \{c, r\}$. We will use the method of testing equations (1.26)–(1.27) with $m q_{j,k_j}^{m-1}$, which is typically employed for equations with nonlinear diffusion, see e.g. [22]. Integrating by parts, the transport operator vanishes as before, and for the diffusion terms we have

$$\begin{aligned}\int_{\mathcal{M}} q_{j,k_j}^{m-1}\mathcal{D}^{q_j} q_j \, d\mathcal{M} \;&=\; \mu_{q_j}\int_{\Gamma_\ell} q_{j,k_j}^{m-1}\partial_n q_j d\Gamma_\ell + \nu_{q_j}\int_{\mathcal{M}'}\left(\frac{gp}{R\bar{T}}\right)^2 q_{j,k_j}^{m-1}\partial_p q_j\Big|_{p_1}^{p_0} d\mathcal{M}' \\ &\quad - \int_{\mathcal{M}}\left(\mu_{q_j}\nabla_h q_j\cdot\nabla_h q_{j,k_j}^{m-1} + \nu_{q_j}\left(\frac{gp}{R\bar{T}}\right)^2\partial_p q_j\partial_p q_{j,k_j}^{m-1}\right) d\mathcal{M}\,.\end{aligned} \tag{3.54}$$

Since, by definition, k_j is larger than the given boundary functions, we obtain

$$\text{on } \Gamma_\ell: \quad q_{j,k_j}^{m-1}\partial_n q_j = ((q_j-k_j)^+)^{m-1}\alpha_{\ell j}(q_{b\ell j}-q_j) \le 0\,, \tag{3.55}$$

$$\text{on } \Gamma_0: \quad q_{j,k_j}^{m-1}\partial_p q_j = ((q_j-k_j)^+)^{m-1}\alpha_{0j}(q_{b0j}-q_j) \le 0\,. \tag{3.56}$$

Moreover since $m \ge 2$ we can reformulate

$$\int_{\mathcal{M}}\nabla_h q_j\cdot\nabla_h q_{j,k_j}^{m-1} d\mathcal{M} = (m-1)\int_{\mathcal{M}} q_{j,k_j}^{m-2}|\nabla_h q_{j,k_j}|^2 d\mathcal{M} = \frac{4(m-1)}{m^2}\|\nabla_h q_{j,k_j}^{\frac{m}{2}}\|^2 \tag{3.57}$$

with an analogous calculation for the p-derivatives.

We now start with the derivation of the uniform boundedness of q_c by testing equation (1.26) with mq_{c,k_c}^{m-1} for any $m \ge 2$. Using the preceding computations and the Young inequality, we obtain directly from the uniform boundedness of q_v derived before:

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} q_{c,k_c}^m d\mathcal{M} &\le -4\frac{m-1}{m}(\mu_{q_c}\|\nabla_h q_{c,k_c}^{\frac{m}{2}}\|^2 + \nu_{q_c}\|\partial_p q_{c,k_c}^{\frac{m}{2}}\|_w^2) \\ &\quad + m\int_{\mathcal{M}}\big(C_{cd}(q_v-q_{vs})q_c + C_{cn}(q_v-q_{vs})^+ - C_{ac}(q_c-q_{ac}^*)^+ - C_{cr}q_cq_r\big)q_{c,k_c}^{m-1}d\mathcal{M} \\ &\le -4\frac{m-1}{m}(\mu_{q_c}\|\nabla_h q_{c,k_c}^{\frac{m}{2}}\|^2 + \nu_{q_c}\|\partial_p q_{c,k_c}^{\frac{m}{2}}\|_w^2) + Cm\int_{\mathcal{M}}(q_{c,k_c}^m + q_{c,k_c}^{m-1})d\mathcal{M} \\ &\le Cm\int_{\mathcal{M}}(1+q_{c,k_c}^m)d\mathcal{M} \le Cm(1+\|q_{c,k_c}\|_{L^m(\mathcal{M})}^m),\end{aligned} \tag{3.58}$$

from which, by the Gronwall inequality and the definition of k_c, one obtains

$$\|q_{c,k_c}\|_{L^m(\mathcal{M})}^m(t) \le e^{Cmt}(Cmt + \|q_{c,k_c}(0)\|_{L^m(\mathcal{M})}^m) = e^{Cmt}Cm(\mathcal{T}+1), \tag{3.59}$$

for any $t \in (0,\mathcal{T})$. Thanks to this estimate, we have

$$\|q_{c,k_c}\|_{L^m(\mathcal{M})}(t) \le e^{Ct}(C(\mathcal{T}+1))^{\frac{1}{m}}m^{\frac{1}{m}}, \tag{3.60}$$

from which, by taking $m \to \infty$, one gets $\|q_{c,k_c}\|_{L^\infty(\mathcal{M})}(t) \le e^{Ct}$ for all $t \in [0,\mathcal{T}]$, and thus

$$\|q_c\|_{L^\infty(\mathcal{M}\times(0,\mathcal{T}))} \le k_c + e^{C\mathcal{T}} =: q_c^*. \tag{3.61}$$

We next apply the same method to derive also uniform boundedness of q_r by employing the test function mq_{r,k_r}^{m-1}. The main difference to the previous estimates constitutes the additional vertical transport term of q_r, which we shall bound as follows:

$$\begin{aligned}-Vm\int_{\mathcal{M}} q_{r,k_r}^{m-1}\partial_p\left(\frac{p}{\bar{T}}q_r\right)d\mathcal{M} &= -Vm\int_{\mathcal{M}} q_{r,k_r}^{m-1}\left[q_r\partial_p\left(\frac{p}{\bar{T}}\right) + \frac{p}{\bar{T}}\partial_p q_r\right]d\mathcal{M} \\ &= -Vm\int_{\mathcal{M}} q_{r,k_r}^{m-1}(q_{r,k_r}+k_r)\partial_p\left(\frac{p}{\bar{T}}\right)d\mathcal{M} - 2V\int_{\mathcal{M}}\frac{p}{\bar{T}}q_{r,k_r}^{\frac{m}{2}}\partial_p q_{r,k_r}^{\frac{m}{2}}d\mathcal{M} \\ &\le Cm\int_{\mathcal{M}}\left(1+q_{r,k_r}^m\right)d\mathcal{M} + \nu_{q_r}\frac{2(m-1)}{m}\|\partial_p q_{r,k_r}^{\frac{m}{2}}\|_w^2\,,\end{aligned} \tag{3.62}$$

where we applied the Cauchy-Schwarz and Young inequalities. Then the estimate for q_r becomes

$$\begin{aligned}\frac{d}{dt}\int_{\mathcal{M}} q_{r,k_r}^m d\mathcal{M} &\le -4\frac{m-1}{m}\big(\mu_{q_r}\|\nabla_h q_{r,k_r}^{\frac{m}{2}}\|^2 + \frac{\nu_{q_r}}{2}\|\partial_p q_{r,k_r}^{\frac{m}{2}}\|_w^2\big) + Cm\int_{\mathcal{M}}(1+q_{r,k_r}^m)d\mathcal{M}\\ &+m\int_{\mathcal{M}}\big(C_{ac}(q_c-q_{ac}^*)^+ + C_{cr}q_cq_r - C_{ev}T(q_r^+)^\beta(q_{vs}-q_v)^+\big)q_{r,k_r}^{m-1}d\mathcal{M}\\ &\le -2\frac{m-1}{m}(\mu_{q_r}\|\nabla_h q_{r,k_r}^{\frac{m}{2}}\|^2 + \nu_{q_r}\|\partial_p q_{r,k_r}^{\frac{m}{2}}\|_w^2) + Cm\int_{\mathcal{M}}(1+q_{r,k_r}^m)d\mathcal{M}\,, \end{aligned}\tag{3.63}$$

where we have used the nonnegativity of T and the uniform boundedness of q_c. We thus obtain

$$\frac{d}{dt}\int_{\mathcal{M}} q_{r,k_r}^m d\mathcal{M} \le Cm(1+\|q_{r,k_r}\|_{L^m(\mathcal{M})}^m).\tag{3.64}$$

By the same argument as that for q_c above, we get the following estimate for q_r:

$$\|q_r\|_{L^\infty(\mathcal{M}\times(0,\mathcal{T}))} \le k_r + e^{C\mathcal{T}} =: q_r^*.\tag{3.65}$$

(iv) Boundedness of θ. We finally derive a similar estimate for θ. As before we set

$$\theta_{k_\theta} = (\theta-k_\theta)^+ \quad\text{with}\quad k_\theta = \sup_{t\in[0,\mathcal{T}]}(\|\theta_{b\ell}\|_{L^\infty(\Gamma_\ell)} + \|\theta_{b0}\|_{L^\infty(\partial\mathcal{M}')} + \|\theta_0\|_{L^\infty(\mathcal{M})}),\tag{3.66}$$

where $\theta_{b\ell} = \big(\frac{p_0}{p}\big)^{R/c_p}T_{b\ell}$ and $\theta_{b0} = \big(\frac{p_0}{p}\big)^{R/c_p}T_{b0}$ accordingly. Similar to above, one can deduce by integration by parts and using the boundary condition (2.7) that

$$\int_{\mathcal{M}}\theta_{k_\theta}^{m-1}\Delta_h\theta d\mathcal{M} \le -\frac{4(m-1)}{m}\|\nabla_h\theta_{k_\theta}^{\frac{m}{2}}\|^2 \le 0.\tag{3.67}$$

While for the diffusion term in the p-direction, the calculations are more involved. Recalling that $\theta = T\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}$, it follows from the boundary conditions (2.5)–(2.6) that

$$\begin{aligned}\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\partial_p\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta\right) &= \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\partial_pT = \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\alpha_{0T}(T_{b0}-T)\\ &= \theta_{k_\theta}^{m-1}\alpha_{0T}(\theta_{b0}-\theta) \le 0, \qquad \text{on } \Gamma_0,\end{aligned}\tag{3.68}$$

and

$$\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\partial_p\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta\right) = \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\partial_pT = 0, \quad \text{on } \Gamma_1.\tag{3.69}$$

Thus, due to integration by parts,

$$\begin{aligned}&\int_{\mathcal{M}}\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\partial_p\left(\left(\frac{gp}{R\bar{T}}\right)^2\partial_p\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta\right)\right)\theta_{k_\theta}^{m-1}d\mathcal{M}\\ =\;& -\int_{\mathcal{M}}\left(\frac{gp}{R\bar{T}}\right)^2\partial_p\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta\right)\partial_p\left(\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\right)d\mathcal{M}\\ &+\int_{\mathcal{M}'}\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}}\theta_{k_\theta}^{m-1}\left(\frac{gp}{R\bar{T}}\right)^2\partial_p\left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}}\theta\right)d\mathcal{M}'\Big|_{p_1}^{p_0}\end{aligned}$$

$$\leq \quad -\int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right) \partial_p \left(\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \theta_{k_\theta}^{m-1} \right) d\mathcal{M}. \tag{3.70}$$

Direct calculations yield

$$\begin{aligned}
& \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right) \partial_p \left(\left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \theta_{k_\theta}^{m-1} \right) \\
= \quad & (m-1)\theta_{k_\theta}^{m-2}(\partial_p \theta_{k_\theta})^2 + \frac{R}{c_p p}\left((m-1)\theta_{k_\theta}^{m-2}\theta - \theta_{k_\theta}^{m-1} \right) \partial_p \theta_{k_\theta} - \left(\frac{R}{c_p p}\right)^2 \theta \theta_{k_\theta}^{m-1} \\
= \quad & (m-1)\theta_{k_\theta}^{m-2} \left[(\partial_p \theta_{k_\theta})^2 + \frac{R}{c_p p}\left(\theta - \frac{\theta_{k_\theta}}{m-1}\right) \partial_p \theta_{k_\theta} \right] - \left(\frac{R}{c_p p}\right)^2 \theta \theta_{k_\theta}^{m-1} \\
= \quad & (m-1)\theta_{k_\theta}^{m-2} \left(\partial_p \theta_{k_\theta} + \frac{R}{2c_p p}\left(\theta - \frac{\theta_{k_\theta}}{m-1}\right) \right)^2 \\
& - \left(\frac{R}{c_p p}\right)^2 \left[\frac{m-1}{4}\left(\theta - \frac{\theta_{k_\theta}}{m-1}\right)^2 + \theta \theta_{k_\theta} \right] \theta_{k_\theta}^{m-2}.
\end{aligned} \tag{3.71}$$

Plugging this relation into the previous inequality, one obtains by the Young inequality

$$\begin{aligned}
& \int_{\mathcal{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \partial_p \left(\left(\frac{gp}{R\bar{T}}\right)^2 \partial_p \left(\left(\frac{p}{p_0}\right)^{\frac{R}{c_p}} \theta \right) \right) \theta_{k_\theta}^{m-1} d\mathcal{M} \\
\leq \quad & \int_{\mathcal{M}} \left(\frac{gp}{R\bar{T}}\right)^2 \left(\frac{R}{c_p p}\right)^2 \left[\frac{m-1}{4}\left(\theta - \frac{\theta_{k_\theta}}{m-1}\right)^2 + \theta \theta_{k_\theta} \right] \theta_{k_\theta}^{m-2} d\mathcal{M} \\
= \quad & \left(\frac{g}{c_p \bar{T}}\right)^2 \int_{\mathcal{M}} \left[\frac{m-1}{4}\left(\theta - \frac{\theta_{k_\theta}}{m-1}\right)^2 + \theta \theta_{k_\theta} \right]\Bigg|_{\theta \geq k_\theta} \theta_{k_\theta}^{m-2} d\mathcal{M} \\
= \quad & \left(\frac{g}{c_p \bar{T}}\right)^2 \int_{\mathcal{M}} \left[\frac{m-1}{4}\left(\frac{m-2}{m-1}\theta_{k_\theta} + k_\theta\right)^2 + \theta_{k_\theta}^2 + k_\theta \theta_{k_\theta} \right]\Bigg|_{\theta \geq k_\theta} \theta_{k_\theta}^{m-2} d\mathcal{M} \\
\leq \quad & C \int_{\mathcal{M}} [m(\theta_{k_\theta} + k_\theta)^2 + \theta_{k_\theta}^2 + k_\theta \theta_{k_\theta}] \theta_{k_\theta}^{m-2} d\mathcal{M} \leq Cm \int_{\mathcal{M}} (1 + \theta_{k_\theta}^m) d\mathcal{M}.
\end{aligned} \tag{3.72}$$

Combing the above estimate with (3.67) yields

$$\int_{\mathcal{M}} \tilde{\mathcal{D}}^\theta \theta \theta_{k_\theta}^{m-1} d\mathcal{M} \leq Cm \int_{\mathcal{M}} (1 + \theta_{k_\theta}^m) d\mathcal{M}. \tag{3.73}$$

We now test equation (1.31) with $m\theta_{k_\theta}^{m-1}$ and integrate by parts

$$\begin{aligned}
& \frac{d}{dt} \int_{\mathcal{M}} \theta_{k_\theta}^m d\mathcal{M} \leq Cm \int_{\mathcal{M}} \left(\theta_{k_\theta}^m + 1 \right) d\mathcal{M} \\
& \quad + m\frac{L}{c_p} \int_{\mathcal{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} \left(C_{cd}(q_v - q_{vs})q_c + C_{cn}(q_v - q_{vs})^+ \right) \theta_{k_\theta}^{m-1} d\mathcal{M} \\
& \quad - m\frac{L}{c_p} \int_{\mathcal{M}} \left(\frac{p_0}{p}\right)^{\frac{R}{c_p}} C_{ev} T (q_r^+)^\beta (q_{vs} - q_v)^+ \theta_{k_\theta}^{m-1} d\mathcal{M} \\
& \leq Cm \int_{\mathcal{M}} (\theta_{k_\theta}^m + 1) d\mathcal{M} \,,
\end{aligned} \tag{3.74}$$

where we made use of (3.73), the boundedness of q_v, q_c and the nonnegativity of T. By the same argument as before for q_c and q_r, it follows that

$$\|\theta\|_{L^\infty(\mathcal{M}\times(0,\mathcal{T}))} \le k_\theta + e^{C\mathcal{T}}, \tag{3.75}$$

proving the upper bound for T, on the interval $[0,\mathcal{T}]$. □

4. Global existence and well-posedness

In this section we prove our main result, i.e., the global existence and well-posedness, i.e., the uniqueness and the continuous dependence with respect to the initial data, of the strong solutions to system (1.25)–(1.30), or equivalently system (1.25)–(1.27) with (1.31), subject to the boundary conditions (2.5)–(2.7).

The uniqueness, and the Lipschitz continuity of the solutions on the initial data, is guaranteed by the following proposition, which, as we already mentioned in the Introduction, requires the Lipschitz continuity of the saturation mixing stated in (1.16).

Proposition 4.1. *Let $(T_i, q_{vi}, q_{ci}, q_{ri})$ for $i \in \{1,2\}$ be two strong solutions, on the interval $[0,\mathcal{T}]$, of (1.25)–(1.30), subject to (2.5)–(2.7), with initial data $(T_i^0, q_{vi}^0, q_{ci}^0, q_{ri}^0) \in (L^\infty(\mathcal{M}))^4 \cap (H^1(\mathcal{M}))^4$. Then, the following estimate holds*

$$\sup_{t\in[0,\mathcal{T}]}\Big(\|T_1-T_2\|^2 + \sum_{j\in\{v,c,r\}}\|q_{j1}-q_{j2}\|^2\Big) \le Ce^{C_0\mathcal{T}}\Big(\|T_1^0-T_2^0\|^2 + \sum_{j\in\{v,c,r\}}\|q_{j1}^0-q_{j2}^0\|^2\Big)$$

for a positive constant C_0, implying in particular the uniqueness of the solutions.

Proof. The main difficulty in the proof of the uniqueness and the continuous dependence on the initial data of the solutions is caused by the evaporation term S_{ev} if the exponent $\beta \in (0,1)$. This problem can be circumvented by introducing the following new unknowns

$$Q = q_v + q_r, \qquad H = T - \frac{L}{c_p}(q_c + q_r)\,. \tag{4.1}$$

These quantities resemble the ones used in Hernandez-Duenas et al. [19]. However, in [19] the cloud water was not taken into account. In particular, c_pH corresponds to the liquid water enthalpy, see e.g. Emanuel [13]. In the following we prove uniqueness by deriving typical L^2-estimates for the differences of the solutions in terms of estimates for the quantities

$$Q, q_c, q_r, H\,. \tag{4.2}$$

We start with the estimate for Q, whose evolution is governed by the equation:

$$\begin{aligned}\partial_t Q + \mathbf{v}_h\cdot\nabla_h Q + \omega\partial_p Q = &-V\partial_p\left(\frac{p}{\overline{T}}q_r\right) + S_{ac} + S_{cr} - S_{cd} + \mu_{q_v}\Delta_h Q + (\mu_{q_r}-\mu_{q_v})\Delta_h q_r\\ &+ \nu_{q_v}\partial_p\Big(\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p Q\Big) + (\nu_{q_r}-\nu_{q_v})\partial_p\Big(\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p q_r\Big)\,.\end{aligned} \tag{4.3}$$

Let Q_1, Q_2 be two solutions of (4.3). We multiply the equation for the difference (Q_1-Q_2) by (Q_1-Q_2), and integrate over $\mathcal{M}$. Recalling (2.10)–(2.11), it follows from integration by parts and using the relevant boundary conditions, that

$$\int_{\mathcal{M}}(\partial_t(Q_1-Q_2)+\mathbf{v}_h\cdot\nabla_h(Q_1-Q_2)+\omega\partial_p(Q_1-Q_2))(Q_1-Q_2)d\mathcal{M}=\frac{1}{2}\frac{d}{dt}\|Q_1-Q_2\|^2. \tag{4.4}$$

By the boundary conditions (2.5)–(2.7), one can check that

$$\text{on } \Gamma_\ell: \quad \partial_n(Q_1-Q_2)+\alpha_{\ell v}(Q_1-Q_2)=(\alpha_{\ell 0}-\alpha_{\ell r})(q_{r1}-q_{r2}), \tag{4.5}$$

$$\text{on } \Gamma_0: \quad \partial_p(Q_1-Q_2)+\alpha_{0v}(Q_1-Q_2)=(\alpha_{0v}-\alpha_{0r})(q_{r1}-q_{r2}), \tag{4.6}$$

$$\text{on } \Gamma_1: \quad \partial_p(Q_1-Q_2)=\partial_p(q_{r1}-q_{r2})=0, \tag{4.7}$$

$$\text{on } \Gamma_\ell: \quad \partial_n(q_{r1}-q_{r2})=\alpha_{\ell r}(q_{r2}-q_{r1}), \quad \text{on } \Gamma_0: \quad \partial_p(q_{r1}-q_{r2})=\alpha_{0r}(q_{r2}-q_{r1}). \tag{4.8}$$

For shortening the expressions, we use hereafter the notation

$$\begin{aligned}&\mathrm{Diff}\,(Q,q_r):= \\ &\mu_{q_v}\Delta_h Q+(\mu_{q_r}-\mu_{q_v})\Delta_h q_r+\nu_{q_v}\partial_p\Big(\Big(\frac{gp}{R\bar{T}}\Big)^2\partial_p Q\Big)+(\nu_{q_r}-\nu_{q_v})\partial_p\Big(\Big(\frac{gp}{R\bar{T}}\Big)^2\partial_p q_r\Big)\end{aligned} \tag{4.9}$$

for the diffusion terms in equation (4.3) for Q. It then follows from integration by parts, using the boundary conditions (4.5)-(4.8), that

$$\begin{aligned}
&\int_{\mathcal{M}}(\mathrm{Diff}(Q_1,q_{r1})-\mathrm{Diff}(Q_2,q_{r2}))(Q_1-Q_2)d\mathcal{M}\\
=\;&\mu_{q_v}\int_{\Gamma_\ell}(Q_1-Q_2)\partial_n(Q_1-Q_2)d\Gamma_\ell-\mu_{q_v}\|\nabla_h(Q_1-Q_2)\|^2\\
&+\nu_{q_v}\int_{\mathcal{M}'}\Big(\frac{gp}{R\bar{T}}\Big)^2\partial_p(Q_1-Q_2)(Q_1-Q_2)d\mathcal{M}'\Big|_{p_1}^{p_0}-\nu_{q_v}\|\partial_p(Q_1-Q_2)\|_w^2\\
&+(\mu_{q_r}-\mu_{q_v})\int_{\Gamma_\ell}\partial_n(q_{r1}-q_{r2})(Q_1-Q_2)d\Gamma_\ell\\
&+(\nu_{q_r}-\nu_{q_v})\int_{\mathcal{M}'}\Big(\frac{gp}{R\bar{T}}\Big)^2\partial_p(q_{r1}-q_{r2})(Q_1-Q_2)d\mathcal{M}'\Big|_{p_1}^{p_0}\\
&-(\mu_{q_r}-\mu_{q_v})\int_{\mathcal{M}}\nabla_h(q_{r1}-q_{r2})\cdot\nabla_h(Q_1-Q_2)d\mathcal{M}\\
&-(\nu_{q_r}-\nu_{q_v})\int_{\mathcal{M}}\Big(\frac{gp}{R\bar{T}}\Big)^2\partial_p(q_{r1}-q_{r2})\partial_p(Q_1-Q_2)d\mathcal{M},
\end{aligned} \tag{4.10}$$

Applying Young's inequality to the integrals over $\mathcal{M}$ and using the boundary conditions (4.5)-(4.8) to the boundary integrals, one obtains, after some manipulations,

$$\begin{aligned}
&\int_{\mathcal{M}}(\mathrm{Diff}(Q_1,q_{r1})-\mathrm{Diff}(Q_2,q_{r2}))(Q_1-Q_2)d\mathcal{M}\\
\leq\;&-\frac{\mu_{q_v}}{2}\|\nabla_h(Q_1-Q_2)\|^2+\frac{(\mu_{q_r}-\mu_{q_v})^2}{2\mu_{q_v}}\|\nabla_h(q_{r1}-q_{r2})\|^2-\frac{\nu_{q_v}}{2}\|\partial_p(Q_1-Q_2)\|_w^2\\
&+\frac{(\nu_{q_r}-\nu_{q_v})^2}{2\nu_{q_v}}\|\partial_p(q_{r1}-q_{r2})\|_w^2-\mu_{q_v}\int_{\Gamma_\ell}\alpha_{\ell v}(Q_1-Q_2)^2d\Gamma_\ell\\
&+\int_{\Gamma_\ell}(\mu_{q_r}\alpha_{\ell r}-\mu_{q_v}\alpha_{\ell v})(q_{r2}-q_{r1})(Q_1-Q_2)d\Gamma_\ell
\end{aligned}$$

$$
\begin{aligned}
&-\nu_{q_v}\int_{\mathcal{M}'}\left(\frac{gp}{R\overline{T}}\right)^2\alpha_{0v}(Q_1-Q_2)^2d\mathcal{M}'\Big|_{p=p_0}\\
&+\int_{\mathcal{M}'}(\nu_{q_r}\alpha_{0r}-\nu_{q_v}\alpha_{0v})(q_{r2}-q_{r1})(Q_1-Q_2)d\mathcal{M}'\Big|_{p=p_0}. \qquad (4.11)
\end{aligned}
$$

Since we do not want to make any restrictions on the boundary data (i.e.,we also want to allow that, e.g., $\alpha_{0r}=0$, but $\alpha_{0v}>0$), we apply Young's inequality with $\varepsilon>0$ sufficiently small to the boundary integrals to finally estimate them using the boundedness of the trace map $H^1(\mathcal{M})\mapsto H^{\frac{1}{2}}(\partial\mathcal{M})\hookrightarrow L^2(\partial\mathcal{M})$, see, e.g., [12], as follows

$$
\begin{aligned}
&\int_{\Gamma_\ell}(\mu_{q_r}\alpha_{\ell r}-\mu_{q_v}\alpha_{\ell v})(q_{r2}-q_{r1})(Q_1-Q_2)d\Gamma_\ell\\
&\quad+\int_{\mathcal{M}'}(\nu_{q_r}\alpha_{0r}-\nu_{q_v}\alpha_{0v})(q_{r2}-q_{r1})(Q_1-Q_2)d\mathcal{M}'\Big|_{p=p_0}\\
\leq\;&\varepsilon\int_{\partial\mathcal{M}}(Q_1-Q_2)^2d(\partial\mathcal{M})+C(\varepsilon)\int_{\partial\mathcal{M}}(q_{r1}-q_{r2})^2d(\partial\mathcal{M})\\
\leq\;&\frac{\mu_{q_v}}{4}\|\nabla_h(Q_1-Q_2)\|^2+\frac{\nu_{q_v}}{4}\|\partial_p(Q_1-Q_2)\|_w^2+C(\|Q_1-Q_2\|^2+\|q_{r1}-q_{r2}\|^2)\\
&+C_Q(\mu_{q_r}\|\nabla_h(q_{r1}-q_{r2})\|^2+\nu_{q_r}\|\partial_p(q_{r1}-q_{r2})\|_w^2). \qquad (4.12)
\end{aligned}
$$

For the L^2-estimate we therefore obtain

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Q_1-Q_2\|^2+\frac{\mu_{q_v}}{4}\|\nabla_h(Q_1-Q_2)\|^2+\frac{\nu_{q_v}}{4}\|\partial_p(Q_1-Q_2)\|_w^2\\
\leq\;&C_Q(\mu_{q_r}\|q_{r1}-q_{r2}\|^2+\nu_{q_r}\|\partial_p(q_{r1}-q_{r2}\|_w^2)+C(\|Q_1-Q_2\|^2+\|q_{r1}-q_{r2}\|^2)\\
&-V\int_{\mathcal{M}}\left((q_{r1}-q_{r2})\partial_p\left(\frac{p}{\overline{T}}\right)+\frac{p}{\overline{T}}\partial_p(q_{r1}-q_{r2})\right)(Q_1-Q_2)d\mathcal{M}\\
&+C_{ac}\int_{\mathcal{M}}((q_{c1}-q_{ac}^*)^+-(q_{c2}-q_{ac}^*)^+)(Q_1-Q_2)d\mathcal{M}\\
&+C_{cr}\int_{\mathcal{M}}\big((q_{c1}-q_{c2})q_{r1}+q_{c2}(q_{r1}-q_{r2})\big)(Q_1-Q_2)d\mathcal{M}\\
&-C_{cd}\int_{\mathcal{M}}((q_{v1}-q_{v2})q_{c1}+q_{v2}(q_{c1}-q_{c2}))(Q_1-Q_2)d\mathcal{M}\\
&+C_{cd}\int_{\mathcal{M}}(q_{vs}(T_1)(q_{c1}-q_{c2})+q_{c2}(q_{vs}(p,T_1)-q_{vs}(p,T_2))(Q_1-Q_2)d\mathcal{M}\\
&-C_{cn}\int_{\mathcal{M}}((q_{v1}-q_{vs}(p,T_1))^+-(q_{v2}-q_{vs}(p,T_2))^+)(Q_1-Q_2)d\mathcal{M}. \qquad (4.13)
\end{aligned}
$$

The Lipschitz continuity property of the saturation mixing ratio (1.16) implies

$$
\begin{aligned}
&|(q_{v1}-q_{vs}(p,T_1))^+-(q_{v2}-q_{vs}(p,T_2))^+|\\
&\quad\leq|(q_{v1}-q_{vs}(p,T_1))^+-(q_{v2}-q_{vs}(p,T_1))^+|+|(q_{v2}-q_{vs}(p,T_1))^+-(q_{v2}-q_{vs}(p,T_2))^+|\\
&\quad\leq|q_{v1}-q_{v2}|+C|T_1-T_2|. \qquad (4.14)
\end{aligned}
$$

Using additionally the uniform boundedness of all moisture quantities as well as Young's inequality and rewriting q_v and T in terms of the quantities in (4.2), we obtain:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|Q_1-Q_2\|^2+\frac{\mu_{q_v}}{4}\|\nabla_h(Q_1-Q_2)\|^2+\frac{\nu_{q_v}}{4}\|\partial_p(Q_1-Q_2)\|_w^2\\
\leq\ & C\left(\|Q_1-Q_2\|^2+\|q_{r1}-q_{r2}\|^2+\|q_{c1}-q_{c2}\|^2+\|H_1-H_2\|^2\right)\\
&+C_Q(\mu_{q_r}\|\nabla_h(q_{r1}-q_{r2})\|^2+\nu_{q_r}\|\partial_p(q_{r1}-q_{r2})\|_w^2). \qquad (4.15)
\end{aligned}$$

We next estimate the difference of the two rain water mixing ratios. We thereby bound the diffusion and the vertical transport term with terminal velocity V as follows:

$$\begin{aligned}
&\int_{\mathcal{M}}(q_{r1}-q_{r2})\mathcal{D}^{q_r}(q_{r1}-q_{r2})d\mathcal{M}-V\int_{\mathcal{M}}(q_{r1}-q_{r2})\partial_p\left(\frac{p}{\overline{T}}(q_{r1}-q_{r2})\right)d\mathcal{M}\\
&=-\mu_{q_r}\int_{\Gamma_\ell}\alpha_{\ell r}(q_{r1}-q_{r2})^2d\Gamma_\ell-\mu_{q_r}\|\nabla_h(q_{r_1}-q_{r_2})\|^2\\
&\quad-\nu_{q_r}\int_{\mathcal{M}'}\left(\frac{gp}{R\overline{T}}\right)^2\alpha_{0r}(q_{r1}-q_{r2})^2d\mathcal{M}'\Big|_{p=p_0}-\nu_{q_r}\|\partial_p(q_{r1}-q_{r2})\|_w^2\\
&\quad-V\int_{\mathcal{M}}\left[(q_{r1}-q_{r2})^2\partial_p\left(\frac{p}{\overline{T}}\right)+\frac{p}{\overline{T}}(q_{r1}-q_{r2})\partial_p(q_{r1}-q_{r2})\right]d\mathcal{M}\\
&\leq-\mu_{q_r}\|\nabla_h(q_{r_1}-q_{r_2})\|^2-\frac{\nu_{q_r}}{2}\|\partial_p(q_{r1}-q_{r2})\|_w^2+C\|q_{r_1}-q_{r2}\|^2, \qquad (4.16)
\end{aligned}$$

and we get for the L^2-estimate of $q_{r1}-q_{r2}$:

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q_{r1}-q_{r2}\|^2+\mu_{q_r}\|\nabla_h(q_{r1}-q_{r2})\|^2+\frac{\nu_{q_r}}{2}\|\partial_p(q_{r1}-q_{r2})\|_w^2\\
\leq\ & C\|q_{r_1}-q_{r2}\|^2+C_{ac}\int_{\mathcal{M}}((q_{c1}-q_{ac}^*)^+-(q_{c2}-q_{ac}^*)^+)(q_{r1}-q_{r2})d\mathcal{M}\\
&+C_{cr}\int_{\mathcal{M}}\left[(q_{c1}-q_{c2})q_{r1}(q_{r1}-q_{r2})+q_{c2}(q_{r1}-q_{r2})^2\right]d\mathcal{M}\\
&-C_{ev}\int_{\mathcal{M}}(T_1-T_2)q_{r1}^\beta(q_{vs}(p,T_1)-q_{v1})^+(q_{r1}-q_{r2})d\mathcal{M}\\
&-C_{ev}\int_{\mathcal{M}}T_2(q_{r1}^\beta-q_{r2}^\beta)(q_{vs}(p,T_1)-q_{v1})^+(q_{r1}-q_{r2})d\mathcal{M}\\
&-C_{ev}\int_{\mathcal{M}}T_2q_{r2}^\beta((q_{vs}(p,T_1)-q_{v1})^+-(q_{vs}(p,T_2)-q_{v2})^+)(q_{r1}-q_{r2})d\mathcal{M}\,. \qquad (4.17)
\end{aligned}$$

We can now use the monotonicity property in the evaporation term $(q_{r1}^\beta-q_{r2}^\beta)(q_{r1}-q_{r2})\geq 0$ and recall (4.14) to estimate further replacing T in terms of H, q_c and q_r

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|q_{r1}-q_{r2}\|^2+\mu_{q_r}\|\nabla_h(q_{r1}-q_{r2})\|^2+\frac{\nu_{q_r}}{2}\|\partial_p(q_{r1}-q_{r2})\|_w^2\\
&\quad\leq C\left(\|Q_1-Q_2\|^2+\|q_{r1}-q_{r2}\|^2+\|q_{c1}-q_{c2}\|^2+\|H_1-H_2\|^2\right). \qquad (4.18)
\end{aligned}$$

We now turn to the difference of the cloud water mixing ratios. We estimate the boundary terms arising from partial integration of the diffusion operator as above, using (4.14) and the Lipschitz continuity of q_{vs}, (1.16), and obtain

$$\frac{1}{2}\frac{d}{dt}\|q_{c1}-q_{c2}\|^2+\mu_{q_c}\|\nabla_h(q_{c1}-q_{c2})\|^2+\nu_{q_c}\|\partial_p(q_{c1}-q_{c2})\|_w^2$$

$$
\begin{aligned}
&\leq \; C_{cd}\int_{\mathcal{M}}(q_{v1}-q_{v2}-(q_{vs}(p,T_1)-q_{vs}(p,T_2))q_{c1}(q_{c1}-q_{c2})d\mathcal{M}\\
&\quad +C_{cd}\int_{\mathcal{M}}(q_{v2}-q_{vs}(p,T_2))(q_{c1}-q_{c2})(q_{c1}-q_{c2})d\mathcal{M}\\
&\quad +C_{cn}\left((q_{v1}-q_{vs}(p,T_1))^+-(q_{v2}-q_{vs}(p,T_2))^+\right)(q_{c1}-q_{c2})d\mathcal{M}\\
&\quad -C_{ac}\int_{\mathcal{M}}((q_{c1}-q_{ac}^*)^+-(q_{c2}-q_{ac}^*)^+)(q_{c1}-q_{c2})d\mathcal{M}\\
&\quad -C_{cr}\int_{\mathcal{M}}\left[(q_{c1}-q_{c2})^2q_{r1}+q_{c2}(q_{r1}-q_{r2})(q_{c1}-q_{c2})\right]d\mathcal{M}\\
&\leq \; C\left(\|Q_1-Q_2\|^2+\|q_{r1}-q_{r2}\|^2+\|q_{c1}-q_{c2}\|^2+\|H_1-H_2\|^2\right). \qquad (4.19)
\end{aligned}
$$

To close the estimates it remains to bound (H_1-H_2). The equation for H reads

$$
\begin{aligned}
&\partial_t H+\mathbf{v}_h\cdot\nabla_h H+\omega\partial_p H=\\
&=\frac{RT}{c_pp}\omega-\frac{L}{c_p}V\partial_p\Big(\frac{p}{\overline{T}}q_r\Big)+\mu_T\Delta_h H+\nu_T\partial_p\Big(\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p H\Big)\\
&\quad-\frac{L}{c_p}\left((\mu_{q_c}-\mu_T)\Delta_h q_c+(\nu_{q_c}-\nu_T)\partial_p\Big(\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p q_c\Big)\right)\\
&\quad-\frac{L}{c_p}\left((\mu_{q_r}-\mu_T)\Delta_h q_r+(\nu_{q_r}-\nu_T)\partial_p\Big(\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p q_r\Big)\right). \qquad (4.20)
\end{aligned}
$$

As it can be seen here from the thermodynamic equation the antidissipative term containing the vertical velocity ω appears on the right hand side. In previous estimates we could circumvent estimations involving this term by switching to the potential temperature θ. However, this is not possible here, since for an alternative definition of the form $\widetilde{H}=\theta-\frac{L}{c_p}\big(\frac{p_0}{p}\big)^{\frac{R}{c_p}}(q_c+q_r)$, which would also allow for a cancellation of the source terms from phase changes, the vertical velocity term would appear again from the vertical advection term due to the pressure function multiplying the moisture quantities. We therefore stick to the previously introduced quantity H.

Testing again the equation for the difference (H_1-H_2) against (H_1-H_2) and treating the boundary terms arising from the diffusion operators and the additional vertical transport term in a similar way as for (Q_1-Q_2) above, we obtain:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|H_1-H_2\|^2\leq-\mu_T\|\nabla_h(H_1-H_2)\|^2-\nu_T\|\partial_p(H_1-H_2)\|_w^2\\
&\quad+C(\|q_{c1}-q_{c2}\|^2_{L^2(\partial\mathcal{M})}+\|q_{r1}-q_{r2}\|^2_{L^2(\partial\mathcal{M})})+\varepsilon\|H_1-H_2\|^2_{L^2(\partial\mathcal{M})}\\
&\quad-\frac{L}{c_p}\int_{\mathcal{M}}\nabla_h(H_1-H_2)\cdot\big((\mu_{q_c}-\mu_T)\nabla_h(q_{c1}-q_{c2})+(\mu_{q_r}-\mu_T)\nabla_h(q_{r1}-q_{r2})\big)d\mathcal{M}\\
&\quad-\frac{L}{c_p}\int_{\mathcal{M}}\Big(\frac{gp}{R\overline{T}}\Big)^2\partial_p(H_1-H_2)\cdot\big((\nu_{q_c}-\nu_T)\partial_p(q_{c1}-q_{c2})+(\nu_{q_r}-\nu_T)\partial_p(q_{r1}-q_{r2})\big)d\mathcal{M}\\
&\quad-\frac{L}{c_p}\int_{\mathcal{M}}\left[V\frac{p}{\overline{T}}(q_{r1}-q_{r2})\partial_p(H_1-H_2)+\frac{R\omega}{c_pp}(T_1-T_2)(H_1-H_2)\right]d\mathcal{M}\\
&\leq-\frac{\mu_T}{2}\|\nabla_h(H_1-H_2)\|^2-\frac{\nu_T}{2}\|\partial_p(H_1-H_2)\|_w^2+C_H\mu_{q_c}\|\nabla_h(q_{c1}-q_{c2})\|^2\\
&\quad+C_H(\mu_{q_r}\|\nabla_h(q_{r1}-q_{r2})\|^2+\nu_{q_c}\|\partial_p(q_{c1}-q_{c2})\|_w^2+\nu_{q_r}\|\partial_p(q_{r1}-q_{r2})\|_w^2)
\end{aligned}
$$

$$+ C\left(\|H_1 - H_2\|^2 + \|q_{r1} - q_{r2}\|^2 + \|q_{c1} - q_{c2}\|^2\right), \tag{4.21}$$

where ε is chosen sufficiently small; we have also used here the boundedness of the trace map $H^1(\mathcal{M}) \mapsto L^2(\partial\mathcal{M})$ and Young's inequality. Moreover, since according to (2.8) we have $\omega \in L^\infty(0,\mathcal{T}; L^2(\mathcal{M}))$, we have also used above the following bound for the integral containing the vertical velocity component ω:

$$\begin{aligned}
&\left|\int_{\mathcal{M}} \frac{R\omega}{c_p p}(T_1 - T_2)(H_1 - H_2)d\mathcal{M}\right| \\
\leq & C\Big(\|H_1 - H_2\|_{L^4(\mathcal{M})}^2 + \sum_{j\in\{c,r\}} \|q_{j1} - q_{j2}\|_{L^4(\mathcal{M})}^2\Big) \\
\leq & C\Big(\|H_1 - H_2\|_{H^1(\mathcal{M})}^{\frac{3}{2}}\|H_1 - H_2\|^{\frac{1}{2}} + \sum_{j\in\{c,r\}} \|q_{j1} - q_{j2}\|_{H^1(\mathcal{M})}^{\frac{3}{2}}\|q_{j1} - q_{j2}\|^{\frac{1}{2}}\Big) \\
\leq & \varepsilon\Big(\|\nabla(H_1 - H_2)\|^2 + \sum_{j\in\{c,r\}} \|\nabla(q_{j1} - q_{j2})\|^2\Big) \\
& + C_\varepsilon\Big(\|H_1 - H_2\|^2 + \sum_{j\in\{c,r\}} \|q_{j1} - q_{j2}\|^2\Big).
\end{aligned} \tag{4.22}$$

We now combine the estimates (4.15), (4.18), (4.19) and (4.21) by introducing

$$J(t) = \frac{1}{2}\left(A\|Q_1 - Q_2\|^2 + \|q_{c1} - q_{c2}\|^2 + \|q_{r1} - q_{r2}\|^2 + B\|H_1 - H_2\|^2\right), \tag{4.23}$$

where A and B are positive constants chosen accordingly (e.g., $A = \frac{1}{2C_Q}$ and $B = \frac{1}{2C_H}$), such that for some $C_0 > 0$

$$\frac{dJ}{dt} \leq C_0 J \tag{4.24}$$

and we conclude the proof by applying the Gronwall inequality. □

We are now ready to prove our main result, Theorem 2.1.

Proof of Theorem 2.1. The uniqueness and continuous dependence on the initial data is an immediate corollary of Proposition 4.1. Therefore, it remains to prove the global existence. By Proposition 3.1, under the assumptions in Theorem 2.1, there is a unique local strong solution (q_v, q_c, q_r, θ), with

$$(q_v, q_c, q_r, \theta) \in C([0, \mathcal{T}_0]; H^1(\mathcal{M})) \cap L^2(0, \mathcal{T}_0; H^2(\mathcal{M})). \tag{4.25}$$

We extend the unique strong solution (q_v, q_c, q_r, θ) to the maximal time of existence $\mathcal{T}_*$. If $\mathcal{T}_* = \infty$, then one obtains a global strong solution. Suppose that $\mathcal{T}_* < \infty$, then one obviously has

$$\lim_{\mathcal{T}\to\mathcal{T}_*^-} \|(q_v, q_c, q_r, \theta)\|_{L^\infty(0,\mathcal{T};H^1(\mathcal{M}))\cap L^2(0,\mathcal{T},H^2(\mathcal{M}))} = \infty. \tag{4.26}$$

By Proposition 3.2, we have $\|(q_v, q_c, q_r, \theta)\|_{L^\infty(\mathcal{M}\times(0,\mathcal{T}_*))} \leq C$, for a positive constant C that depends continuously on $\mathcal{T}_*$. Let ε be a small enough positive time to be specified later. We will estimate the norms to the solution in the time interval $(\mathcal{T}_* - \varepsilon, \mathcal{T})$, for $\mathcal{T} \in (\mathcal{T}_* - \varepsilon, \mathcal{T}_*)$. For simplicity, we denote $\mathbf{U} = (q_v, q_c, q_r, \theta)$. Applying Corollary A.1 in the appendix, below, to system (1.25)–(1.27) with (1.31), and recalling that $\|(q_v, q_c, q_r, \theta)\|_{L^\infty(\mathcal{M}\times(0,\mathcal{T}_*))} \leq C$, we have

$$\begin{aligned} &\|\mathbf{U}\|_{L^\infty(\mathcal{T}_*-\varepsilon,\mathcal{T};H^1(\mathcal{M}))\cap L^2(\mathcal{T}_*-\varepsilon,\mathcal{T};H^2(\mathcal{M}))} \qquad (4.27)\\ \leq\ & C(1+\|\mathbf{v}_h\cdot\nabla_h\mathbf{U}\|_{L^2(\mathcal{M}\times(\mathcal{T}_*-\varepsilon,\mathcal{T}))}+\|\omega\partial_p\mathbf{U}\|_{L^2(\mathcal{M}\times(\mathcal{T}_*-\varepsilon,\mathcal{T}))}+\|\partial_p\mathbf{U}\|_{L^2(\mathcal{M}\times(\mathcal{T}_*-\varepsilon,\mathcal{T}))}), \end{aligned}$$

for a positive constant C independent of $\mathcal{T} < \mathcal{T}_*$.

The same argument as for (3.11) yields

$$\begin{aligned} \|\mathbf{v}_h\cdot\nabla_h\mathbf{U}\|_{L^2(\mathcal{M}\times(\mathcal{T}_*-\varepsilon,\mathcal{T}))} \leq\ & C(\mathcal{T}-\mathcal{T}_*+\varepsilon)^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}}\|\mathbf{v}_h\|_{L^r(\mathcal{T}_*-\varepsilon,\mathcal{T};L^\sigma(\mathcal{M}))}\\ &\cdot\|\mathbf{U}\|_{L^\infty(\mathcal{T}_*-\varepsilon,\mathcal{T};H^1(\mathcal{M}))\cap L^2(\mathcal{T}_*-\varepsilon,\mathcal{T};H^2(\mathcal{M}))}\\ \leq\ & C\varepsilon^{\frac{\sigma-3}{2\sigma}-\frac{1}{r}}\|\mathbf{U}\|_{L^\infty(\mathcal{T}_*-\varepsilon,\mathcal{T};H^1(\mathcal{M}))\cap L^2(\mathcal{T}_*-\varepsilon,\mathcal{T};H^2(\mathcal{M}))}, \quad (4.28) \end{aligned}$$

for a positive constant C independent of $\mathcal{T} < \mathcal{T}_*$, with an analogous estimate for $\omega\partial_p\mathbf{U}$. By the Hölder inequality one moreover gets

$$\begin{aligned} \|\partial_p\mathbf{U}\|_{L^2(\mathcal{M}\times(\mathcal{T}_*-\varepsilon,\mathcal{T}))} \leq\ & C(\mathcal{T}-\mathcal{T}_*+\varepsilon)^{\frac{1}{2}}\|\mathbf{U}\|_{L^\infty(\mathcal{T}-\mathcal{T}_*+\varepsilon,\mathcal{T};H^1(\mathcal{M}))}\\ \leq\ & C\varepsilon^{\frac{1}{2}}\|\mathbf{U}\|_{L^\infty(\mathcal{T}-\mathcal{T}_*+\varepsilon,\mathcal{T};H^1(\mathcal{M}))}, \quad (4.29) \end{aligned}$$

for a positive constant C independent of $\mathcal{T} < \mathcal{T}_*$.

Plugging the above two estimates into (4.27), and choosing ε small enough, one concludes

$$\|\mathbf{U}\|_{L^\infty(\mathcal{T}_*-\varepsilon,\mathcal{T};H^1(\mathcal{M}))\cap L^2(\mathcal{T}_*-\varepsilon,\mathcal{T};H^2(\mathcal{M}))} \leq C, \quad (4.30)$$

for a positive constant C independent of $\mathcal{T} < \mathcal{T}_*$, which contradicts (4.26). Therefore, we must have $\mathcal{T}_* = \infty$, in other words, the solution $(q_v, q_c, q_r, \theta)(t)$ can be extended to be a global solution. □

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Appendix A. Elliptic and parabolic problems on cylindrical domains

As mentioned in the introduction, this technical section is devoted to establishing the existence and uniqueness of solutions (weak or strong) to some elliptic or parabolic problems, subject to the Robin boundary conditions on the cylindrical domains. Since the results stated in this section hold for any finite dimensional space, and in order to state and prove the results in the general settings, the notations used in this section are independent of those used in the previous sections.

Let $M \subseteq \mathbb{R}^N$ be a bounded domain with smooth boundary, L_0 and L_1 two numbers, with $L_0 < L_1$, and set the cylindrical domain $\Omega = M \times (L_0, L_1)$. For a spatial variable $x \in \Omega$, we denote by $x' \in M$ the first N components of x, that is $x' = (x^1, \cdots, x^N)$, and by z the last component of x, that is $z = x^{N+1}$. We divide the boundary into the lateral boundary Γ_ℓ, the upper boundary Γ_1 and the lower boundary Γ_0, that is $\partial\Omega = \Gamma_\ell \cup \Gamma_1 \cup \Gamma_0$, where

$$\Gamma_\ell = \partial M \times [L_0, L_1], \quad \Gamma_0 = M \times \{L_0\}, \quad \Gamma_1 = M \times \{L_1\}, \tag{A.1}$$

and set $\Gamma_{01} = \Gamma_0 \cup \Gamma_1$. Note that the unit outward normal vector field on the boundary $\partial\Omega$ is piecewise smooth. In view of this, we use different notations to distinguish the normal vectors on different portions of the boundary: we denote by n the unit outward normal vector on Γ_ℓ, while the unit outward normal vector on Γ_{01} is denoted by ν.

A.1. **Linear elliptic problem.**

In this subsection we establish the existence, uniqueness and regularity of weak solutions to the following elliptic problem:

$$\begin{cases} -\Delta_h u - \partial_z(a\partial_z u) + bu = f, & \text{in } \Omega, \\ \partial_n u + \alpha u = \varphi, & \text{on } \Gamma_\ell, \\ \partial_\nu u + \beta u = \psi, & \text{on } \Gamma_{01}, \end{cases} \tag{A.2}$$

where $a, b, \alpha, \beta, f, \varphi$ and ψ are given functions. As above ∇_h and Δ_h denote the gradient and Laplacian in the horizontal coordinates $(x^1, \ldots, x^N)$.

Weak solutions to (A.2) are defined as follows.

Definition A.1. *A function $u \in H^1(\Omega)$ is called a weak solution to the elliptic problem (A.2), if the following equation holds*

$$\int_\Omega (\nabla_h u \cdot \nabla_h \phi + a\partial_z u \partial_z \phi + bu\phi)dx + \int_{\Gamma_\ell} \alpha u \phi d\Gamma_\ell + \int_{\Gamma_{01}} a\beta u \phi dx' = \int_\Omega f\phi dx + \int_{\Gamma_\ell} \varphi \phi d\Gamma_\ell + \int_{\Gamma_{01}} a\psi\phi dx' \tag{A.3}$$

holds for any $\phi \in H^1(\Omega)$. *If* $b \equiv \alpha \equiv \beta \equiv 0$, *we ask for the additional constraint* $\int_\Omega u dx = 0$ *on* u, *and assume that the following compatibility condition holds*

$$\int_\Omega f dx + \int_{\Gamma_\ell} \varphi d\Gamma_\ell + \int_{\Gamma_{01}} a\psi dx' = 0. \tag{A.4}$$

Existence and uniqueness of weak solutions to (A.2) is stated in the following proposition, which can be proven in the standard way by the Lax-Milgram Lemma [23].

Proposition A.1 (Existence and uniqueness of weak solutions). *Assume that the functions* $a, b, \alpha, \beta, f, \varphi$ *and* ψ *satisfy*

$$a, b \in L^\infty(\Omega), \quad \alpha \in L^\infty(\Gamma_\ell), \quad \beta \in L^\infty(\Gamma_{01}), \quad \lambda \leq a \leq \Lambda, \quad 0 \leq b, \alpha, \beta \leq \Lambda, \tag{A.5}$$

$$f \in L^2(\Omega), \quad \varphi \in L^2(\Gamma_\ell), \quad \psi \in L^2(\Gamma_{01}), \tag{A.6}$$

for some positive numbers λ *and* Λ. *Then, there is a unique weak solution* u *to (A.2), satisfying*

$$\|u\|_{H^1(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\varphi\|_{L^2(\Gamma_\ell)} + \|\psi\|_{L^2(\Gamma_{01})}), \tag{A.7}$$

for a positive constant C *depending only on* Ω *and the coefficients* a, b, α *and* β.

We are going to study the H^2 regularity of the weak solutions established in Proposition A.1. To this end, we start with the corresponding result for the elliptic problem of a special case stated in the next lemma.

Lemma A.1 (H^2 regularity: a special case). *Assume that* a, f *and* φ *satisfy the assumptions in Proposition A.1, and let* $u \in H^1(\Omega)$ *be a weak solution to*

$$\begin{cases} -\Delta_h u - \partial_z(a\partial_z u) = f, & in\ \Omega, \\ \partial_n u = \varphi, & on\ \Gamma_\ell, \\ \partial_\nu u = 0, & on\ \Gamma_{01}. \end{cases} \tag{A.8}$$

Suppose in addition that $a \in C(\overline{\Omega}), \partial_z a \in L^\infty(\Omega)$ *and* $\varphi \in H^{\frac{1}{2}}(\Gamma_\ell)$.

Then, $u \in H^2(\Omega)$, *and the following estimate holds*

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|u\|_{H^1(\Omega)}), \tag{A.9}$$

where C *is a positive constant depending only on* Ω, *the modulus of continuity of* a, $\|a\|_{L^\infty(\Omega)}$ *and* $\|\partial_z a\|_{L^\infty(\Omega)}$.

Proof. Our strategy to prove the conclusion is as follows: we extend the weak solution appropriately to a larger domain, multiply it by some cutoff function, and show that the resultant satisfies some elliptic equations subject to the Neumann boundary conditions in a smooth domain, for which the classical regularity and elliptic estimates apply.

We first extend the boundary function φ to the whole domain Ω. By the trace theorem, one can find an extension $\varphi_{ext} \in H^1(\Omega)$ of φ to the domain Ω, such that $\varphi_{ext} = \varphi$ on Γ_ℓ, in the sense of trace, and

$$c\|\varphi_{ext}\|_{H^1(\Omega)} \leq \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} \leq C\|\varphi_{ext}\|_{H^1(\Omega)}, \tag{A.10}$$

for two positive constants c and C depending only on Ω. Thanks to this, in the rest of the proof we always suppose that the boundary function φ has already been extended to the whole domain Ω, such that $\varphi \in H^1(\Omega)$ and the above estimate holds with $\varphi_{ext} = \varphi$.

We first extend u, so that it satisfies a similar elliptic equation in the extended domain. To this end, let $\tilde{\Omega} = M \times (2L_0 - L_1, 2L_1 - L_0)$, and extend u in z evenly, with respect to the plane $z = L_0$ and $z = L_1$, that is we define

$$\tilde{u}(x', z) = \begin{cases} u(x', 2L_0 - z), & z \in (2L_0 - L_1, L_0), \\ u(x', z), & z \in [L_0, L_1], \\ u(x', 2L_1 - z), & z \in (L_1, 2L_1 - L_0), \end{cases} \tag{A.11}$$

for $x = (x', z) \in \tilde{\Omega}$. The extended functions $\tilde{a}, \tilde{f}, \tilde{\alpha}, \tilde{\varphi}$ are defined in the similar way as $\tilde{u}$. Then one can check that $\tilde{u} \in H^1(\tilde{\Omega})$ is a weak solution to

$$\begin{cases} -\Delta_h \tilde{u} - \partial_z(\tilde{a}\partial_z \tilde{u}) = \tilde{f}, & \text{in } \tilde{\Omega}, \\ \partial_n \tilde{u} = \tilde{\varphi}, & \text{on } \tilde{\Gamma}_\ell, \\ \partial_\nu \tilde{u} = 0, & \text{on } \tilde{\Gamma}_{01}, \end{cases} \tag{A.12}$$

where $\tilde{\Gamma}_\ell = \partial M \times (2L_0 - L_1, 2L_1 - L_0)$ and $\tilde{\Gamma}_{01} = M \times \{2L_0 - L_1, 2L_1 - L_0\}$.

We will introduce some appropriate truncation function of $\tilde{u}$, and show that the truncation satisfies some elliptic equation, subject to the Neumann boundary condition, in a smooth domain. To this end, let us first take a smooth bounded domain $\mathcal{O}$, with $\Omega \subseteq \mathcal{O} \subseteq \tilde{\Omega}$, $\partial\mathcal{O} \cap \tilde{\Omega} \cap \Gamma_{01} = \emptyset$ and $\partial\mathcal{O} \cap \tilde{\Omega} \cap \tilde{\Gamma}_{01} = \emptyset$, and choose a function $\eta \in C_0^\infty(\mathbb{R}^{N+1})$, with $\eta \equiv 1$ on Ω, and $\eta \equiv 0$ on $\tilde{\Omega} \setminus \mathcal{O}$. Since $\tilde{u}$ is a weak solution to (A.12), choosing $\phi\eta$ as a testing function yields

$$\int_{\tilde{\Omega}} [\nabla_h \tilde{u} \cdot \nabla_h(\phi\eta) + \tilde{a}\partial_z \tilde{u}\partial_z(\phi\eta)]dx = \int_{\tilde{\Omega}} \tilde{f}\phi\eta dx + \int_{\tilde{\Gamma}_\ell} \tilde{\varphi}\eta\phi d\Gamma_\ell, \tag{A.13}$$

for any $\phi \in H^1(\tilde{\Omega})$. Noticing that $\partial\mathcal{O} = (\partial\mathcal{O} \cap \tilde{\Omega}) \cup (\partial\mathcal{O} \cap \tilde{\Gamma}_s)$, and $\nabla_h \eta = 0$ on $\partial\mathcal{O} \cap \tilde{\Omega}$, integration by parts yields

$$\begin{aligned} \int_{\tilde{\Omega}} \nabla_h \tilde{u} \cdot \nabla_h(\phi\eta) dx &= \int_{\mathcal{O}} [\nabla_h(\tilde{u}\eta) \cdot \nabla_h \phi + (\nabla_h \cdot (\tilde{u}\nabla_h \eta) + \nabla_h \tilde{u} \cdot \nabla_h \eta)\phi]dx \\ &\quad - \int_{\partial\mathcal{O} \cap \tilde{\Gamma}_\ell} \tilde{u}\partial_n \eta\phi d\Gamma_\ell. \end{aligned} \tag{A.14}$$

Noticing that $\partial_z \eta = 0$ on $\partial\mathcal{O} \cap \tilde{\Omega}$, it follows from integration by parts that

$$\int_{\tilde{\Omega}} a\partial_z \tilde{u}\partial_z(\phi\eta)dx = \int_{\mathcal{O}} a\partial_z(\tilde{u}\eta)\partial_z \phi dx + \int_{\mathcal{O}} (\partial_z(a\partial_z \eta\tilde{u}) + a\partial_z \tilde{u}\partial_z \eta)\phi dx. \tag{A.15}$$

Plugging (A.14) and (A.15) into (A.13), and noticing that $\eta = 0$ on $\tilde{\Gamma}_\ell \setminus \partial\mathcal{O}$, we deduce

$$\begin{aligned} &\int_{\mathcal{O}} [\nabla_h(\tilde{u}\eta) \cdot \nabla_h \phi + a\partial_z(\tilde{u}\eta)\partial_z \phi]dx \\ = &\int_{\mathcal{O}} [\tilde{f}\eta - (\nabla_h \cdot (\tilde{u}\nabla_h \eta) + \nabla_h \tilde{u} \cdot \nabla_h \eta) - (\partial_z(a\partial_z \eta\tilde{u}) \\ &+ a\partial_z \tilde{u}\partial_z \eta)]\phi dx + \int_{\partial\mathcal{O} \cap \tilde{\Gamma}_\ell} (\tilde{\varphi}\eta + \tilde{u}\partial_n \eta)\phi d\Gamma_\ell, \end{aligned} \tag{A.16}$$

for any $\phi \in H^1(\tilde{\Omega})$.

Define the truncation $U = \tilde{u}\eta$, and denote

$$F = \tilde{f}\eta - (\nabla_h \cdot (\tilde{u}\nabla_h\eta) + \nabla_h\tilde{u} \cdot \nabla_h\eta) - (\partial_z(a\partial_z\eta\tilde{u}) + a\partial_z\tilde{u}\partial_z\eta), \tag{A.17}$$

and

$$\Phi = \begin{cases} \tilde{u}\partial_n\eta + \tilde{\varphi}\eta, & \text{on } \partial\mathcal{O} \cap \tilde{\Gamma}_\ell, \\ 0, & \text{on } \partial\mathcal{O} \setminus \tilde{\Gamma}_\ell. \end{cases} \tag{A.18}$$

Then, noticing that any function from $H^1(\mathcal{O})$ can be extended to be a function in $H^1(\tilde{\Omega})$, (A.16) implies that $U \in H^1(\mathcal{O})$ is a weak solution to

$$\begin{cases} -\Delta_h U - \partial_z(\tilde{a}\partial_z U) = F, & \text{in } \mathcal{O}, \\ \partial_{\mathcal{N}} U = \Phi, & \text{on } \partial\mathcal{O}, \end{cases} \tag{A.19}$$

where $\mathcal{N}$ is the unit outward normal vector at the boundary $\partial\mathcal{O}$.

Since $\mathcal{O}$ is a smooth domain, one can now apply the classic regularity and elliptic estimates to the truncation U. Then, by the assumptions, one can check that $F \in L^2(\Omega)$, and $\Phi \in H^{\frac{1}{2}}(\partial\mathcal{O})$, by the trace theorem. Therefore, it follows from the regularity and elliptic estimates for the elliptic equations subject to the Neumann boundary conditions in the bounded smooth domains, that $U \in H^2(\mathcal{O})$ and the following estimate holds

$$\|U\|_{H^2(\mathcal{O})} \leq C(\|F\|_{L^2(\mathcal{O})} + \|\Phi\|_{H^{\frac{1}{2}}(\partial\mathcal{O})} + \|U\|_{H^1(\mathcal{O})}), \tag{A.20}$$

where the constant C depends only on $\mathcal{O}$, the modulus of continuity of a, λ and $\|\partial_z a\|_{L^\infty(\mathcal{O})}$.

We can now derive the regularity and elliptic estimate for u from those for U. Recalling that $\eta \equiv 1$ on Ω, one obtains $u \in H^2(\Omega)$, and it is obvious that

$$\|u\|_{H^2(\Omega)} \leq \|U\|_{H^2(\mathcal{O})}. \tag{A.21}$$

Note that

$$\|F\|_{L^2(\mathcal{O})} \leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)}), \tag{A.22}$$

for a positive constant C depending only on Ω, $\|a\|_{L^\infty(\Omega)}$ and $\|\partial_z a\|_{L^\infty(\Omega)}$,

$$\|U\|_{H^1(\mathcal{O})} \leq C\|u\|_{H^1(\Omega)}, \tag{A.23}$$

for a positive constant C depending only on Ω, and by the trace inequality

$$\begin{aligned} \|\Phi\|_{H^{\frac{1}{2}}(\mathcal{O})} &\leq C\|\Phi\|_{H^1(\mathcal{O})} \leq C(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^1(\Omega)}) \\ &\leq C(\|u\|_{H^1(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)}), \end{aligned} \tag{A.24}$$

for a positive constant C depending only on Ω. In the above inequality, one recalls that the boundary function φ has already been extended to the whole domain, at the beginning of the proof.

Thanks to the above estimates, it follows from (A.20) that

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|u\|_{H^1(\Omega)}), \tag{A.25}$$

for a positive constant C depending only on Ω, the modulus of continuity of a, λ, $\|a\|_{L^\infty(\Omega)}$ and $\|\partial_z a\|_{L^\infty(\mathcal{O})}$. This completes the proof of Lemma A.1. $\square$

Proposition A.2 (H^2 regularity: the general case)**.** *Assume, in addition to the assumptions on a, f, φ in Lemma A.1, that*

$$b \in L^\infty(\Omega), \quad \alpha, \beta \in W^{1,\infty}(\Omega), \quad \psi \in H^{\frac{1}{2}}(\Gamma_{01}), \tag{A.26}$$

where α and β have been extended to the whole domain Ω.

Then, the unique weak solution u stated in Proposition A.1 belongs to $H^2(\Omega)$, and the following estimate holds

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{01})} + \|u\|_{H^1(\Omega)}), \tag{A.27}$$

for a positive constant C depending only on Ω, the modulus continuity of a, λ, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|b\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

Proof. We set

$$\varphi_0 = \varphi - \alpha u, \quad \psi_0 = \psi - \beta u, \quad f_0 = f - bu. \tag{A.28}$$

Then it is obvious that u is a weak solution to

$$\begin{cases} -\Delta_h u - \partial_z(a\partial_z u) = f_0, & \text{in } \Omega, \\ \partial_n u = \varphi_0, & \text{on } \Gamma_\ell, \\ \partial_\nu u = \psi_0, & \text{on } \Gamma_{01}. \end{cases} \tag{A.29}$$

Since $\psi_0 \in H^{\frac{1}{2}}(\Gamma_{01})$ by the trace theorem there is a function $\Psi_0 \in H^2(\Omega)$, such that $\partial_\nu \Psi_0 = \psi_0$ on Γ_{01}, and $\|\Psi_0\|_{H^2(\Omega)} \leq C\|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}$ for a positive constant C depending only on Ω. Setting $u_1 = u - \Psi_0$, one can easily check that u_1 is a weak solution to the following elliptic problem

$$\begin{cases} -\Delta_h u_1 - \partial_z(a\partial_z u_1) = f_1, & \text{in } \Omega, \\ \partial_n u_1 = \varphi_1, & \text{on } \Gamma_\ell, \\ \partial_\nu u_1 = 0, & \text{on } \Gamma_{01}, \end{cases} \tag{A.30}$$

where f_1 and φ_1 are given by

$$f_1 = f_0 + \Delta_h \Psi_0 + \partial_z(a\partial_z \Psi_0), \quad \varphi_1 = \varphi_0 - \partial_n \Psi_0. \tag{A.31}$$

Note that $f_1 \in L^2(\Omega)$ and $\varphi_1 \in H^{\frac{1}{2}}(\Gamma_\ell)$, by Lemma A.1, one has $u_1 \in H^2(\Omega)$, and the following estimate holds

$$\|u_1\|_{H^2(\Omega)} \leq C(\|f_1\|_{L^2(\Omega)} + \|\varphi_1\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|u_1\|_{H^1(\Omega)}), \tag{A.32}$$

for a positive constant C depending only on Ω, the modulus continuity of a, λ, $\|a\|_{L^\infty(\Omega)}$ and $\|\partial_z a\|_{L^\infty(\Omega)}$. Thus, recalling that $\Psi_0 \in H^2(\Omega)$, it is clear that $u = u_1 + \Psi_0 \in H^2(\Omega)$. Recalling that $\|\Psi_0\|_{H^2(\Omega)} \leq C\|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}$, for a positive constant C depending only on Ω, one deduces

$$\begin{aligned} \|u\|_{H^2(\Omega)} &\leq \|u_1\|_{H^2(\Omega)} + \|\Psi_0\|_{H^2(\Omega)} \leq \|u_1\|_{H^2(\Omega)} + C\|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}, && \text{(A.33)} \\ \|u_1\|_{H^1(\Omega)} &\leq \|u\|_{H^1(\Omega)} + \|\Psi_0\|_{H^1(\Omega)} \leq \|u\|_{H^1(\Omega)} + C\|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}, && \text{(A.34)} \\ \|\varphi_1\|_{H^{\frac{1}{2}}(\Gamma_\ell)} &\leq \|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\partial_n \Psi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} \leq \|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + C\|\Psi_0\|_{H^2(\Omega)} \end{aligned}$$

$$\leq C(\|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}), \tag{A.35}$$

for a positive constant C depending only on Ω, and

$$\begin{aligned} \|f_1\|_{L^2(\Omega)} &\leq \|f\|_{L^2(\Omega)} + C\|u\|_{H^1(\Omega)} + C\|\Psi_0\|_{H^2(\Omega)} \\ &\leq C(\|f\|_{L^2(\Omega)} + \|u\|_{H^1(\Omega)} + \|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}), \end{aligned} \tag{A.36}$$

for a positive constant C depending only on Ω, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$ and $\|b\|_{L^\infty(\Omega)}$. Thanks to the above estimates, one derives from (A.32) that

$$\|u\|_{H^2(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})} + \|u\|_{H^1(\Omega)}), \tag{A.37}$$

for a positive constant C depending only on Ω, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$ and $\|b\|_{L^\infty(\Omega)}$. One still need to estimate $\|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})}$, which is done as follows. By the trace inequality, one has

$$\begin{aligned} &\|\varphi_0\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi_0\|_{H^{\frac{1}{2}}(\Gamma_{01})} \\ \leq\ &\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{01})} + \|\alpha u\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\beta u\|_{H^{\frac{1}{2}}(\Gamma_{01})} \\ \leq\ &\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{01})} + C(\|\alpha u\|_{H^1(\Omega)} + \|\beta u\|_{H^1(\Omega)}) \\ \leq\ &\|\varphi\|_{H^{\frac{1}{2}}(\Gamma_\ell)} + \|\psi\|_{H^{\frac{1}{2}}(\Gamma_{01})} + C\|u\|_{H^1(\Omega)}, \end{aligned} \tag{A.38}$$

for a positive constant C depending only on Ω, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$. Plugging the above estimate into (A.37) yields the conclusion. ☐

A.2. **Linear parabolic equations.** Given a positive time $\mathcal{T}$, set $Q_{\mathcal{T}} = \Omega \times (0, \mathcal{T})$. Consider the parabolic problem

$$\begin{cases} \partial_t u + \mathcal{L}u = f, & \text{in } Q_{\mathcal{T}}, \\ \mathcal{B}u = \Phi, & \text{on } \partial\Omega \times (0, \mathcal{T}), \\ u(\cdot, 0) = u_0, & \text{on } \Omega, \end{cases} \tag{A.39}$$

where the elliptic operator $\mathcal{L}$ is given by

$$\mathcal{L}u = -\Delta_h u - \partial_z(a\partial_z u), \tag{A.40}$$

the boundary operator $\mathcal{B}$ is given by

$$\mathcal{B}u = \begin{cases} \partial_n u + \alpha u, & \text{on } \Gamma_\ell, \\ \partial_\nu u + \beta u, & \text{on } \Gamma_{01}, \end{cases} \tag{A.41}$$

and the boundary function Φ is given by

$$\Phi = \begin{cases} \varphi, & \text{on } \Gamma_\ell, \\ \psi, & \text{on } \Gamma_{01}, \end{cases} \tag{A.42}$$

for some functions φ and ψ.

Throughout this subsection, we assume that the coefficients a, α and β in the elliptic operator $\mathcal{L}$ and the boundary operator $\mathcal{B}$ are independent of the time variable t, while the nonhomogeneous functions φ and ψ are allowed to depend on t. Assume that the coefficients a, α and β satisfy

$$a \in C(\overline{\Omega}), \quad \partial_z a \in L^\infty(\Omega), \quad \lambda \leq a(x) \leq \Lambda, \quad 0 \leq \alpha, \beta \in W^{1,\infty}(\Omega), \tag{A.43}$$

for some positive constants λ and Λ, where the functions α, β, defined on the boundary $\partial\Omega$, have been extended to the whole domain Ω. We assume moreover that the boundary functions φ, ψ and Φ satisfy

$$\varphi \in L^2(0, T; H^{\frac{1}{2}}(\Gamma_\ell)), \quad \psi \in L^2(0, T; H^{\frac{1}{2}}(\Gamma_{01})), \quad \partial_t \Phi \in L^2(\partial\Omega \times (0, T)). \tag{A.44}$$

We are going to prove the existence and uniqueness of weak and strong solutions (see the definitions below) to (A.39).

Definition A.2. *Given a positive time $\mathcal{T} \in (0, \infty)$ and a function $u_0 \in L^2(\Omega)$. Assume that (A.43) and (A.44) hold, and $f \in L^2(Q_\mathcal{T})$. A function u is called a weak solution to (A.39), if $u \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H^1(\Omega))$, and the following integral equality holds*

$$\int_0^T [-(u, \partial_t \xi) + \langle u, \xi \rangle_a] dt = \int_0^T b(f, \varphi, \psi, \xi) dt + (u_0, \xi(\cdot, 0)), \tag{A.45}$$

for any $\xi \in H^1(Q_\mathcal{T})$, with $\xi(\cdot, \mathcal{T}) \equiv 0$, where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product,

$$\langle u, \xi \rangle_a = \int_\Omega (\nabla_h u \cdot \nabla \xi + a \partial_z u \partial_z \xi) dx + \int_{\Gamma_\ell} \alpha u \xi d\Gamma_\ell + \int_{\Gamma_{01}} a \beta u \xi dx', \tag{A.46}$$

and

$$b(f, \varphi, \psi, \xi) = \int_\Omega f \xi dx + \int_{\Gamma_\ell} \varphi \xi d\Gamma_\ell + \int_{\Gamma_{01}} \psi \xi dx'. \tag{A.47}$$

Definition A.3. *Given a positive time $\mathcal{T} \in (0, \infty)$ and a function $u_0 \in H^1(\Omega)$. Assume that (A.43) and (A.44) hold, and let $f \in L^2(Q_\mathcal{T})$. A function u is called a strong solution to (A.39), if*

$$u \in C([0, \mathcal{T}]; H^1(\Omega)) \cap L^2(0, \mathcal{T}; H^2(\Omega)), \quad \partial_t u \in L^2(0, \mathcal{T}; L^2(\Omega)), \tag{A.48}$$

the equation in (A.39) is satisfied, a.e. in $Q_\mathcal{T}$, the boundary condition in (A.39) is satisfied in the sense of trace, and the initial condition in (A.39) is fulfilled.

Let us first transform the nonhomogeneous boundary value problem to the homogeneous one. For each $t \in [0, \mathcal{T}]$, we define $\mathcal{U}_\Phi(\cdot, t)$ as the unique solution to

$$\begin{cases} \mathcal{L}\mathcal{U}_\Phi = 0, & \text{in } \Omega, \\ \mathcal{B}\mathcal{U}_\Phi = \Phi, & \text{on } \partial\Omega. \end{cases} \tag{A.49}$$

Noticing that $\Phi \in C([0, \mathcal{T}]; L^2(\partial\Omega))$, by applying Proposition A.1 and Proposition A.2, one can see that $\mathcal{U}_\Phi \in C([0, \mathcal{T}]; H^1(\Omega)) \cap L^2(0, \mathcal{T}; H^2(\Omega))$, and

$$\|\mathcal{U}_\Phi\|_{L^\infty(0, \mathcal{T}; H^1(\Omega))} + \|\mathcal{U}_\Phi\|_{L^2(0, \mathcal{T}; H^2(\Omega))} \leq C\big(\|\varphi\|_{L^2(0, \mathcal{T}; H^{\frac{1}{2}}(\Gamma_\ell))} + \|\psi\|_{L^2(0, \mathcal{T}; H^{\frac{1}{2}}(\Gamma_{01}))}\big),$$

for a positive constant C depending only on Ω, λ, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$. Note that $\partial_t\mathcal{U}_\Phi$ satisfies

$$\begin{cases} \mathcal{L}\partial_t\mathcal{U}_\Phi = 0, & \text{in } \Omega, \\ \mathcal{B}\partial_t\mathcal{U}_\Phi = \partial_t\Phi, & \text{on } \partial\Omega, \end{cases} \tag{A.50}$$

by Proposition A.1, it follows

$$\|\partial_t\mathcal{U}_\Phi\|_{L^2(0,T;H^1(\Omega))} \leq C\|\partial_t\Phi\|_{L^2(0,T;L^2(\partial\Omega))}. \tag{A.51}$$

Therefore, we have

$$\begin{aligned} &\|\mathcal{U}_\Phi\|_{L^\infty(0,\mathcal{T};H^1(\Omega))} + \|\mathcal{U}_\Phi\|_{L^2(0,\mathcal{T};H^2(\Omega))} + \|\partial_t\mathcal{U}_\Phi\|_{L^2(0,\mathcal{T};H^1(\Omega))} \\ \leq\ & C\big(\|\varphi\|_{L^2(0,\mathcal{T};H^{\frac12}(\Gamma_\ell))} + \|\psi\|_{L^2(0,\mathcal{T};H^{\frac12}(\Gamma_{01}))} + \|\partial_t\Phi\|_{L^2(0,\mathcal{T};L^2(\partial\Omega))}\big), \end{aligned} \tag{A.52}$$

for a positive constant C depending only on Ω, λ, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

Suppose that u is a weak (strong) solution to (A.39), by setting $v = u - \mathcal{U}_\Phi$, one can easily verify that v is a weak (strong) solution to the following problem

$$\begin{cases} \partial_t v + \mathcal{L}v = g, & \text{in } Q_\mathcal{T}, \\ \mathcal{B}v = 0, & \text{on } \partial\Omega\times(0,\mathcal{T}), \\ v(\cdot,0) = v_0, & \text{on } \Omega, \end{cases} \tag{A.53}$$

where

$$g = f - \partial_t\mathcal{U}_\Phi, \quad v_0 = u_0 - \mathcal{U}_\Phi(\cdot,0). \tag{A.54}$$

Conversely, if v is a weak (strong) solution to (A.53), the $u = v + \mathcal{U}_\Phi$ is a weak (strong) solution to (A.39). Therefore, to prove the existence and uniqueness of weak (strong) solutions to (A.39), it suffices to prove the corresponding results for (A.53).

We are going to construct a sequence of approximated solutions v_N to the parabolic problem (A.53) by discretizing in time. Let $g \in L^2(Q_\mathcal{T})$. We fix an arbitrary positive integer N, and let $h = \frac{\mathcal{T}}{N}$. We set

$$g^{k+1} = \frac{1}{h}\int_{kh}^{(k+1)h} g(t)dt, \quad k = 0,1,\cdots,N-1. \tag{A.55}$$

Due to the assumption, it is obvious that $g^{k+1} \in L^2(\Omega)$ for $k = 0,1,\cdots,N-1$. It follows from the Minkowski and Hölder inequalities that

$$\begin{aligned} h\sum_{k=0}^{N-1}\|g^{k+1}\|_{L^2(\Omega)} &= \sum_{k=0}^{N-1}\left\|\int_{kh}^{(k+1)h} g(t)dt\right\|_{L^2(\Omega)} \leq \sum_{k=0}^{N-1}\int_{kh}^{(k+1)h}\|g(\cdot,t)\|_{L^2(\Omega)}dt \\ &= \int_0^\mathcal{T}\|g(\cdot,t)\|_{L^2(\Omega)}dt = \|g\|_{L^1(0,\mathcal{T};L^2(\Omega))} \leq \sqrt{\mathcal{T}}\|g\|_{L^2(Q_\mathcal{T})}, \end{aligned} \tag{A.56}$$

and

$$h\sum_{k=0}^{N-1}\|g^{k+1}\|_{L^2(\Omega)}^2 = \frac{1}{h}\sum_{k=0}^{N-1}\left\|\int_{kh}^{(k+1)h} g(t)dt\right\|_{L^2(\Omega)}^2$$

$$
\begin{aligned}
&\le \frac{1}{h}\sum_{k=0}^{N-1}\left(\int_{kh}^{(k+1)h}\|g(\cdot,t)\|_{L^2(\Omega)}dt\right)^2 \\
&\le \int_0^{\mathcal{T}}\|g(\cdot,t)\|^2_{L^2(\Omega)}dt = \|g\|^2_{L^2(Q_{\mathcal{T}})}. \qquad \text{(A.57)}
\end{aligned}
$$

We set $v^0 = v_0$ and define v^{k+1} for $k = 0, 1, \cdots, N-1$ successively as the unique solution to the elliptic problem

$$
\begin{cases}
\frac{v^{k+1}-v^k}{h} + \mathcal{L}v^{k+1} = g^{k+1}, & \text{in } \Omega, \\
\mathcal{B}v^{k+1} = 0, & \text{on } \partial\Omega.
\end{cases} \qquad \text{(A.58)}
$$

According to Proposition A.1 and Proposition A.2 there is a unique solution $v^{k+1} \in H^2(\Omega)$ to (A.58) for $k = 0, 1, \cdots, N-1$. Define the approximate solution v_N as

$$
v_N(t) = \sum_{k=0}^{N-1}\chi_{[kh,(k+1)h)}(t)v^{k+1}, \quad t \in [0, T). \qquad \text{(A.59)}
$$

Some a priori estimates on v_N are stated in the following lemma.

Lemma A.2. *Assume that (A.43) holds. Set $v^0 = v_0 \in L^2(\Omega)$, and let v^{k+1}, $k = 0, 1, \cdots, N-1$, be the functions defined by (A.58), and v_N by (A.59). Then, the following estimates hold:*

(i) There is a positive constant C depending only on λ and T, such that

$$
\|v_N\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} + \|v_N\|_{L^2(0,\mathcal{T};H^1(\Omega))} \le C(\|g\|_{L^2(Q_{\mathcal{T}})} + \|v_0\|_{L^2(\Omega)}); \qquad \text{(A.60)}
$$

(ii) If we assume in addition that $v_0 \in H^1(\Omega)$, then we have

$$
\|v_N\|_{L^\infty(0,\mathcal{T};H^1(\Omega))} + \|v_N\|_{L^2(0,\mathcal{T};H^2(\Omega))} \le C(\|g\|_{L^2(Q_{\mathcal{T}})} + \|v_0\|_{H^1(\Omega)}), \qquad \text{(A.61)}
$$

for a positive constant C depending only on Ω, λ, T, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

Proof. (i) Testing (A.58) by v^{k+1}, and summing the results with respect to k from 0 to M yields

$$
h\sum_{k=0}^{M}\langle v^{k+1}, v^{k+1}\rangle_a + \sum_{k=0}^{M}(v^{k+1} - v^k, v^{k+1}) = h\sum_{k=0}^{M}(g^{k+1}, v^{k+1}), \qquad \text{(A.62)}
$$

for $M = 0, 1, \cdots, N-1$. Straightforward calculations yield

$$
\begin{aligned}
\sum_{k=0}^{M}(v^{k+1} - v^k, v^{k+1}) &= \frac{1}{2}\left(\|v^{M+1}\|^2_{L^2(\Omega)} - \|v_0\|^2_{L^2(\Omega)} + \sum_{k=0}^{M}\|v^{k+1} - v^k\|^2_{L^2(\Omega)}\right) \\
&\ge \frac{1}{2}\left(\|v^{M+1}\|^2_{L^2(\Omega)} - \|v_0\|^2_{L^2(\Omega)}\right). \qquad \text{(A.63)}
\end{aligned}
$$

It follows from the Cauchy inequality that

$$
\sum_{k=0}^{M}(g^{k+1}, v^{k+1}) \le \sup_{0\le k\le M}\|v^{k+1}\|_{L^2(\Omega)}\sum_{k=0}^{M}\|g^{k+1}\|_{L^2(\Omega)}
$$

$$\leq \sup_{0\leq k\leq N-1}\|v^{k+1}\|_{L^2(\Omega)}\sum_{k=0}^{N-1}\|g^{k+1}\|_{L^2(\Omega)}. \tag{A.64}$$

Thanks to (A.63) and (A.64) it follows from (A.62) using (A.56) that

$$\begin{aligned}
& h\sum_{k=0}^{M}\langle v^{k+1},a^{k+1}\rangle_a+\frac{1}{2}\|v^{M+1}\|_{L^2(\Omega)}^2 \\
\leq\ & h\sup_{0\leq k\leq N-1}\|v^{k+1}\|_{L^2(\Omega)}\sum_{k=0}^{N-1}\|g^{k+1}\|_{L^2(\Omega)}+\frac{1}{2}\|v_0\|_{L^2(\Omega)}^2 \\
\leq\ & \sup_{0\leq k\leq N-1}\|v^{k+1}\|_{L^2(\Omega)}\sqrt{\mathcal{T}}\|g\|_{L^2(Q_\mathcal{T})}+\frac{1}{2}\|v_0\|_{L^2(\Omega)}^2,
\end{aligned} \tag{A.65}$$

for $M=0,1,\cdots,N-1$. Setting

$$A=\sup_{0\leq M\leq N-1}\left(h\sum_{k=0}^{M}\langle v^{k+1},v^{k+1}\rangle_a+\frac{1}{2}\|v^{M+1}\|_{L^2(\Omega)}^2\right), \tag{A.66}$$

it follows from the Young inequality and (A.65) that

$$\begin{aligned}
A\ \leq\ & \sqrt{2A}\sqrt{\mathcal{T}}\|g\|_{L^2(Q_\mathcal{T})}+\frac{1}{2}\|v_0\|_{L^2(\Omega)}^2 \\
\leq\ & \frac{1}{2}A+\mathcal{T}\|g\|_{L^2(Q_\mathcal{T})}^2+\frac{1}{2}\|v_0\|_{L^2(\Omega)}^2,
\end{aligned} \tag{A.67}$$

and thus

$$A\leq 2\mathcal{T}\|g\|_{L^2(Q_\mathcal{T})}^2+\|v_0\|_{L^2(\Omega)}^2. \tag{A.68}$$

Thanks to this, and recalling that $\langle v,v\rangle_a\geq\lambda\|\nabla v\|_{L^2(\Omega)}^2$, one obtains

$$h\lambda\sum_{k=0}^{N-1}\|\nabla v^{k+1}\|_{L^2(\Omega)}^2+\sup_{0\leq k\leq N-1}\|v^{k+1}\|_{L^2(\Omega)}^2\leq 8\mathcal{T}\|g\|_{L^2(Q_\mathcal{T})}^2+4\|v_0\|_{L^2(\Omega)}^2, \tag{A.69}$$

from which, recalling the definition of v_N, (i) follows.

(ii) Testing (A.58) by $v^{k+1}-v^k$, and summing the resultants with respect to k from 0 to M, for $M=0,1,\cdots,N-1$, yields

$$\sum_{k=0}^{M}\left(\langle v^{k+1},v^{k+1}-v^k\rangle_a+\frac{1}{h}\|v^{k+1}-v^k\|_{L^2(\Omega)}^2\right)=\sum_{k=0}^{M}(g^{k+1},v^{k+1}-v^k), \tag{A.70}$$

from which, noticing that

$$\sum_{k=0}^{M}\langle v^{k+1},v^{k+1}-v^k\rangle_a=\frac{1}{2}\left(\langle v^{M+1},v^{M+1}\rangle_a-\langle v_0,v_0\rangle_a+\sum_{k=0}^{M}\langle v^{k+1}-v^k,v^{k+1}-v^k\rangle_a\right), \tag{A.71}$$

one obtains

$$\langle v^{M+1},v^{M+1}\rangle_a+\frac{2}{h}\sum_{k=0}^{M}\|v^{k+1}-v^k\|_{L^2(\Omega)}^2\leq 2\sum_{k=0}^{M}(g^{k+1},v^{k+1}-v^k)+\langle v_0,v_0\rangle_a, \tag{A.72}$$

for $M = 0, 1, \cdots, N-1$. Applying the Cauchy inequality to the righthand side of the above equality, and using (A.57), one obtains

$$\begin{aligned} 2\sum_{k=0}^{M}(g^{k+1}, v^{k+1} - v^k) &\leq 2\sum_{k=0}^{M}\|g^{k+1}\|_{L^2(\Omega)}\|v^{k+1} - v^k\|_{L^2(\Omega)} \\ &\leq \frac{1}{h}\sum_{k=0}^{M}\|v^{k+1} - v^k\|^2_{L^2(\Omega)} + h\sum_{k=0}^{M}\|g^{k+1}\|^2_{L^2(\Omega)} \\ &\leq \frac{1}{h}\sum_{k=0}^{M}\|v^{k+1} - v^k\|^2_{L^2(\Omega)} + \|g\|^2_{L^2(Q_\mathcal{T})}, \end{aligned} \tag{A.73}$$

which, plugged into the previous inequality, yields

$$\langle v^{M+1}, v^{M+1}\rangle_a + h\sum_{k=0}^{M}\left\|\frac{v^{k+1} - v^k}{h}\right\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(Q_\mathcal{T})} + \langle v_0, v_0\rangle_a, \tag{A.74}$$

for $M = 0, 1, \cdots, N-1$, and thus

$$\sup_{1\leq k\leq N}\langle v^k, v^k\rangle_a + h\sum_{k=0}^{N-1}\left\|\frac{v^{k+1} - v^k}{h}\right\|^2_{L^2(\Omega)} \leq \|g\|^2_{L^2(Q_\mathcal{T})} + \langle v_0, v_0\rangle_a. \tag{A.75}$$

Combining (A.57), (A.69) and (A.75), and applying the H^2 estimate to the elliptic operator $\mathcal{L}$, there is a positive constant C depending only on Ω, λ, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$, such that

$$\begin{aligned} h\sum_{k=1}^{N-1}\|v^{k+1}\|^2_{H^2(\Omega)} &\leq Ch\sum_{k=1}^{N-1}(\|\mathcal{L}v^{k+1}\|^2_{L^2(\Omega)} + \|v^{k+1}\|^2_{H^1(\Omega)}) \\ &\leq Ch\sum_{k=1}^{N-1}\left(\left\|g^{k+1} - \frac{v^{k+1} - v^k}{h}\right\|^2_{L^2(\Omega)} + \|v^{k+1}\|^2_{H^1(\Omega)}\right) \\ &\leq C(\mathcal{T}^2 + 1)(\|g\|^2_{L^2(Q_\mathcal{T})} + \|v_0\|^2_{H^1(\Omega)}), \end{aligned} \tag{A.76}$$

from which, recalling the definition of v_N and (A.75), (ii) follows. □

The existence and uniqueness of weak and strong solutions to (A.53) is now stated in the following proposition.

Proposition A.3. *Given a positive time $\mathcal{T} \in (0, \infty)$ and the initial data $v_0 \in L^2(\Omega)$, we assume that (A.43) holds, and that $g \in L^2(Q_\mathcal{T})$.*

Then, there is a unique weak solution to (A.53), satisfying

$$\|v\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} + \|v\|_{L^2(0,\mathcal{T};H^1(\Omega))} \leq C(\|g\|_{L^2(Q_\mathcal{T})} + \|v_0\|_{L^2(\Omega)}),$$

for a positive constant C depending only on λ and $\mathcal{T}$.

Moreover, if we assume in addition that $v_0 \in H^1(\Omega)$, then the unique weak solution is a strong one, and satisfies

$$\|v\|_{L^\infty(0,\mathcal{T};H^1(\Omega))} + \|v\|_{L^2(0,\mathcal{T};H^2(\Omega))} + \|\partial_t v\|_{L^2(Q_\mathcal{T})} \leq C(\|g\|_{L^2(Q_\mathcal{T})} + \|v_0\|_{H^1(\Omega)}), \tag{A.77}$$

for a positive constant C depending only on Ω, λ, $\mathcal{T}$, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

Proof. Let g^{k+1}, $k=0,1,\cdots,N-1$, be the functions defined by (A.55), and set

$$g_N(t)=\sum_{k=0}^{N-1}\chi_{[kh,(k+1)h)}(t)g^{k+1}. \tag{A.78}$$

Recalling that $g\in L^2(0,\mathcal{T};L^2(\Omega))$, one can verify that $g_N\to g$, in $L^2(0,\mathcal{T};L^2(\Omega))$, as $N\to\infty$. Let v^{k+1}, $k=0,1,\cdots,N-1$, be the functions defined by (A.58). Let v_N be the function given by (A.59).

By Lemma A.2, the following estimate holds

$$\|v_N\|_{L^\infty(0,\mathcal{T};L^2(\Omega))}+\|v_N\|_{L^2(0,\mathcal{T};H^1(\Omega))}\leq C(\|g\|_{L^2(Q_\mathcal{T})}+\|v_0\|_{L^2(\Omega)}), \tag{A.79}$$

for a positive constant C depending only on λ and $\mathcal{T}$; and if we assume in addition that $v_0\in H^1(\Omega)$, then the following additional estimate holds

$$\|v_N\|_{L^\infty(0,\mathcal{T};H^1(\Omega))}+\|v_N\|_{L^2(0,\mathcal{T};H^2(\Omega))}\leq C(\|g\|_{L^2(Q_\mathcal{T})}+\|v_0\|_{H^1(\Omega)}), \tag{A.80}$$

for a positive constant C depending only on Ω, λ, $\mathcal{T}$, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$. Thanks to the above estimates, there is a subsequence, still denoted by $\{v_N\}_{N=1}^\infty$, and a function v, with $v\in L^\infty(0,\mathcal{T};L^2(\Omega))\cap L^2(0,\mathcal{T};H^1(\Omega))$, such that

$$\|v\|_{L^\infty(0,\mathcal{T};L^2(\Omega))}+\|v\|_{L^2(0,\mathcal{T};H^1(\Omega))}\leq C(\|g\|_{L^2(Q_\mathcal{T})}+\|v_0\|_{L^2(\Omega)}), \tag{A.81}$$

for a positive constant C depending only on λ and $\mathcal{T}$, and

$$v_N\overset{*}{\rightharpoonup}v \text{ in } L^\infty(0,\mathcal{T};L^2(\Omega)), \text{ and } v_N\rightharpoonup v \text{ in } L^2(0,\mathcal{T};H^1(\Omega)). \tag{A.82}$$

Moreover, if we assume in addition that $v_0\in H^1(\Omega)$, then the above v has the additional regularity that $v\in L^2(0,\mathcal{T};H^2(\Omega))\cap L^\infty(0,\mathcal{T};H^1(\Omega))$, and

$$\|v\|_{L^\infty(0,\mathcal{T};H^1(\Omega))}+\|v\|_{L^2(0,\mathcal{T};H^2(\Omega))}\leq C(\|g\|_{L^2(Q_\mathcal{T})}+\|v_0\|_{H^1(\Omega)}), \tag{A.83}$$

for a positive constant C depending only on Ω, λ, $\mathcal{T}$, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

We take an arbitrary function $\eta\in C^1(\overline{Q_\mathcal{T}})$ with $\eta(\cdot,t)\equiv0$ when t is close enough to $\mathcal{T}$, and set $\eta^k=\eta(\cdot,hk)$, $k=0,1,\cdots,N-1$. Then, for large enough N, one has $\eta^{N-1}=\eta(\cdot,(1-\frac1N)\mathcal{T})\equiv0$. Define two functions η_N and $\bar\partial_t\eta_N$ as

$$\eta_N=\sum_{k=0}^{N-1}\chi_{[kh,(k+1)h)}(t)\eta^k,\quad \bar\partial_t\eta_N=\sum_{k=0}^{N-1}\chi_{[kh,(k+1)h)}(t)\frac{\eta^{k+1}-\eta^k}{h}. \tag{A.84}$$

Then, using $\eta\in C^1(\overline{Q_\mathcal{T}})$, one can show that

$$\eta_N\to\eta, \text{ in } L^2(0,\mathcal{T};H^1(\Omega)),\quad\text{and}\quad \bar\partial_t\eta_N\to\partial_t\eta, \text{ in } L^2(Q_\mathcal{T}). \tag{A.85}$$

Taking η^k as the testing function, and summing the resultant with respect to k from 0 to $N-1$ yields

$$h\sum_{k=0}^{N-1}\langle v^{k+1},\eta^k\rangle_a+\sum_{k=0}^{N-1}(v^{k+1}-v^k,\eta^k)=h\sum_{k=0}^{N-1}(g^{k+1},\eta^k). \tag{A.86}$$

Direct calculations yield

$$\begin{aligned}\sum_{k=0}^{N-1}(v^{k+1}-v^k,\eta^k) &= (v^N,\eta^{N-1})-(v_0,\eta^0)-\sum_{k=1}^{N-1}(v^k,\eta^k-\eta^{k-1})\\ &= -(v_0,\eta^0)-h\sum_{k=1}^{N-1}(v^k,\frac{\eta^k-\eta^{k-1}}{h})\\ &= -(v_0,\eta^0)-h\sum_{k=0}^{N-2}(v^{k+1},\frac{\eta^{k+1}-\eta^k}{h}),\end{aligned} \tag{A.87}$$

from which, noticing that $\eta^N=\eta^{N-1}\equiv 0$, one obtains

$$\sum_{k=0}^{N-1}(v^{k+1}-v^k,\eta^k)=-(v_0,\eta^0)-h\sum_{k=0}^{N-1}(v^{k+1},\frac{\eta^{k+1}-\eta^k}{h}). \tag{A.88}$$

Thanks to the above equality, it follows from (A.86) that

$$-h\sum_{k=0}^{N-1}(v^{k+1},\frac{\eta^{k+1}-\eta^k}{h})+h\sum_{k=0}^{N-1}\langle v^{k+1},\eta^k\rangle_a=h\sum_{k=0}^{N-1}(g^{k+1},\eta^k)+(v_0,\eta^0). \tag{A.89}$$

Recalling the definitions of v_N, g_N, η_N and $\partial_{\bar{t}}\eta_N$, one can verify that the above equality is equivalent to

$$\int_0^T(-(v_N,\bar{\partial}_t\eta_N)+\langle v_N,\eta_N\rangle_a)dt=\int_0^T(g_N,\eta_N)dt+(v_0,\eta(\cdot,0)), \tag{A.90}$$

from which by letting $N\to\infty$ due to $\eta_N\to\eta$ in $L^2(0,\mathcal{T};H^1(\Omega))$ and $\bar{\partial}_t\eta_N\to\partial_t\eta$ in $L^2(Q_{\mathcal{T}})$ one obtains

$$\int_0^T(-(v,\bar{\partial}_t\eta)+\langle v,\eta\rangle_a)dt=\int_0^T(g,\eta)dt+(v_0,\eta(\cdot,0)). \tag{A.91}$$

Therefore, v is a weak solution to (A.53) fulfilling the estimate (A.81). Moreover, if in addition $v_0\in H^1(\Omega)$, one has the additional regularities that $v\in L^2(0,\mathcal{T};H^2(\Omega))\cap L^\infty(0,\mathcal{T};H^1(\Omega))$, and the estimate (A.83) holds. This proves the existence part of the proposition, while the uniqueness can be proven in the standard way, see e.g. Ladyzhenskaya et al. [24].

We now prove that the weak solutions just established are strong ones and satisfy the corresponding estimate, if $v_0\in H^1(\Omega)$. Recall that, in this case, one has $v\in L^2(0,\mathcal{T};H^2(\Omega))\cap L^\infty(0,\mathcal{T};H^1(\Omega))$, and (A.83) holds. Thanks to this fact, and noticing that the equation in (A.53) is satisfied in the sense of distribution, one obtains $\partial_t v\in L^2(Q_{\mathcal{T}})$, which, along with $v\in L^2(0,\mathcal{T};H^2(\Omega))$ and $g\in L^2(Q_{\mathcal{T}})$, in turn implies that the equation in (A.53) is satisfied, a.e. in $Q_{\mathcal{T}}$. Thus, recalling the estimate (A.83), one obtains the desired $L^2(Q_{\mathcal{T}})$ estimate for $\partial_t v$, stated in the proposition. The regularities

$v \in L^2(0,\mathcal{T};H^2(\Omega))$ and $\partial_t v \in L^2(Q_\mathcal{T})$ imply $v \in C([0,\mathcal{T}];H^1(\Omega))$, from which, using the weak formula of weak solution to (A.53), one can see that $v(\cdot,0)=v_0$. Hence the initial condition is fulfilled. Thanks to the regularities of v, by integration by parts in the weak formula of (A.53), one can further see that the boundary conditions are satisfied in the sense of trace. Therefore, v is a strong solution to (A.53), and satisfies the desired estimate. This completes the proof of Proposition A.3. □

Recalling (A.52), as a direct corollary of Proposition A.3, one obtains the existence and uniqueness of weak and strong solutions to (A.39), which is stated in the following:

Corollary A.1. *Given a positive time $\mathcal{T}\in(0,\infty)$ and the initial data $u_0 \in L^2(\Omega)$. We assume that (A.43) and (A.44) hold, and that $f\in L^2(Q_\mathcal{T})$. Set*

$$M_0 = \|\varphi\|_{L^2(0,\mathcal{T};H^{\frac12}(\Gamma_\ell))} + \|\psi\|_{L^2(0,\mathcal{T};H^{\frac12}(\Gamma_{01}))} + \|\partial_t\Phi\|_{L^2(0,\mathcal{T};L^2(\partial\Omega))}. \tag{A.92}$$

Then, there is a unique weak solution to (A.39), satisfying

$$\|u\|_{L^\infty(0,\mathcal{T};L^2(\Omega))} + \|u\|_{L^2(0,\mathcal{T};H^1(\Omega))} \le C(\|f\|_{L^2(Q_\mathcal{T})} + \|u_0\|_{L^2(\Omega)} + M_0), \tag{A.93}$$

for a positive constant C depending only on λ and $\mathcal{T}$.

Moreover, if we assume in addition that $u_0\in H^1(\Omega)$, then the unique weak solution is a strong one, and satisfies

$$\|v\|_{L^\infty(0,\mathcal{T};H^1(\Omega))} + \|v\|_{L^2(0,\mathcal{T};H^2(\Omega))} + \|\partial_t v\|_{L^2(Q_\mathcal{T})} \le C(\|f\|_{L^2(Q_\mathcal{T})} + \|u_0\|_{H^1(\Omega)} + M_0), \tag{A.94}$$

for a positive constant C depending only on Ω, λ, $\mathcal{T}$, the modulus of continuity of a, $\|a\|_{L^\infty(\Omega)}$, $\|\partial_z a\|_{L^\infty(\Omega)}$, $\|\alpha\|_{W^{1,\infty}(\Omega)}$ and $\|\beta\|_{W^{1,\infty}(\Omega)}$.

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