Adaptive Finite Element Methods for Variational Inequalities

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Abstract

In this paper we are concerned with the numerical solution of stationary variational inequalities of obstacle type associated with second order elliptic differential operators in two or three space dimensions. In particular, we present adaptive finite element techniques featuring multilevel iterative solvers and a posteriori error estimators for local refinement of the triangulations. The algorithms rely on an outer-inner iterative scheme with an outer active set strategy and inner multilevel preconditioned cg-iterations involving variants of the hierarchical and the BPX-preconditioner which are derivded in the framework of multilevel additive Schwarz iterations. For the a posteriori error estimation in the energy norm three error estimators are presented which are based on the approximate solution of a quasivariational inequality satisfied by a piecewise quadratic approximation of the global discretization error. Finally, the performance of the preconditioners and the error estimators is illustrated by numerical results for a wide variety of stationary free boundary problems.

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1 Introduction

We consider stationary variational inequalities associated with second order elliptic differential operators: Given a bounded polygonal resp. polyhedral domain Ω in $I\!R^2$ resp. $I\!R^3$ with piecewise smooth boundary $\Gamma = \partial\Omega$, a closed subspace $V \subset H^1(\Omega)$ and a closed, convex set $K \subset V$, find $u \in K$ such that

$$a(u, v-u) \geq l(v-u), \quad v \in K \tag{1.1}$$

where l is a bounded linear functional on V and $a(\cdot,\cdot)$ is a symmetric, V-elliptic bilinear form

$$a(v,w) = \sum_{i,j=1}^{d} a_{ij}\partial_i v\, \partial_j w, \quad v,w \in V,\, d \in \{2,3\}$$

with coefficients $a_{ij} \in L^\infty(\Omega)$ satisfying for almost all $x \in \Omega$

$$\begin{array}{l} a_{ij}(x) = a_{ji}(x), \quad 1 \leq i,j \leq d, \\ \alpha_0|\xi|^2 \leq \sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \leq \alpha_1|\xi|^2, \quad \xi \in I\!R^d, 0 \leq \alpha_0 \leq \alpha_1. \end{array} \tag{1.2}$$

In particular, we confine ourselves to obstacle type problems where the constraint set K is given by means of an obstacle function $\psi \in H^1(\Omega)$

$$K := \{v \in V | v \leq \psi \text{ a.e. in } \Omega\}. \tag{1.3}$$

Instead of an upper obstacle we may likewise consider a lower obstacle in which case the inequality sign in (1.3) has to be reversed. However, for notational convenience and simplicity, in the theoretical part of this paper we will only treat upper obstacle problems and stick to the case of homogenous Dirichlet boundary conditions, i.e., we assume $V = H_0^1(\Omega)$. Note that variational inequalities of obstacle type can be encountered in numerous applications oriented problems. We refer to Baiocchi/Capelo (1984), Crank (1987), Duvaut/Lions (1976), Elliot/Ockendon (1982), Friedmann (1988), Kikuchi/Oden (1988), Rodrigues (1987) for problems in engineering and to Bensoussan (1982), Bensoussan/Lions (1982, 1984), Cottle et al. (1992) for applications in operations research.

While the existence and uniqueness of a solution to (1.1) is a classical result in convex analysis (cf. e.g. Ekeland/Temam (1976), Kinderlehrer/Stampacchia (1980)), for the numerical solution based on the discretization by either finite difference or finite element methods there is a wide variety of techniques such as projected SOR, gradient projection, penalization, duality methods and augmented Lagrangians (for an overview see e.g. Glowinski et al. (1981)). With regard to the numerical efficiency these iterative schemes suffer from rapidly deteriorating convergence rates for decreasing step sizes which motivated the use of multigrid techniques in order to overcome this deficiency (cf. e.g. Boyer/Martinet (1986), Brandt/Cryer (1983), Hackbusch/Mittelmann (1983), Hoppe (1987a, 1987b, 1988, 19990), Mandel (1984a, 1984b), Smoch (1990)). However, all these multigrid methods are based on a hierarchy of grids generated by successive uniform refinement of an initial coarse grid and have not incorporated adaptive concepts (except for the adaptively chosen grid transfers in Hoppe (loc. cit.)).

On the other hand, in the unconstrained case, i.e., for linear elliptic boundary value problems, adaptive multilevel finite element techniques are well established (cf. e.g. Bank (1990), Deuflhard et al. (1989), Johnson (1987), Mesztenyi/Rheinboldt (1987), Szabó/Babuska (1991)). In particular, the techniques proposed in Deuflhard et al. (1989) and its recent extensions feature multilevel preconditioned cg-iterations with preconditioners of hierarchical or BPX-type (cf. Yserentant (1986), Bramble et al. (1990)) combined with an edge-oriented a posteriori error estimator for the global discretization error whose local contributions serve as an indicator for local refinement of the triangulations. For obstacle type problems such an approach has been recently undertaken by Hoppe and Kornhuber (1993) based on an outer-inner iterative scheme. Specifically, the outer iteration consists in an active set strategy requiring the solution of a linear algebraic system with a symmetric, positive definite coefficient matrix whose structure changes at each iteration step. These systems are taken care of by hierarchically preconditioned cg-iterations constituting the inner iterations. Note that Yserentant's original hierarchical preconditioner has to be modified appropriately by a truncation of the hierarchical basis functions in order to cope with the special structure of the linear systems caused by the active constraints. Moreover, the a posteriori error estimation is not as simple as in the unconstrained case due to the fact that the piecewise quadratic approximation of the global discretization error satisfies a quasivariational inequality. In particular, a semi-local and a local error estimator have been investigated in Hoppe and Kornhuber (1993).

The purpose of this paper is threefold: Firstly, we extend the results in Hoppe and Kornhuber (1993) in so far as we include two variants of the BPX-preconditioner. Note that both the hierarchical and the BPX-preconditioner are closely related to multilevel additive Schwarz methods which allows to derive condition number estimates by taking advantage of the well developed theory of domain decomposition techniques (cf. Bornemann (1991), Xu (1992), Zhang (1992)). It should also be emphasized that the BPX-preconditioner is the only alternative for problems in higher than two dimensions with regard to the poor performance of the hierarchical preconditioner in such cases (cf. Go Ong (1989)). Secondly, we present a further a posteriori error estimator based on a two-sided approximation of the quasivariational inequality satisfied by the piecewise quadratic approximation of the global discretization error. Finally, we will illustrate the performance of the preconditioners and the error estimators for a wide variety of obstacle type problems including lubrication in infinite journal bearings, elastic-plastic torsion of cylinders with simply and multiply connected cross sections and stationary porous media flow as in the dam problem and the axialsymmetric water cone problem in oil reservoir simulation.

2 Finite Element Discretization and Basic Iterative Strategy

Using continuous, piecewise linear finite elements with respect to a regular triangulation $\mathcal{T}_h$ of the given domain Ω we denote by S_h the associated finite element space spanned by the nodal basis functions ψ_p^h with supporting point $p \in \mathcal{N}_h$ where $\mathcal{N}_h$ stands for the set of interior nodal points. We further assume that $\varphi_h \in S_h$ is a suitable approximation of the given obstacle φ, e.g. the S_h-interpolate $\Pi_h \varphi$ if $\varphi \in C(\bar{\Omega})$, and we refer to K_h as the constraint set

$$K_h := \{v_h \in S_h |\ v_h(p) \leq \varphi_h(p), \quad p \in \mathcal{N}_h\}. \tag{2.1}$$

We then consider the following finite element approximation of the variational inequality (1.1):
Find $u_h \in K_h$ satisfying

$$a(u_h, v_h - u_h) \geq l(v_h - u_h), \quad v_h \in K_h, \tag{2.2}$$

or equivalently

$$J(u_h) = \inf_{v_h \in K_h} J(v_h) \tag{2.3}$$

where $J(v_h) := \frac{1}{2} a(v_h, v_h) - l(v_h),\ v_h \in S_h$.

For the numerical solution of the finite dimensional constrained minimization problem (2.3) we will use a special active set strategy originally proposed by P.L. Lions and Mercier (1980). In particular, each iteration step requires the unconstrained minimization of the energy functional J with respect to a subspace of S_h specified by active constraints. Given an iterate $u_h^{(\nu)} \in S_h$, $\nu \geq 0$, the specification of the set of active nodal points is motivated by the fact that if we proceed in descent direction $-\nabla J(u_h^{(\nu)})$ then

$$u_h^{(\nu)} - \nabla J(u_h^{(\nu)}) \in \operatorname{int} K_h$$

holds true if and only if

$$(u_h^{(\nu)} - \varphi_h)(p) < a(u_h^{(\nu)}, \psi_p^h) - l(\psi_p^h), \quad p \in \mathcal{N}_h. \tag{2.4}$$

Consequently, a constraint is said to be active in $p \in \mathcal{N}_h$ if (2.4) is violated and we refer to

$$\mathcal{N}_h^2 := \{p \in \mathcal{N}_h |\ (u_h^{(\nu)} - \varphi_h)(p) \geq a(u_h^{(\nu)}, \psi_p^h) - l(\psi_p^h)\}$$

as the set of active nodal points and to its complement $\mathcal{N}_h^1 := \mathcal{N}_h \setminus \mathcal{N}_h^2$ as the set of inactive nodal points. We set

$$\tilde{S}_h := \{v_h \in S_h |\ v_h(p) = \varphi_h(p),\ p \in \mathcal{N}_h^2\}$$

and compute a new iterate $u_h^{(\nu+1)} \in \tilde{S}_h$ as the solution of the unconstrained minimization problem

$$u_h^{(\nu+1)} = \inf_{v_h \in \tilde{S}_h} J(v_h). \tag{2.5}$$

Setting

$$S_h^i := \text{span}\,\{\psi_p^h|\; p \in \mathcal{N}_h^i\}, \quad 1 \le i \le 2,$$

and denoting by Π_h^i, $1 \le i \le 2$, the S_h^i-interpolation operators, we have $u_h^{(\nu+1)} = \Pi_h^1 u_h^{(\nu+1)} + \Pi_h^2 \varphi_h$ and it is easy to see that (2.5) reduces to the computation of $\tilde{u}_h^{(\nu+1)} := \Pi_h^1 u_h^{(\nu+1)}$ as the solution of the variational equation

$$a(\tilde{u}_h^{(\nu+1)}, v_h) = l(v_h) - a(\Pi_h^2 \varphi_h, v_h), \quad v_h \in S_h^1. \tag{2.6}$$

Assuming that (2.6) is solved exactly, it has been shown in Hoppe (1987a, 1987b) that for any startiterate $u_h^{(0)} \in S_h$ the sequence $(u_h^{(\nu)})_{\nu \ge 1}$ of iterates is a monotonically decreasing sequence converging after a finite number of steps to the unique solution of (2.2). Of course, this result is merely of theoretical interest, since we do not use exact solvers. With regard to the iterative solution we note that algebraically (2.6) represents a linear algebraic system with a coefficient matrix $\tilde{A}_h$ being a principal submatrix of the stiffness matrix A_h associated with $a|_{S_h \times S_h}$. Hence, $\tilde{A}_h$ is symmetric, positive definite and (2.6) can be solved by a preconditioned conjugate gradient iteration. The following section exclusively deals with the construction of efficient multilevel preconditioners.

3 The Multilevel Preconditioners

We specify an initial coarse simplicial triangulation $\mathcal{T}_0$ of the given computational domain Ω and generate a sequence of triangulations $(\mathcal{T}_k)_{k=0}^j$, $j > 0$, by successive local refinement based on a posteriori error estimators that will be described in the next section. We use the meanwhile standard refinement process due to Bank et al. in the 2-D case (see e.g. Bank et al. (1983), Bank (1990)) and its extension for 3-D domains as developed by Bey (1991) and Go Ong (1989) (cf. also Bänsch (1991) and Zhang (1988) for related concepts). Note that these refinement strategies have been implemented in adaptive finite element codes, namely in the 2-D codes PLTMG (cf. Bank (1990)) and KASKADE (cf. Deuflhard et al. (1989)) and in the 3-D code 3-D ELLKASK (see Bornemann et al. (1993); cf. also Erdmann/Roitzsch (1993), Leinen (1990), Roitzsch (1989a, 1989b) for a comprehensive description of the underlying data structures). Since the refinement processes have been extensively described in the references cited above, for details the reader is referred to these sources.

As a consequence of the refinement rules the spaces S_k, $0 \le k \le j$, of continous, piecewise linear finite element functions associated with $\mathcal{T}_k$, constitute a nested sequence of subspaces of $H_0^1(\Omega)$, i.e., $S_0 \subset S_1 \subset \ldots \subset S_j$. Nodal points and edges of elements $\tau \in \mathcal{T}_k$ are called interior, if they are situated in Ω and not on $\partial\Omega$. For $0 \le k \le j$ we denote by $\mathcal{N}_k$ the set of interior nodal points, by $\mathcal{E}_k$ the set of interior edges and by $\mathcal{M}_k$ the set of midpoints of interior edges $e \in \mathcal{E}_k$. Further, we refer to $\psi_p^k \in S_k$ as the nodal basis function with supporting point $p \in \mathcal{N}_k$, i.e., $\psi_p^k(q) = \delta_{pq}$, $q \in \mathcal{N}_k$.

Having provided the nested hierarchy $(S_k)_{k=0}^j$, we now consider the solution of the discretized obstacle problem on level j by the active set strategy described in the previous section. In particular, given an iterate $u_j^{(\nu)}$, $\nu \ge 0$, the sets $\mathcal{N}_j^1$, $\mathcal{N}_j^2$ of inactive and active nodal points and the associated finite element spaces S_j^1, S_j^2, we have to compute $\tilde{u}_j^{(\nu+1)} \in S_j^1$ as the solution of the variational equation (cf. (2.6))

$$a(\tilde{u}_j^{(\nu+1)}, v_j) = l(v_j) - a(\Pi_j^2 \varphi_j . v_j), \quad v_j \in S_j^1. \tag{3.1}$$

If we attempt to solve (3.1) iteratively by a multilevel preconditioned cg-iteration, we first have to mimic the level j decomposition $\mathcal{N}_j = \mathcal{N}_j^1 \cup \mathcal{N}_j^2$ on the lower levels $0 \le k < j$ and to specify the corresponding spaces S_k^i, $1 \le i \le 2$. This can be easily achieved by means of

$$\mathcal{N}_k^i := \mathcal{N}_k \cap \mathcal{N}_j^i, \quad S_k^i := \text{span}\,\{\psi_p^k |\, p \in \mathcal{N}_k^i\}, \quad 1 \le i \le 2. \tag{3.2}$$

With regard to the construction of multilevel preconditioners of hierarchical and BPX-type we are faced with the problem that in contrast to the unconstrained case the sequence $(S_k^1)_{k=0}^j$ does not constitute a nested hierarchy of subspaces of S_j^1. The reason is that a level k nodal basis function ψ_p^k with supporting point $p \in \mathcal{N}_k^1$ does not vanish in active level $l > k$ nodal points $q \in \mathcal{N}_l^2$ within the interior of the support of ψ_p^k. For ease of visualization such a situation is depicted in Figure 3.1(a) below.

In particular, an inactive nodal point $p \in \mathcal{N}_k^1$ will be called regular, if $\text{int supp}\, \psi_p^k \cap \mathcal{N}_j^2 = \emptyset$, and irregular, otherwise. We denote by $N_k^{1,reg}$ and $N_k^{1,irr}$

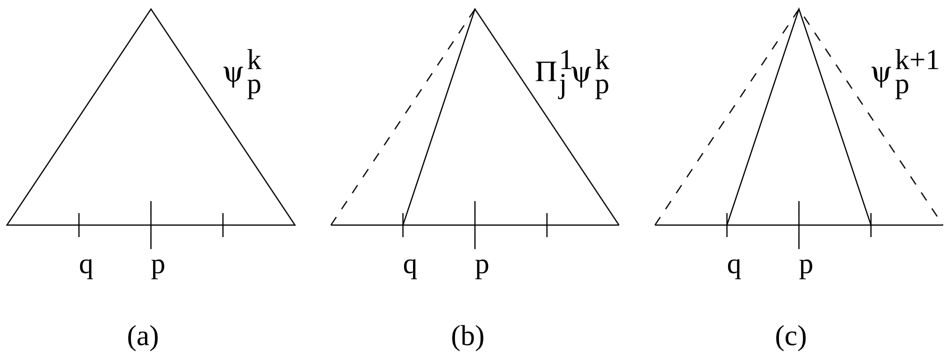

Fig. 3.1 Truncation of nodal basis functions

the set of regular and irregular inactive level k nodal points, respectively. A convenient remedy to get rid of the nonnestedness of S_k^1, $0 \leq k \leq j$, consists in an appropriate modification of basis functions ψ_p^k, $p \in N_k^{1,irr}$ by truncation. We consider two different truncation processes. The first one is that we replace ψ_p^k by its S_j^1-interpolate $\Pi_j^1 \psi_p^k$ which in general corresponds to a "nonsymmetric" truncation (cf. Figure 3.1(b)). The second approach is to replace ψ_p^k by the higher level nodal basis function $\psi_p^{l_p}$ where $l_p := \min\{l \geq k | \text{ int supp } \psi_p^l \cap \mathcal{N}_j^2 = \emptyset\}$ which can be interpreted as a "symmetric" truncation (cf. Figure 3.1(c)). Correspondingly, for $0 \leq k \leq j$ we define

$$S_k^{1,NS} := \text{span}\{\Pi_j^1 \psi_p^k | \; p \in \mathcal{N}_k^1\}, \tag{3.3}$$

$$S_k^{1,S} := \text{span}\{\psi_p^k | \; p \in N_k^{1,reg}\} \tag{3.4}$$

where the upper indices "NS" and "S" refer to the "nonsymmetric" and "symmetric" truncation process, respectively. Note that by construction both sequences $(S_k^{1,NS})_{k=0}^j$ and $(S_k^{1,S})_{k=0}^j$ represent a hierarchy of nested subspaces of $S_j^{1,NS} = S_j^{1,S} = S_j^1$.

Collecting the level k nodal basis functions associated with new, inactive nodal points, we obtain the hierarchical basis of $S_j^{1,NS}$ and $S_j^{1,S}$, respectively. In particular we set

$$\Psi_H^{NS} := \bigcup_{k=1}^{j} \Psi_{H,NS}^k \quad , \quad \Psi_{H,NS}^k := \{\Pi_j^1 \psi_p^k | \; p \in \mathcal{N}_k^1 \setminus \mathcal{N}_{k-1}^1\}, \tag{3.5}$$

$$\Psi_H^{S} := \bigcup_{k=1}^{j} \Psi_{H,S}^k \quad , \quad \Psi_{H,S}^k := \{\psi_p^k | \; p \in N_k^{1,reg} \setminus N_{k-1}^{1,reg}\}. \tag{3.6}$$

Likewise we sample the new level k nodal basis functions of depth k according to

$$\Psi_B^{NS} := \bigcup_{k=1}^{j} \Psi_{B,NS}^k \quad , \quad \Psi_{B,NS}^k := \{\Pi_j^1 \psi_p^k \in S_k^{1,NS} \setminus S_{k-1}^{1,NS}\}, \tag{3.7}$$

$$\Psi_B^{S} := \bigcup_{k=1}^{j} \Psi_{BS}^k \quad , \quad \Psi_{B,S}^k := \{\psi_p^k \in S_k^{1,S} \setminus S_{k-1}^{1,S}\} \tag{3.8}$$

to provide the underlying structure of the BPX-preconditioner (cf. Bornemann (1991) and Bornemann et al. (1993)).

As has been shown by Bornemann (1991) and Zhang (1992) in the unconstrained case, the hierarchical and the BPX preconditioner can be derived within the framework of multilevel additive Schwarz iterations. In case of the hierarchical preconditioner we start from the decomposition of $S_j^{1,D}$, $D \in \{NS, S\}$ into the direct sum

$$S_j^{1,D} = V_0^D \oplus \bigoplus_{\psi \in \Psi_H^D} V_\psi^D \tag{3.9}$$

of the subspaces $V_0^D := S_0^{1,D}$ and $V_\psi^D := \operatorname{span}\{\psi\}$, $\psi \in \Psi_H^D$. On the other hand, for the BPX preconditioner the underlying subspace decomposition is given by

$$S_j^{1,D} := V_0^D + \sum_{\psi \in \Psi_B^D} V_\psi^D \tag{3.10}$$

where $V_\psi^D := \operatorname{span}\{\psi\}$, $\psi \in \Psi_B^D$. Both (3.9) and (3.10) induce an associated additive Schwarz iteration with iteration operator

$$M_C^D := I - (P_0 + \sum_{\psi \in \Psi_C^D} P_\psi), \quad C \in \{B, H\},\ D \in \{NS, S\}$$

where P_0 and P_ψ are the elliptic projections onto V_0^D and V_ψ^D, respectively. Denoting by A_j the representation operator of $a|_{S_j^1 \times S_j^1}$, the operator $N_C^D := (I - M_C^D)A_j^{-1}$ of the so-called second normal form of the additive Schwarz iteration is a natural candidate for the inverse of the wanted preconditioner H_C^D, i.e., $N_C^D = (H_C^D)^{-1}$. Referring to Q_0^D and Q_ψ^D, $\psi \in \Psi_C^D$, as the L^2-projections $Q_0^D : S_j^1 \to V_0^D$, $Q_\psi^D : S_j^1 \to V_\psi^D$ and to $A_0^D : V_0^D \to V_0^D$, $A_\psi^D : V_\psi^D \to V_\psi^D$ as the representation operators of $a|_{V_0^D \times V_0^D}$ and $a|_{V_\psi^D \times V_\psi^D}$, respectively, we obtain

$$(H_C^D)^{-1} = (A_0^D)^{-1} Q_0^D + \sum_{\psi \in \Psi_C^D} (A_\psi^D)^{-1} Q_\psi^D .$$

Evaluation of $(A_\psi^D)^{-1} Q_\psi^D$ finally results in

$$(H_C^D)^{-1} u = (A_0^D)^{-1} Q_0^D u + \sum_{\psi \in \Psi_C^D} \frac{(u, \psi)_0}{a(\psi, \psi)} \psi, \quad u \in S_j^1, \tag{3.11}$$

where $(\cdot, \cdot)_0$ stands for the usual L^2 inner product.

It should be noted that in the nonsymmetric case the preconditioner (3.11) is not yet suited for actual computations. The reason is that due to the nonsymmetric truncation process the entries of the level 0 stiffness matrix A_0^{NS} and the scaling factors $a(\psi, \psi)$, $\psi \in \Psi_C^{NS}$, in general change with each outer iteration which may cause considerable computational efforts. Therefore, we simply replace the nonsymmetrically truncated basis functions by their nontruncated originals. In particular, denoting by Ψ_C, $C \in \{B, H\}$, the collection of basis functions as in (3.6), (3.8) with $\Pi_j^1 \psi_p^k$ replaced by ψ_p^k, we thus end up with

$$(\tilde{H}_C^{NS})^{-1} u = \Pi_j^1 A_0^{-1} Q_0 u + \sum_{\psi \in \Psi_C} \frac{(u, \Pi_j^1 \psi)_0}{a(\psi, \psi)} \Pi_j^1 \psi \tag{3.12}$$

where Q_0 stands for the L^2-projection onto S_0^1 and A_0 is the representation operator of $a(\cdot,\cdot)|_{S_0^1\times S_0^1}$. On the other hand, it should be emphasized that in the symmetric case the preconditioner (3.11) can be implemented without modification, since in view of (3.6), (3.8) the active set has no impact on the shape but only on the selection of the basis functions.

The rest of this section will be devoted to condition number estimates for $(H_C^D)^{-1}A_j$. Throughout the following, for any measurable $U \subset \Omega$ we will refer to $(\cdot,\cdot)_{0;U}$ and $(\cdot,\cdot)_{1;U}$ as the standard inner product and semi-inner product on $L^2(U)$ and $H^1(U)$, respectively. Correspondingly, we refer to $|\cdot|_{\nu,\mu;U}$, $\nu,\mu \in \{0,1\}$, as the associated operator norm, i.e., $|A|_{\nu,\mu;U} := \sup\{|Av|_{\nu;U} | \, |v|_{\mu;U} \le 1\}$. In particular, if $U = \Omega$ the lower index U will be dropped. Further, we will denote by c and C generic positive constants depending only on the ellipticity of $a(\cdot,\cdot)$ and the shape regularity of the initial triangulation $\mathcal{T}_0$.

The condition number estimates can be derived by means of the following fundamental result from the theory of multilevel additive Schwarz methods (cf. e.g. Bornemann (1991), Xu (1992), Zhang (1992) and Yserentant (1993)):

Lemma 3.1 *Let Ψ be a collection of level $k \le j$ basis functions and denote by H_j the preconditioner corresponding to the associated multilevel additive Schwarz iteration. Then the following two assertions hold true:*

1. *If for all $v \in S_j^1$ there is a splitting $v = v_0 + \sum_{\psi\in\Psi} v_\psi$ such that for some $c > 0$*
$$a(v_0,v_0) + \sum_{\psi\in\Psi} a(v_\psi,v_\psi) \le c^{-1}a(v,v), \tag{3.13}$$
then c is a lower bound for the condition number of $H_j^{-1}A_j$, i.e.
$$ca(v,v) \le a(H_j^{-1}A_j v, v), \quad v \in S_j^1.$$

2. *If for all splittings $v = v_0 + \sum_{\psi\in\Psi} v_\psi$ of $v \in S_j^1$ there exists a constant $C > 0$ such that*
$$a(v,v) \le C[a(v_0,v_0) + \sum_{\psi\in\Psi} a(v_\psi,v_\psi)], \tag{3.14}$$
then C is an upper bound for the condition number of $H_j^{-1}A_j$, i.e.
$$a(H_j^{-1}A_j v, v) \le Ca(v,v), \quad v \in S_j^1.$$

Note that part (1) of the preceding result is commonly referred to as the lemma of P.L. Lions (cf. P.L. Lions (1988)) while (3.14) can frequently be established by means of a strengthened Cauchy-Schwarz inequality exhibiting an asymptotic orthogonality of the subspaces V_0 and $V_\psi = \text{span}\{\psi\}$, $\psi \in \Psi$. In case of the nonsymmetric preconditioners this asymptotic orthogonality can only be guaranteed if the subsequent truncation of level k basis functions terminates after a finite number of steps being independent of the refinement level. In particular, we assume

(A_1) There exists a constant integer $k_0 \ge 0$ independent of k such that
$$\Pi_j^1 v = \Pi_{k+k_0}^1 v, \quad v \in S_k^1, \ k + k_0 \le j. \tag{3.15}$$

Remark 3.1 Since nonsymmetric truncation only takes place in a vicinity of the discrete free boundary, from a heurestical point of view the condition (A_1) can be interpreted as a regularity assumption on the free boundary. A typical situation which will be frequently encountered in the applications is that the free boundary represents a smooth lower dimensional manifold that is approximated of sufficient accuracy up to a certain level while then refinement only occurs in a subregion of the coincidence set and the free boundary (cf. section 5).

It is well known from the unconstrained case (cf. e.g. Yserentant (1986, 1990)) that for the hierarchical preconditioner the derivation of a lower bound for the condition number along the lines of Lemma 3.1 (1) relies on the H^1-stability and an approximation-of-unity property of the interpolation operators $I_k : S_j \to S_k$, $0 \leq k \leq j$, given by $(I_k v)(p) = v(p)$, $p \in \mathcal{N}_k$, that are used in the hierarchical splitting $v = I_0 v + \sum_{k=1}^{j} (I_k v - I_{k-1} v)$. Since $|I_0|_{1,1}$ grows linearly in the refinement level j in the 2-D case but exponentially in j in 3-D, the practical use of the hierarchical preconditioner is restricted to 2-D applications (cf. e.g. Go Ong (1989), Oswald (1990) and Dahmen et al. (1993)). Therefore, the following result which has been established by Hoppe and Kornhuber (1993) only holds true for 2-D problems:

Theorem 3.1 *Let $H_j \in \{H_H^S, H_H^{NS}, \tilde{H}_H^{NS}\}$ denote the symmetric or nonsymmetric hierarchical preconditioner associated with the hierarchical multilevel splitting (3.9) of $S_j^1 \subset H_0^1(\Omega)$ where Ω is a bounded polygonal domain in $\mathbb{R}^2$. Assume that in the nonsymmetric case condition (A_1) is satisfied. Then there exist positive constants c_0, c_1 depending only on the ellipticity of $a(\cdot, \cdot)$, the shape regularity of $\mathcal{T}_0$ and in the nonsymmetric case on the integer k_0 from (3.15) such that*

$$c_0 (j+1)^{-2} a(v,v) \leq a(H_j^{-1} A_j v, v) \leq c_1 a(v,v), \quad v \in S_j^1. \tag{3.16}$$

Proof. In the nonsymmetric case the proof of the upper bound follows from Lemma 3.1 (2) by verifying (3.14) based on a strengthened Cauchy-Schwarz inequality for truncated hierarchical basis functions $v_k \in \operatorname{span}\Psi_{H,NS}^k$, $v_l \in \operatorname{span}\Psi_{H,NS}^l$, $|l-k| \geq k_0$

$$(v_l, v_k)_1 \leq c(k_0) 2^{-(|l-k|-k_0)/2} |v_k|_1 |v_l|_1 \tag{3.17}$$

and the elementary norm equivalence

$$c(k_0) \sum_{p \in \mathcal{N}_k \cap \tau} |v(p)|^2 \leq |\Pi_l^1 v|_{1;\tau}^2 \leq c(k_0) \sum_{p \in \mathcal{N}_k \cap \tau} |v(p)|^2, \quad \tau \in \mathcal{T}_k$$

which holds true for all $v \in \operatorname{span}\{\psi_p^k | \, p \in \mathcal{N}_k^1 \setminus \mathcal{N}_{k-1}^1\}$.

The lower bound is a consequence of (3.13) in Lemma 3.1 (1). In particular, considering the unique splitting $v = v_0 + \sum_{\psi \in \Psi_H^{NS}} v_\psi$ for some fixed $v \in S_j^1$, we construct auxiliary functions $\hat{v}_0 \in \operatorname{span}\{\psi_p^0 | \, p \in \mathcal{N}_0^1\}$, $\hat{v}_\psi \in \operatorname{span}\{\psi_p^k | \, p \in \mathcal{N}_k^1 \setminus \mathcal{N}_{k-1}^1\}$ satisfying $v_0 = \Pi_j^1 \hat{v}_0$ and $v_\psi = \Pi_j^1 \hat{v}_\psi$. Then we are able to establish (3.13) for $\hat{v}_0$, $\hat{v}_\psi$ essentially using the H^1-stability $|I_0|_{1,1} = O(j)$ and the approximation-of-unity property $|I - I_k|_{0,1} = O(4^{-k}(j+k-1))$ of the interpolation operators in much the same way as has been done by Yserentant (1986). Note that the case $\mathcal{N}_{k^*}^1 \neq \emptyset$, $\mathcal{N}_k^1 = \emptyset$, $0 \leq k < k^*$, for some $k^* > 0$ deserves special attention (cf.

Theorem 3.2 below).

In the symmetric case, (3.17) can be shown to hold true with $k_0 = 0$ and for the proof of (3.13) we do not have to resort to the auxiliary functions $\hat{v}_0$, $\hat{v}_\psi$. (For details the reader is referred to Theorems 3.1, 3.2 in Hoppe and Kornhuber (1993)). □

Finally, we focus our attention on the BPX-type preconditioners and remind that in the unconstrained case (cf. e.g. Yserentant (1990)) the lower bound for the condition number can be again derived with regard to Lemma 3.1 (1). This time the proof of (3.13) follows from a splitting $v = Q_0 v + \sum_{k=1}^{j}(Q_k v - Q_{k-1}v)$, $v \in S_j$, where the $Q_k : S_j \to S_k$ are the L^2-projections given by

$$(Q_k v, w)_0 = (v, w)_0, \quad w \in S_k,\ 0 \le k \le j. \tag{3.18}$$

The essential tools in the proof are a dimensionally independent H^1-stability and an approximation-of-unity property of the projections Q_k. In particular, denoting by $U(\tau, k) = \bigcup\{\tau' \in \mathcal{T}_k |\ \tau' \cap \tau \neq \emptyset\}$, $\tau \in \mathcal{T}_k$, the union of all level k elements intersecting τ and by $h(\tau)$ the diameter of τ, these properties rely on the Poincaré inequality

$$|v|_{0;U(\tau,k)} \le C\, h(\tau)|v|_{1;U(\tau,k)}, \quad v \in S_j^1,\ \tau \cap \partial\Omega \neq \emptyset \tag{3.19}$$

(cf. Lemma 4.1 in Yserentant (1990)). Note that (3.19) can be established by local transformations to a finite number of reference configurations, since $\partial\Omega$ consists of faces of level 0 elements $\tau \in \mathcal{T}_0$. However, for the obstacle problems under consideration the reduced problems (3.1) constitute Dirichlet problems on the computational domain $\Omega_j^1 := \bigcup_{p \in \mathcal{N}_j^1} \operatorname{supp} \psi_p^j$ the boundary of which includes the discrete free boundary $\partial\Omega_j^1 \setminus \partial\Omega$. Consequently, if we define

$$U(\tau, k) := \{\tau' \in \mathcal{T}_k |\ \tau' \cap \tau \neq \emptyset,\ \tau' \cap \mathcal{N}_j^1 \neq \emptyset\}, \quad \tau \in \mathcal{T}_k,\ \tau \cap \mathcal{N}_j^1 \neq \emptyset \tag{3.20}$$

and if we do not have a priori information on the shape regularity of $U(\tau, k)$ intersecting the free boundary, we cannot deduce Poincaré's inequality as in the unconstrained case and hence, we make the following assumption:

(A_2) There exists a constant $p > 0$ independent of j such that for all $0 \le k \le j$ and all $U(\tau, k)$ as given by (3.20) there holds

$$|v|_{0;U(\tau,k)} \le p\, h(\tau)|v|_{1;U(\tau,k)}, \quad v \in S_j^1. \tag{3.21}$$

Remark 3.2 With regard to the above considerations the condition (A_2) can be interpreted as a regularity assumption on the free boundary.

As an immediate consequence of (A_2) we have:

Lemma 3.2 *Under the assumption (A_2) there exist positive constants C_1, C_2 depending only on the local geometry of $\mathcal{T}_0$ and the constant p from (3.21) such that for the L^2-projections Q_k, $0 \le k \le j$, given by (3.18) there holds*

$$|Q_k|_{1,1} \le C_1, \tag{3.22}$$
$$|I - Q_k|_{0,1} \le C_2\, 4^{-k}. \tag{3.23}$$

Proof. As already indicated in the discussion leading to (A_2), the proof of the H^1-stability (3.23) and the approximation-of-unity property (3.23) follows directly from Yserentant (1990), if the application of Poincaré's inequality in Lemma 4.1 of Yserentant's paper is replaced by (3.21). □

We emphasize that in contrast to the interpolation operators I_k used for the hierarchical splittings the H^1-stability and approximation-of-unity property of Q_k are not dimensionally dependent. Therefore, the following result which is partly contained in Erdmann et al. (1993) holds true in any space dimension.

Theorem 3.2 *Let $H_j \in \{H_B^S,\, H_B^{NS}\}$ denote the symmetric or nonsymmetric BPX preconditioner based on the multilevel splitting (3.10) of $S_j^1 \subset H_0^1(\Omega)$, Ω being a bounded polygonal resp. polyhedral domain in $\mathbb{R}^2$ resp. $\mathbb{R}^3$. Assume that condition (A_2) is satisfied and that additionally, in the nonsymmetric case assumption (A_1) holds true. Then there exist constants c_0, c_1 depending only on the ellipticity of $a(\cdot,\cdot)$, the shape regularity of $\mathcal{T}_0$, the constant p from (3.21) and in the nonsymmetric case on the integer k_0 from (3.15) such that*

$$c_0\,(j+1)^{-1} a(v,v) \le a(H_j^{-1} A_j v, v) \le c_1\, a(v,v), \quad v \in S_j^1. \tag{3.24}$$

Proof. The proof will only be given for the symmetric BPX preconditioner. The modifications in the nonsymmetric case follow the same arguments as in Theorem 3.1.

The upper bound in (3.24) can be shown by means of Lemma 3.1 (2) in exactly the same way as in the unconstrained case. The essential tools are a decomposition $\Psi_{B,S}^k = \bigcup_{i=1}^M \Psi_{B,S}^{k,i}$ of $\Psi_{B,S}^k$ into a uniformly bounded number M of subsets $\Psi_{B,S}^{k,i}$, $1 \le i \le M$, such that $\operatorname{supp}\psi \cap \operatorname{supp}\psi' = \emptyset$, $\psi, \psi' \in \Psi_{B,S}^{k,i}$, and a strengthened Cauchy-Schwarz inequality

$$(v_k, w_l)_1 \le C\, 2^{-(l-k)/2} |v_k|_1 |w_l|_1$$

for $v_k \in S_k^1$ and $w_l \in \operatorname{span}\Psi_{B,S}^{l,i}$, $l > k$ (cf. Bornemann (1991) and Zhang (1992)).

On the other hand, assuming at first $\mathcal{N}_0^{1,reg} \neq \emptyset$ the lower bound in (3.24) is a consequence of Lemma 3.1 (1) as soon as we have verified

$$a(v_0, v_0) + \sum_{\psi \in \Psi_B^S} a(v_\psi, v_\psi) \le C(j+1)|v|_1^2 \tag{3.25}$$

for the particular splitting $v = v_0 + \sum_{\psi \in \Psi_B^S} v_\psi$ of some fixed $v \in S_j^1$ where $v_0 = Q_0 v$ and the v_ψ, $\psi \in \Psi_B^S$, are uniquely determined by $Q_k v - Q_{k-1} v = \sum_{\psi \in \Psi_{B,S}^k} v_\psi$, $1 \le k \le j$. Using the H^1-stability (3.23) of Q_0, we have

$$a(v_0, v_0) \le \alpha_1 |Q_0 v|_1^2 \le C\,|v|_1^2. \tag{3.26}$$

Further, by means of the inverse inequality $|v_\psi|_{1;\tau}^2 \le C 4^k |v_\psi|_{0;\tau}^2$, $\tau \in \mathcal{T}_k$, and the approximation-of-unity property (3.23) we get

$$\sum_{\psi \in \Psi_B^S} a(v_\psi, v_\psi) \le \alpha_1 \sum_{\psi \in \Psi_B^S} |v_\psi|_1^2 \le$$

$$\le C \sum_{k=1}^j 4^k \sum_{\psi \in \Psi_{B,S}^k} |v_\psi|_0^2 \le C \sum_{k=1}^j 4^k |Q_k v - Q_{k-1} v|_0^2 \le C_j |v|_1^2 \tag{3.27}$$

where we have additionally used the estimate

$$\sum_{\psi \in \Psi^k_{B,S}} |v_\psi|_0^2 \leq C \sum_{\tau \in \mathcal{T}_k \cap \Omega^1_j} \text{area}(\tau) \sum_{p \in \tau} |(Q_k v - Q_{k-1} v)(p)|^2 \leq C |Q_k v - Q_{k-1} v|_0^2.$$

Clearly, (3.26) and (3.27) give the assertion.
We still have to consider the case $\mathcal{N}_0^{1,reg} = \emptyset$ or, more generally, $\mathcal{N}_{k^*}^{1,reg} \neq \emptyset$, $\mathcal{N}_k^{1,reg} = \emptyset$, $0 \leq k \leq k^* - 1$, for some $k^* > 0$ which may occur due to the specification of inactive nodal points at levels $k < j$ prescribed by the outer active set strategy. Then $a(v_0, v_0)$ has to be replaced by $\sum_{\psi \in \Psi^{k^*}_{B,S}} a(v_\psi, v_\psi)$ and (3.25) follows if in lieu of (3.26) we use the inequality

$$\sum_{\psi \in \Psi^{k^*}_{B,S}} a(v_\psi, v_\psi) \leq \alpha_1 \sum_{p \in \mathcal{N}^{1,reg}_{k^*}} |v_{\psi_p^{k^*}}|_1^2 \leq C\, |Q_{k^*} v|_1^2.$$

which is an immediate consequence of Lemma 3.2 of Erdmann et al. (1993). □

4 A Posteriori Error Estimators

A posteriori error estimators for the global discretization error are an important tool for adaptive finite element codes, since its local contributions are used as indicators for local refinement of the triangulations. In the unconstrained case, i.e., for second order elliptic boundary value problems, element-oriented and edge-oriented error estimators have been proposed by Bank and Weiser (1985) and by Deuflhard, Leinen and Yserentant (1989) and have been implemented in the existing adaptive codes PLTMG and KASKADE, respectively (cf. also the KASKADE extension 3-D ELLKASK by Bornemann et al. (1993) in the 3-D case). Both estimators rely on a piecewise quadratic ansatz which is assumed to be of higher accuracy, but they differ by the localization technique for the defect problem. While the element-oriented estimator in PLTMG amounts to the solution of local maximal 3×3 subproblems associated with the elements $\tau \in \mathcal{T}_j$, the edge-oriented estimator in KASKADE merely requires the solution of scalar equations associated with the midpoints $m \in \mathcal{M}_j$ of interior edges. For obstacle type problems a semi-local and a local error estimator based on the edge-oriented approach have been suggested in Kornhuber/Roitzsch (1991,1993), Erdmann et al. (1993) and Hoppe/Kornhuber (1993). Both estimators are based on the approximation of a quasivariational inequality constituting the defect problems. In the sequel we will derive that quasivariational inequality, shortly review the basic results from Hoppe/Kornhuber (1993) and also establish a further estimator resulting from the application of fixed point techniques that are widely used for quasivariational inequalities (cf. e.g. Glowinski et al. (1981)).

Denoting by $\tilde{u}_j$ the piecewise linear approximation obtained by the multilevel iterative solution process, we are interested in an estimator ε of the discretization error $u - \tilde{u}_j$ yielding a lower and an upper bound in the sense that the two-sided estimate

$$\gamma_0 \varepsilon \leq \|u - \tilde{u}_j\| \leq \gamma_1 \varepsilon \tag{4.1}$$

holds true with coefficients $0 < \gamma_0 \leq \gamma_1$ depending only moderately on the refinement level j. In order to provide an easily accessible estimate ε we proceed in two steps. First, we replace the unknown exact solution u by an approximation $\hat{u}_j$ of higher accuracy than u_j and then we try to reduce the computational cost for solving the defect problem satisfied by $\hat{u}_j - \tilde{u}_j$.

We refer to $Q_j \subset H_0^1(\Omega)$ as the subspace of Lagrangian finite elements of degree 2 with respect to $\mathcal{T}_j$ and to

$$K_j^Q := \{v_j \in Q_j \,|\, v_j(q) \leq \varphi_j^Q(q),\ q \in \mathcal{N}_j \cup \mathcal{M}_j\}$$

as the associated constraint set where $\varphi_j^Q \in Q_j$ is a piecewise quadratic approximation of the obstacle φ (e.g. the Q_j-interpolate if $\varphi \in C(\Omega)$). Note that $Q_j = \text{span}\{\psi_q^Q \,|\, q \in \mathcal{N}_j \cup \mathcal{M}_j\}$ where $\psi_q^Q|_\tau$, $\tau \in \mathcal{T}_j$, is a polynomial of degree 2 satisfying $\psi_q^Q(q') = \delta_{qq'}$, $q' \in \mathcal{N}_j \cup \mathcal{M}_j$. Then a natural candidate for $\hat{u}_j$ is the continuous, piecewise quadratic approximation $u_j^Q \in K_j^Q$ satisfying the variational inequality

$$a(u_j^Q, v_j - u_j^Q) \geq l(v_j - u_j^Q), \quad v_j \in K_j^Q.$$

Denoting by $r: \; Q_j \to I\!R$ the residual

$$r(v_j) := l(v_j) - a(\tilde{u}_j, v_j), \quad v_j \in Q_j$$

with respect to $\tilde{u}_j$, it is easy to verify that the defect $e_j := u_j^Q - \tilde{u}_j$ is the unique solution of the variational inequality:
Find $e_j \in K_j^E := \{v_j \in Q_j | \; v_j + \tilde{u}_j \in K_j^Q\}$ such that

$$a(e_j, v_j - e_j) \geq r(v_j - e_j), \quad v_j \in K_j^E. \tag{4.2}$$

Evidently, the computation of e_j is as expensive as that of u_j^Q and therefore, following the approach in the unconstrained case (see e.g. Deuflhard et al. (1989)) we use a decoupling based on the hierarchical splitting

$$Q_j = S_j^L \oplus \bigoplus_{\psi \in S_j^Q} V_\psi, \quad V_\psi := \text{span}\,\{\psi\} \tag{4.3}$$

where $S_j^L := S_j$ and S_j^Q stands for the hierarchical surplus, i.e., $S_j^Q = \text{span}\{\psi_m^Q | \; m \in \mathcal{M}_j\}$. Splitting $v_j \in Q_j$ accordingly, i.e., $v_j = (v_j^L, v_j^Q)$, $v_j^L \in S_j^L$, $v_j^Q \in S_j^Q$, and setting $a^L := a|_{S_j^L \times S_j^L}$, $d^Q = \sum_{\psi \in S_j^Q} a|_{V_\psi \times V_\psi}$, we denote by $\tilde{a}$ the bilinear form

$$\tilde{a}(v_j, w_j) := a^L(v_j^L, w_j^L) + d^Q(v_j^Q, w_j^Q) \tag{4.4}$$

associated with the two-level additive Schwarz iteration induced by the direct sum decomposition (4.3). Note that algebraically (4.4) amounts to a block diagonal splitting of the stiffness matrix representing $a|_{Q_j \times Q_j}$ followed by a further diagonal splitting of the subblock associated with $a|_{S_j^Q \times S_j^Q}$. Based on the decoupling (4.4) we then consider the reduced defect problem:
Find $\tilde{e}_j \in K_j^E$ such that

$$\tilde{a}(\tilde{e}_j, v_j - \tilde{e}_j) \geq r(v_j - \tilde{e}_j), \quad v_j \in K_j^E. \tag{4.5}$$

However, in contrast to the unconstrained case the variational inequality (4.5) does not result in a fully localized problem, since there is still a global coupling caused by the constraints. To see this we denote by T the transformation from the hierarchical to the nodal basis representation of Q_j given by

$$v_j = (v_j^L, v_j^Q) \mapsto T v_j := \sum_{p \in \mathcal{N}_j} v_j^L(p) \psi_p^Q + \sum_{m \in \mathcal{M}_j} (v_j^Q + \pi v_j^L)(m) \psi_m^Q$$

where $(\pi v_j^L)(m) := \frac{1}{2}[v_j^L(p_m^1) + v_j^L(p_m^2)]$ and p_m^1, p_m^2 stand for the vertices of the edge $e \in \mathcal{E}_j$ with midpoint m. Then in terms of the hierarchical basis representation $v_j = (v_j^L, v_j^Q)$ the constraint $v_j \in K_j^E$ reads as follows

$$\begin{aligned} v_j^L(p) \leq \varphi_j^Q(p) - \tilde{u}_j(p) \quad &=: \quad \tilde{\varphi}_j^Q(p), \\ (v_j^Q + \pi v_j^L)(m) \leq (\varphi_j^Q - \pi \tilde{u}_j)(m) \quad &=: \quad \tilde{\varphi}_j^Q(m). \end{aligned}$$

Consequently, defining an operator $M: \; Q_j \to Q_j$ by means of

$$M v_j := \sum_{p \in \mathcal{N}_j} \tilde{\varphi}_j^Q(p) \psi_p^Q + \sum_{m \in \mathcal{M}_j} (\tilde{\varphi}_j^Q - \pi v_j^L)(m) \psi_m^Q,$$

the reduced defect problem (4.5) can be equivalently written as the quasivariational inequality:
Find $\tilde{e}_j \in Q_j$ such that $\tilde{e}_j \le M\tilde{e}_j$ and

$$\tilde{a}(\tilde{e}_j, v_j - \tilde{e}_j) \ge r(v_j - \tilde{e}_j), \quad v_j \le Mv_j. \tag{4.6}$$

For the numerical solution of (4.6) we may further reduce the computational cost, if we replace $a^L(\cdot,\cdot)$ in (4.4) by the available bilinear form $a^L_C(\cdot,\cdot)$ associated with the hierarchical ($C = H$) or the BPX preconditioner ($C = B$) giving

$$\hat{a}_C(v_j, w_j) := a^L_C(v^L_j, w^L_j) + d^Q(v^Q_j, w^Q_j), \quad v_j, w_j \in Q_j.$$

We thus end up with the semi-local defect problem:
Find $\hat{e}_j \le M\hat{e}_j$ such that

$$\hat{a}_C(\hat{e}_j, v_j - \hat{e}_j) \ge r(v_j - \hat{e}_j), \quad v_j \le Mv_j. \tag{4.7}$$

The semi-local defect problem (4.7) will be solved numerically by using the same active set strategy as described in section 2. The following result shows that the semi-local error estimator

$$|\hat{e}_j|_{\hat{a}_C} := \hat{a}_C(\hat{e}_j, \hat{e}_j)^{1/2} \tag{4.8}$$

provides a lower and an upper bound for the discretization error in the sense of (4.1).

Theorem 4.1 *Assume that there are constants* $0 \le q < 1$ *and* $\sigma \ge 0$, $q\sigma < 1$, *independent of the refinement level* j, *such that for* $u^L_j := u_j$, $\tilde{u}_j$ *and* u^Q_j *there holds*

$$\begin{aligned} \|u - u^Q_j\| &\le q\|u - u^L_j\|, \quad j \in I\!N \cup \{0\}, && (4.9)\\ \|u - u^L_j\| &\le \sigma\|u - \tilde{u}_j\|, \quad j \in I\!N \cup \{0\}. && (4.10)\end{aligned}$$

Then there exist positive constants γ_0, γ_1 *depending only on* $q\sigma$, *the ellipticity of* $a(\cdot,\cdot)$ *and the shape regularity of* $\mathcal{T}_0$ *such that*

$$\gamma_0(j+1)^{-s}|\hat{e}_j|_{\hat{a}_C} \le \|u - \tilde{u}_j\| \le \gamma_1(j+1)^s|\hat{e}_j|_{\hat{a}_C} \tag{4.11}$$

where $s = 1$ *in case of the hierarchical preconditioner and the dimension* $d = 2$ *while* $s = 1/2$ *holds for the BPX preconditioner independently of the dimension* d.

Proof (in the hierarchical case cf. Hoppe/Kornhuber (1993)). It follows from Bornemann (1991), Yserentant (1990) and Zhang (1992) that there exist positive constants C_0, C_1 depending only on the ellipticity of $a(\cdot,\cdot)$ and the shape regularity of $\mathcal{T}_0$ such that

$$C_0\hat{a}_C(v_j, v_j) \le a(v_j, v_j) \le C_1(j+1)^{2s}\hat{a}_C(v_j, v_j), \quad v_j \in Q_j. \tag{4.12}$$

Then, in view of (4.2) and (4.7) we have

$$a(e_j, e_j) \le C_1(j+1)^{2s}|\hat{e}_j|^2_{\hat{a}_C} + 2r(e_j - \hat{e}_j). \tag{4.13}$$

Using the Lipschitz-continuous dependence of the solution of the variational inequality (4.7) on the right-hand side with Lipschitz constant C_0^{-1} we find

$$r(e_j - \hat{e}_j) \le |\hat{e}_j|_{\hat{a}_C} |e_j - \hat{e}_j|_{\hat{a}_C} \le C_0^{-1}(1 + C_1(j+1)^{2s})|\hat{e}_j|^2_{\hat{a}_C}. \tag{4.14}$$

Using (4.14) in (4.13) gives

$$a(e_j, e_j) \le \bar{C}_1 (j+1)^{2s} |\hat{e}_j|^2_{\hat{a}_C}, \quad \bar{C}_1 := C_0^{-1}(2(1+C_1) + C_0 C_1).$$

Analogously, one can prove

$$a(e_j, e_j) \ge \bar{C}_0 (j+1)^{-2s} |\hat{e}_j|^2_{\hat{a}_C}, \quad \bar{C}_0 := \frac{1}{2} C_0 (C_1(1+C_0))^{-1}.$$

Finally, taking advantage of (4.9) the assertion follows with $\gamma_0 := C_0^{-1/2}(1+q\sigma)^{-1}$ and $\gamma_1 := C_1^{-1/2}(1-q\sigma)^{-1}$. □

We get a fully local, but less reliable error estimator which has been originally proposed in Kornhuber and Roitzsch (1991), if we decouple the defect problem (4.6) by the application of just one block Gauss-Seidel iteration using $\tilde{e}_j^0 = (\tilde{e}_j^{L,0}, \tilde{e}_j^{Q,0}) = (0,0)$ as a startiterate. Denoting by r^L, r^Q the restrictions of the residual r to S_j^L, S_j^Q, respectively, this amounts to the computation of $\hat{\delta}_j = (\hat{\delta}_j^L, \hat{\delta}_j^Q)$ by successive solution of the variational inequalities:

1. Find $\hat{\delta}_j^L \in D_j^L := S_j^L \cap K_j^E$ such that
$$a^L(\hat{\delta}_j^L, v_j - \hat{\delta}_j^L) \ge r^L(v_j - \hat{\delta}_j^L), \quad v_j \in D_j^L, \tag{4.15}$$
2. Find $\hat{\delta}_j^Q \in D_j^Q(\hat{\delta}_j^L) := \{v_j^Q \in S_j^Q | \; v_j^Q + \hat{\delta}_j^L \in K_j^E\}$ such that
$$d^Q(\hat{\delta}_j^Q, v_j - \hat{\delta}_j^Q) \ge r^Q(v_j - \hat{\delta}_j^Q), \quad v_j \in D_j^Q(\hat{\delta}_j^L). \tag{4.16}$$

If we assume

$$K_j^L \subset K_j^Q \tag{4.17}$$

where $K_j^L := K_j$, it follows readily from (4.15) that $\hat{\delta}_j^L = u_j - \tilde{u}_j$. On the other hand, the computation of $\hat{\delta}_j^Q$ merely requires the solution of scalar inequalities associated with the midpoints $m \in \mathcal{M}_j$ of interior edges. For that reason we refer to

$$|\hat{\delta}_j|_{\tilde{a}} := (|u_j - \tilde{u}_j|^2_{a^L} + |\hat{\delta}_j^Q|^2_{d^Q})^{1/2}, \tag{4.18}$$

where $|v_j|_{a^L} := a^L(v_j, v_j)^{1/2}$, $v_j \in S_j^L$, as a local error estimator. Note that an estimate for the iteration error $|u_j - \tilde{u}_j|_{a^L}$ is easily available from the multilevel iterative solution process.

We may interpret the linear part $\hat{e}_j^L$ of the semi-local estimator as a perturbation of the linear part $\hat{\delta}_j^L$ of the local estimator caused by the global coupling of the constraints. We assume that this perturbation remains local for increasing refinement level j in the sense that there exists a constant $\beta > 0$ independent of j such that

$$|\pi(\hat{e}_j^L - \hat{\delta}_j^L)|_{d^Q} \le \beta |\hat{e}_j^L - \hat{\delta}_j^L|_{a^L}. \tag{4.19}$$

Then the local error estimator provides at least a lower bound for the discretization error as is stated in the following result which we quote from Hoppe and Kornhuber (1993):

Theorem 4.2 *Suppose that (4.9), (4.17) and (4.19) hold true. Then there exists a constant $\gamma_0 > 0$ depending only on $q\sigma$, β, the ellipticity of $a(\cdot,\cdot)$ and the shape regularity of $\mathcal{T}_0$ such that*

$$\gamma_0 |\hat{\delta}_j|_{\hat{a}} \leq \|u - \tilde{u}_j\|. \tag{4.20}$$

Finally, we present a third a posteriori error estimator which is also based on the reduced defect problem (4.7) which is further simplified by replacing the level 0 matrix A_0^D in (3.11) by its diagonal (cf. e.g. Yserentant (1990)). For the computation of $\hat{e}_j$ we use the fact that $\hat{e}_j$ is the unique fixed point of the operator $\Sigma : Q_j \to Q_j$ which assigns to $w_j \in Q_j$ the unique solution $z_j = \Sigma w_j$ of the variational inequality:
Find $z_j \leq M w_j$ such that

$$\hat{a}_C(z_j, v_j - z_j) \geq r(v_j - z_j), \quad v_j \leq M w_j.$$

Starting from an appropriate startiterate $\hat{e}_j^0$ we can show the following:

Theorem 4.3 *Let $\hat{e}_j^0 = (\hat{e}_j^{L,0}, \hat{e}_j^{Q,0})$ with $\hat{e}_j^{L,0}$ being the solution of the variational equation*

$$a_C^L(\hat{e}_j^{L,0}, v_j) = r^L(v_j), \quad v_j \in S_j^L$$

and $\hat{e}_j^{Q,0}$ being given arbitrarily. Then the sequence $(\hat{e}_j^\nu)_{\nu \geq 1}$ of iterates obtained by the fixed point iteration

$$\hat{e}_j^\nu = \Sigma \hat{e}_j^{\nu-1}, \quad \nu \geq 1 \tag{4.21}$$

satisfies

$$\hat{e}_j^{2\nu-1} \leq \hat{e}_j^{2\nu+1} \leq \hat{e}_j \leq \hat{e}_j^{2\nu+2} \leq \hat{e}_j^{2\nu}, \quad \nu \geq 1. \tag{4.22}$$

Moreover, if $\lim_{\nu\to\infty} \hat{e}_j^{2\nu-1} = \lim_{\nu\to\infty} \hat{e}_j^{2\nu} = \hat{e}_j^*$, then $\hat{e}_j^* = \hat{e}_j$ is the unique solution of the reduced defect problem (4.7).

Proof. For the proof of (4.22) it is convenient to rewrite the fixed point iteration (4.21) algebraically as a system of linear complementary problems. Denoting by A_C^L, D^Q and Π the matrix representations of $a_C^L(\cdot,\cdot)$, $d^Q(\cdot,\cdot)$ and the mapping π, respectively, and identifying finite element functions with vectors, (4.21) is equivalent to the successive solution of the linear complementary problems:
Find $\hat{e}_j^\nu = (\hat{e}_j^{L,\nu}, \hat{e}_j^{Q,\nu})$ such that

$$\begin{aligned} &\hat{e}_j^{Q,\nu} \leq \tilde{\varphi}_j^Q - \Pi \hat{e}_j^{L,\nu-1}, \quad D^Q \hat{e}_j^{Q,\nu} \leq r^Q \\ &< \hat{e}_j^{Q,\nu} - (\tilde{\varphi}_j^Q - \Pi \hat{e}_j^{L,\nu-1}), D^Q \hat{e}_j^{Q,\nu} - r^Q >= 0, \end{aligned} \tag{4.23}$$

$$\begin{aligned} &\hat{e}_j^{L,\nu} \leq \tilde{\varphi}_j^L, \quad A_C^L \hat{e}_j^{L,\nu} \leq r^L + \Pi^T(D^Q \hat{e}_j^{Q,\nu} - r^Q) \\ &< \hat{e}_j^{L,\nu} - \tilde{\varphi}_j^L, A_C^L \hat{e}_j^{L,\nu} - (r^L + \Pi^T(D^Q \hat{e}_j^{Q,\nu} - r^Q)) >= 0 \end{aligned} \tag{4.24}$$

where $< \cdot,\cdot >$ stands for the standard inner product in $\mathbb{R}^{n_j^L}$ and $\mathbb{R}^{n_j^Q}$, respectively ($n_j^L := \dim S_j^L$, $n_j^Q := \dim S_j^Q$). Using the fact that under the assumptions of this theorem the solutions of (4.23), (4.24) monotonically depend on the upper obstacles and right-hand sides, respectively, and that D^Q and Π are nonnegative matrices, the proof of (4.22) is by induction on ν. Since $\hat{e}_j^{L,0}$ is the solution of the

unconstrained problem $A_C^L \hat{e}_j^{L,0} = r^L$, we have $\hat{e}_j^L \le \hat{e}_j^{L,0}$. This implies $\Pi \hat{e}_j^L \le \Pi \hat{e}_j^{L,0}$ whence $\hat{e}_j^{Q,1} \le \hat{e}_j^Q$ by means of (4.23). It follows that $\Pi^T D^Q \hat{e}_j^{Q,1} \le \Pi^T D^Q \hat{e}_j^Q$ which in view of (4.24) gives $\hat{e}_j^{L,1} \le \hat{e}_j^L$. Repeating these arguments proves (4.22) for $\nu = 1$. Assuming (4.22) for some $\nu \ge 1$, the proof that the inequalities also hold true for $\nu + 1$ can be given in exactly the same way as in the case $\nu = 1$.
If $\lim_{\nu\to\infty} \hat{e}_j^{2\nu-1} = \lim_{\nu\to\infty} \hat{e}_j^{2\nu} = \hat{e}_j^*$, the continuity of Σ implies that $\hat{e}_j^*$ is a fixed point of Σ and hence, $\hat{e}_j^* = \hat{e}_j$ by uniqueness. □

It should be emphasized that the convergence of the monotonically increasing sequence $(\hat{e}_j^{2\nu-1})_{\nu\ge 1}$ of subsolutions and the monotonically decreasing sequence $(\hat{e}_j^{2\nu})_{\nu\ge 1}$ of supersolutions to the same limit is not guaranteed. Indeed, it may happen that we end up with blockage points

$$\lim_{\nu\to\infty} \hat{e}_j^{2\nu-1} = \hat{e}_j^* < \hat{e}_j < \hat{e}_j^{**} = \lim_{\nu\to\infty} \hat{e}_j^{2\nu}.$$

In this case, based on a purely heuristical argument we use the arithmetic mean $\hat{e}_j^a := \frac{1}{2}(\hat{e}_j^* + \hat{e}_j^{**})$ as a substitute for the solution $\hat{e}_j$. The corresponding error estimator $|\hat{e}_j|_{\hat{a}_C}$ resp. $|\hat{e}_j^a|_{\hat{a}_C}$ will be referred to as the "fixed point" error estimator.

A final remark should be due to the criterion for local refinement of an element $\tau \in \mathcal{T}_j$. In view of the decoupling (4.4) the local contribution of an edge $e \in \mathcal{E}_j$ with midpoint $m \in \mathcal{M}_j$ to the total error is given by $\eta_e = d_m^2 a(\psi_m^Q, \psi_m^Q)$ where d_m stands for the computed nodal value in m. Then, an element $\tau \in \mathcal{T}_j$ will be marked for refinement, if the value η_e of at least one of the edges $e \in \mathcal{E}_j$ of τ exceeds a certain threshold η. In case $j > 0$ such a bound $\eta = \eta^E$ will be computed by local extrapolation. In particular, for all edges $e \in \mathcal{E}_j$ which have been obtained by refinement of a "father" edge $g \in \mathcal{E}_k$, $k < j$, extrapolation to the next level yields $\Theta_e = \eta_e^2 \setminus \eta_g$ while we set $\Theta_e = 0$ if there is not such a "father" edge. We then choose

$$\eta^E := \sigma^E \max\{\Theta_e |\, e \in \mathcal{E}_j\}$$

where $0 < \sigma^E < 1$ is an appropriate safety factor. In case $j = 0$, i.e., for the given coarse triangulation $\mathcal{T}_0$, we compute the arithmetic mean $\Theta_M := |\mathcal{E}_0|^{-1} \sum_{e\in\mathcal{E}_0} \eta_e$ of all local error indicators and use $\eta = \eta^M$ with η^M given by

$$\eta^M := \sigma^M \Theta_M$$

where again $0 < \sigma^M < 1$ is a suitable safety factor.

5 Numerical Results

In this section we will apply the adaptive algorithm to some selected stationary free boundary problems in two and three space dimensions. The problems include the torsion of an elastic, ideally plastic cylindrical bar, lubrication in journal bearings and stationary flow in porous media as in the dam problem and in the axialsymmetric water cone problem in oil reservoir simulation. In each case we will illustrate the adaptive refinement process by displaying the triangulations or clippings thereof at some selected levels and we will compare both the performance of the four multilevel preconditioners and of the three a posteriori error estimators. For notational brevity the symmetric and nonsymmetric hierarchical preconditioners will be denoted by H, S and H, NS and its BPX-type counterparts by B, S and B, NS, respectively. Further, the local, the semi-local and the "fixed point" error estimator will be referred to as Estimator 1, 2 and 3.

If not stated otherwise, the components of the adaptive code are given as follows: Starting with the interpolation of the final iterate from the previous level the outer active set strategy stops as soon as the active set remains invariant. We use the symmetric BPX preconditioner for the preconditioned inner cg-iterations which will be terminated if the estimated iteration error is less than $\kappa = 10^{-2}$. Furthermore, local refinement is based on the semi-local error estimator using the extrapolation strategy with $\sigma^E = 0.5$.

5.1 Elasto-plastic torsion problem

We consider an elastic, ideally plastic isotropic cylindrical bar $Q := \Omega \times (0, l)$ of cross section $\Omega \subset I\!R^2$ and length $l > 0$ with m cylindrical cavities $Q_i := \Omega_i \times (0, l)$ having the same direction of generatrices and cross sections $\Omega_i \subset \Omega$, $1 \leq i \leq m$, such that $\bar{\Omega}_i \cap \bar{\Omega}_j = \emptyset$, $1 \leq i \neq j \leq m$. Denoting by $\partial Q_l := \Omega \times \{l\}$, $\partial Q_0 := \Omega \times \{0\}$ the upper and lower ends of the bar and by $\partial Q_s := \Gamma \times (0, l)$, $\Gamma := \partial\Omega$, the lateral surface, we suppose that at the upper end ∂Q_l the bar is twisted around its longitudinal x_3-axis by a twist-angle $\Theta > 0$ in such a way that the lateral surface ∂Q_s remains stress-free. Under this assumption the stress potential u is independent of the x_3-coordinate so that we are faced with a 2-D problem. In particular, using Hencky's law, modeling the plastic region according to the von Mises yield criterion and normalizing physical constants, it can be shown (cf. e.g. Lanchon (1970)) that u is the unique solution of the variational inequality:

Find $u \in K := \{v \in H_0^1(\Omega) |\ v|_{\bar{\Omega}_i} = c_i,\ 1 \leq i \leq m,\ |\nabla v| \leq 1 \text{ a.e. in } \Omega\}$ such that

$$\int_{\Omega^*} \nabla u \cdot \nabla (v - u)\, dx \geq 2C \int_{\Omega^*} (v - u)\, dx, \quad v \in K \tag{5.1}$$

where $C := \Theta / l$ stands for the torsion angle per unit length, $\Omega^* := \Omega \setminus (\bigcup_{i=1}^m \bar{\Omega}_i)$ is the region effectively occupied by the elastic-plastic material and c_i, $1 \leq i \leq m$, are constants (see below). Consequently, the two subregions $\Omega_E^* := \{x \in \Omega^* |\ |\nabla u| <$ a.e.$\}$ and $\Omega_P^* := \{x \in \Omega^* |\ |\nabla u| = 1$ a.e.$\}$ represent the elastic and plastic region being separated by the free boundary $\Gamma_F^* = \partial\bar{\Omega}_E^* \cap \bar{\Omega}_P^*$.

Setting $\Omega_0 := I\!R^2 \setminus \bar{\Omega}$ and $\Gamma_0 := \partial\Omega_0$, $\Gamma_i := \partial\Omega_i$ we consider the sets Λ_i, $0 \leq i \leq m$, of directed paths from Γ_i to Γ_0 where a path $P \in \Lambda_i$ consists of directed edges $P_{i_j,i_{j+1}}$, $0 \leq j \leq n-1$, $n \in I\!N$, of length $\mathrm{dist}(\Omega_{i_j}, \Omega_{i_{j+1}}$ connecting Ω_{i_j} and $\Omega_{i_{j+1}}$ such that $i_0 = i$, $i_n = 0$ and $i_j \neq i_k$ for $0 \leq j \neq k \leq n$. We denote by d_i the length of the shortest path within Λ_i and define $\varphi: \ \Omega \to I\!R$ as the generalized distance function

$$\varphi(x) = \inf_{0 \leq i \leq m} [\mathrm{dist}(x, \Omega_i) + d_i], \quad x \in \Omega. \tag{5.2}$$

Then it is well known (cf. Lanchon (1970)) that (5.1) is closely related to the obstacle problem:

Find $u \in K^* := \{v \in H_0^1(\Omega) | \ v|_{\bar{\Omega}_i} = c_i, \ 1 \leq i \leq m, \ v \leq \varphi$ a.e. in $\Omega\}$ such that

$$\int_{\Omega^*} \nabla u \cdot \nabla(v-u)\, dx \geq 2C \int_{\Omega^*} (v-u)\, dx, \quad v \in K^*. \tag{5.3}$$

where $c_i := \varphi|_{\bar{\Omega}_i}$. Indeed, in case $m = 0$ of simply connected domains, where φ reduces to $\mathrm{dist}(\cdot, \Gamma)$, the equivalence between (5.1) and (5.2) has been established by Brézis and Sibony (1971). For $m \geq 1$ there is no strict mathematical proof for that equivalence which, however, is strongly supported by various numerical results (cf. e.g. Glowinski/Lanchon (1973) and Hoppe (1988)).

As a first example we have chosen a bar with simply connected cross section $\Omega = (0,1)^2$ and $C = 15$. Figure 5.1a shows the initial coarse triangulation $\mathcal{T}$ while Figures 5.1b, c and d represent adaptively refined triangulations at some selected levels. Note that nodal points within the plastic region Ω_P^* are marked by a black square.

For a comparison of the performance of the preconditioners, in Figure 5.2 we have plotted the average number of inner cg-iterations versus the total number of nodal points. To amplify the different behavior we chose κ unreasonable small, i.e. $\kappa = 10^{-4}$ and the initial iterate is fixed to the obstacle for all inner iterations. We see that preconditioning becomes effective as soon as we have a sufficiently good resolution of the free boundary between the elastic and the plastic region. As predicted by the theoretical findings, all preconditioners asymptotically behave in the same way with the symmetric versions giving better results.

The complete history of the adaptive refinement process is reflected by Table 5.1 containing the level, the depth, the total number of inner nodal points and the number of outer iterations and average number of inner cg-iterations both for the iterative solution and the error estimation. Note that the depth of a triangulation is the maximal number of ancestors of its elements which may be less than the actual level of the refinement process (cf. Deuflhard et al. (1989)).

For a comparison of the performance of the error estimators, in Figure 5.3 we have plotted the "exact" error and the error predicted by the three error estimators versus the total number of nodal points. Note that the error in Figure 5.3 is given in units of 10^{-1}. Moreover, to compute the "exact" error we have performed two uniform refinements of the final triangulation and accepted the corresponding result as the "exact" solution. We see that at the lower levels the local error estimator badly underestimates the exact error but produces acceptable results as soon as the free boundary is resolved by a sufficiently good accuracy. On the other

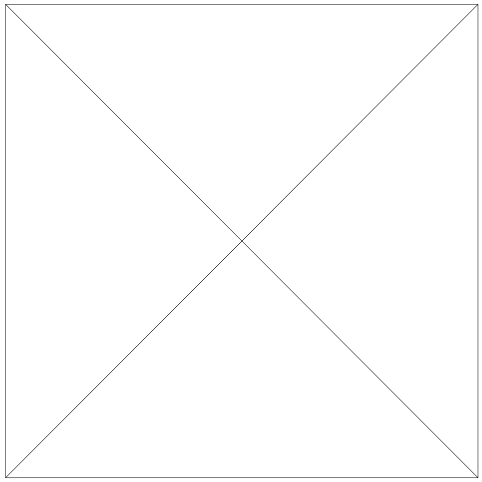

Fig. 5.1a Initial Triangulation

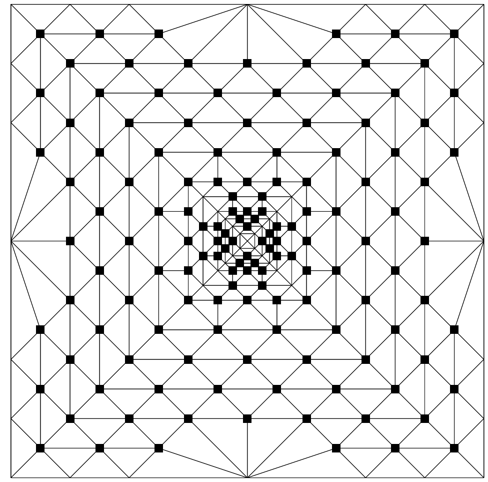

Fig. 5.1b Level 5

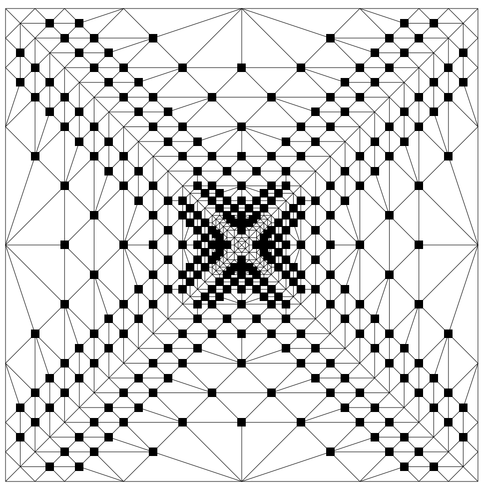

Fig. 5.1c Level 6

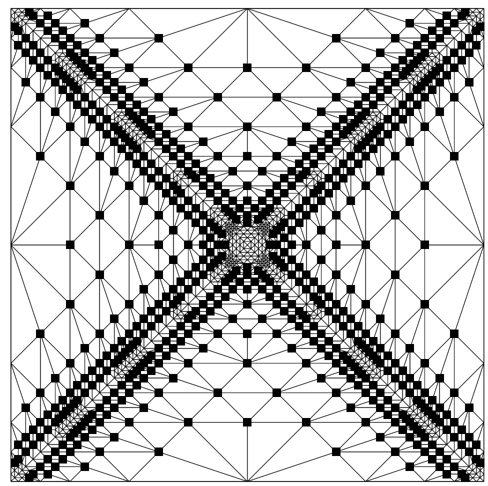

Fig. 5.1d Level 8

Table 1: History of the iterative process

Level	Depth	Nodes	Iterations: Solution	Iterations: Error Est.
0	0	1	2/0.0	1/2.0
1	1	5	1/1.0	1/2.0
2	2	13	1/1.0	2/1.5
3	3	53	1/2.0	2/1.5
4	4	69	1/1.0	2/2.0
5	5	161	2/1.5	2/1.5
6	6	461	2/2.0	2/1.5
7	7	673	2/2.0	1/1.0
8	7	1177	2/3.0	1/1.0
9	7	3153	3/2.3	1/1.0
10	8	9877	1/1.0	1/0.0

hand, at the lower levels we have a pronounced overestimation by the "fixed point" error estimator which results in a uniform refinement on these levels. Obviously, the most reliable estimates are provided by the semi-local error estimator. It should be emphasized, however, that asymptotically all estimators behave in the same way.

We have performed a series of computations for cylindrical bars with multiply connected cross sections. In all cases we did observe a similar behaviour of the preconditioners and the error estimators. As representative examples we present the results for a bar with one hole and torsion number $C = 7.5$ and for a bar with four symmetrically distributed holes and torsion number $C = 5.0$. In particular, Figures 5.4a,b and 5.5a,b represent the initial triangulation and an intermediate level of the refinement process while Tables 5.1, 5.1 reflect the history of the iterative process, respectively (for further details see Frei (1992) and Wiest (1991)).

Table 2: History of the iterative process

			Iterations	
Level	Depth	Nodes	Solution	Error Est.
1	1	12	1/1.0	1/1.0
2	2	50	1/1.0	1/3.0
3	3	130	3/1.6	1/1.0
4	4	417	4/2.0	1/1.0
5	5	776	3/2.0	1/1.0
6	6	1759	4/2.3	1/1.0
7	7	2993	3/2.3	1/1.0
10	9	4053	3/2.3	1/1.0
15	11	5082	2/3.5	1/0.0

Table 3: History of the iterative process

			Iterations	
Level	Depth	Nodes	Solution	Error Est.
0	0	5	1/0.0	1/1.0
1	1	33	3/2.3	1/1.0
2	2	113	1/3.0	1/1.0
3	3	317	2/3.0	1/1.0
4	4	681	3/2.6	1/1.0
6	6	1313	3/3.0	1/1.0
8	8	2541	2/2.5	1/1.0
9	9	4501	3/2.3	1/1.0
10	10	8517	2/2.5	1/0.0

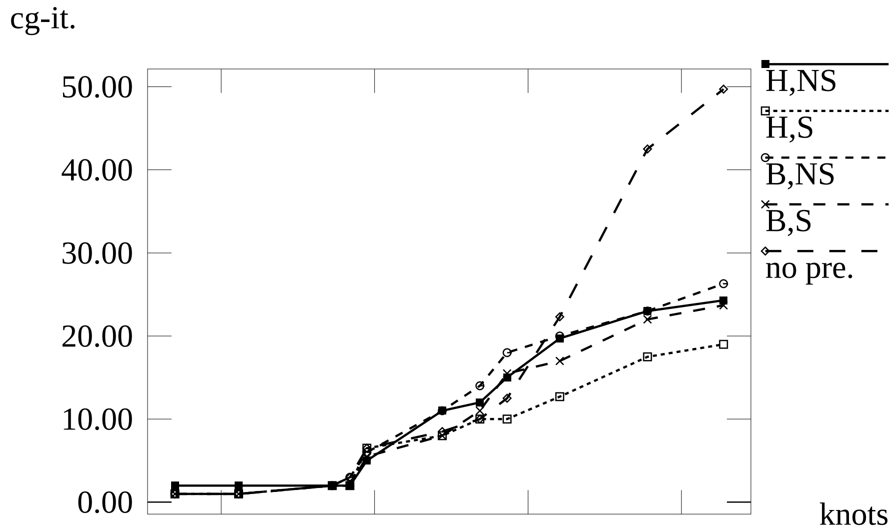

Fig. 5.2 Comparison of the preconditioners

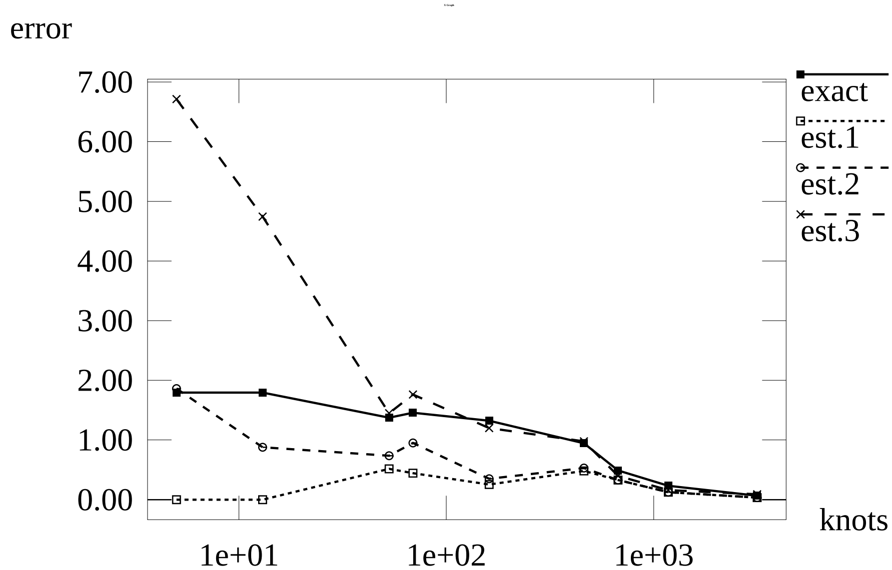

Fig. 5.3 Comparison of the error estimators

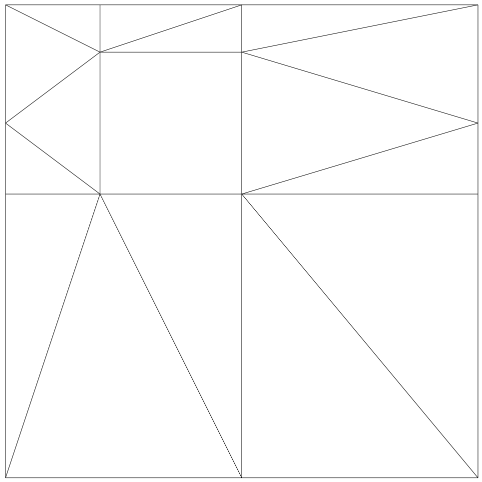

Fig. 5.4a Initial triangulation

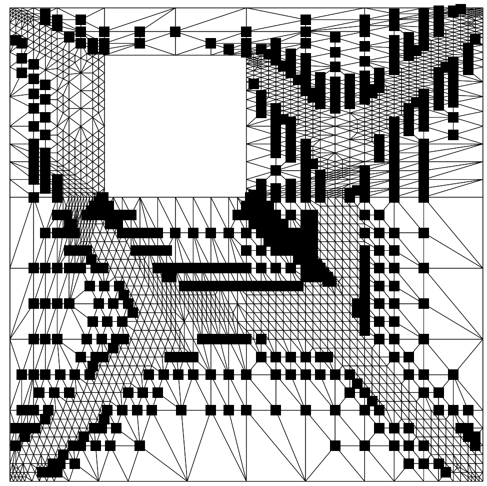

Fig. 5.4b Level 6

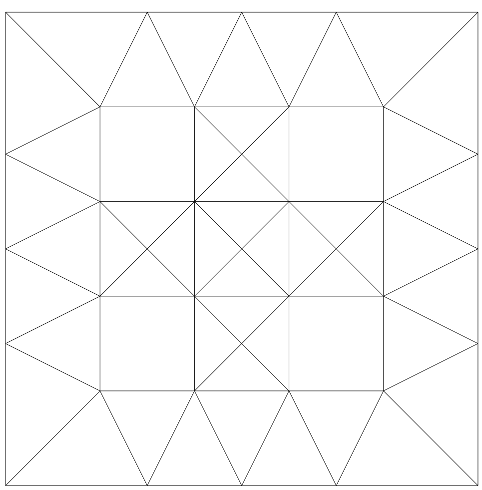

Fig. 5.5a Initial triangulation

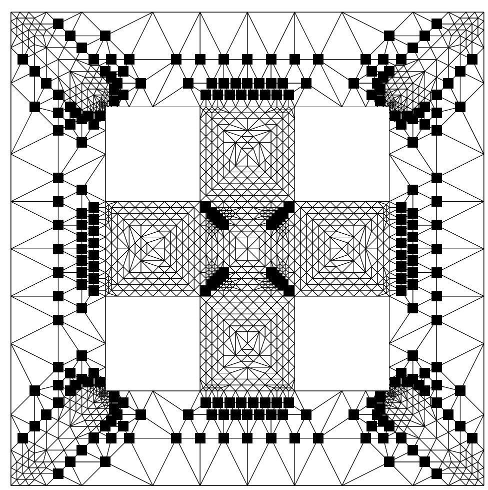

Fig. 5.5b Level 6

5.2 Lubrication in journal bearings

We consider a journal bearing consisting of a rotating cylinder which is separated from the bearing surface by a thin film of lubricating fluid (cf. Figure 5.6a).

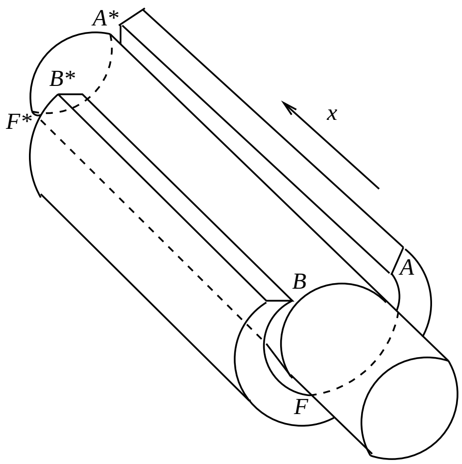

Fig. 5.6a Journal bearings

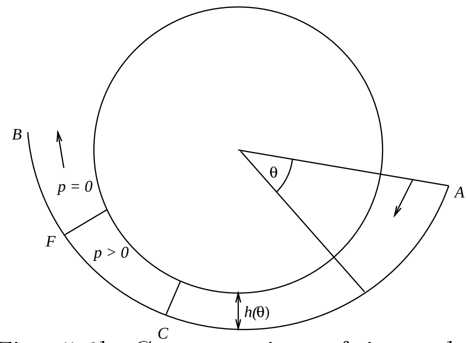

Fig. 5.6b Cross section of journal bearings

The fluid is fed in between A and A^* and flows out between B and B^*. Introducing the angle Θ as an independent variable (cf. Figure 5.6b) and performing the coordinate transformation $x_2 = \Theta/\pi$, the computational domain Ω is given by $\Omega := \{(x_1, x_2) \in I\!R^2 |\ 0 < x_1 < a := \text{dist}(A, A^*),\ 0 < x_2 < b := \Theta_B/\pi\}$, $\Theta_B \le 2\pi$. The width of the film can be modeled by the function $h(x_1, x_2) := (1 + \epsilon \cos(\pi x_2))/\sqrt{\pi}$ where $0 \le \epsilon < 1$ is the eccentricity ratio (cf. e.g.Pinkus and Sternlicht (1961)). Evidently, the width is increasing in the subdomain $[0, a] \times [1, b]$ causing the pressure in the lubricating film to decrease. At the line $S_P := [0, a] \times \{x_2^F\}$, $1 < x_2^F < b$, we assume the pressure to become so low that the fluid vaporizes. The interface S_P represents a free boundary separating the liquid phase $\Omega_L := \{x \in \Omega |\ p(x) > 0\}$ from the gaseous phase $\Omega_G := \{x \in \Omega |\ p(x) = 0\}$ of the fluid.

As shown e.g. in Crank (1987) and Cryer (1971) the problem can be formulated as the following 2-D obstacle problem:

Find $p \in K := \{q \in H_0^1(\Omega) |\ q \ge 0 \text{ a.e. in } \Omega\}$ such that

$$\int_\Omega h^3 \nabla p \cdot \nabla(q - p)\, dx \ge - \int_\Omega \frac{\partial h}{\partial x_2}(q - p)\, dx, \quad q \in K. \tag{5.4}$$

In our computations we have used $a = 4$ and $\Theta_B = 2\pi$. Moreover, the eccentricity ratio ϵ has been chosen as $\epsilon = 0.4$. Figure 5.7a shows the initial triangulation $\mathcal{T}$ while Figures 5.7b and c represent some selected levels during the adaptive refinement process. Again, nodal points within the gaseous phase Ω_G are marked by a black square.

Figures 5.8 and 5.9 illustrate the performance of the preconditioners and the error estimators, respectively, while Table 5.2 covers the history of the adaptive refinement process.

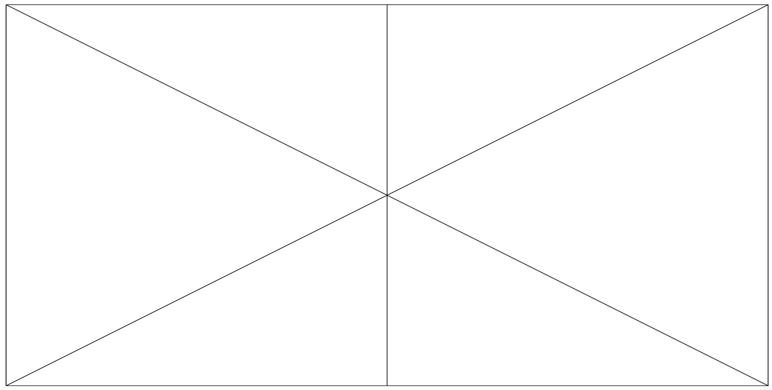

Fig. 5.7a Initial triangulation

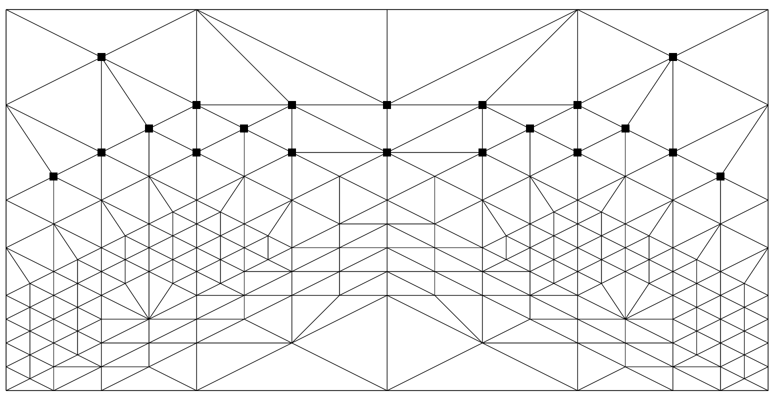

Fig. 5.7b Level 4

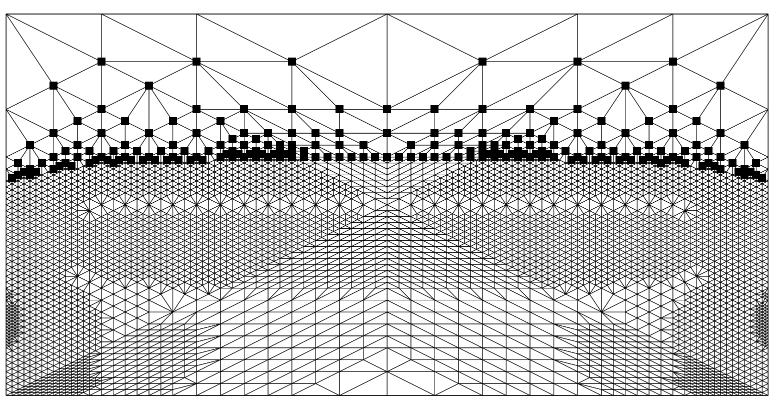

Fig. 5.7c Level 10

As opposed to the elastic-plastic torsion problem, in this application the non-symmetric versions of the preconditioners perform better than their symmetric counterparts which can be partially explained by the fact that there already is a good resolution of the free boundary on the lower levels. This is also reflected by the small difference in the performance of the error estimators.

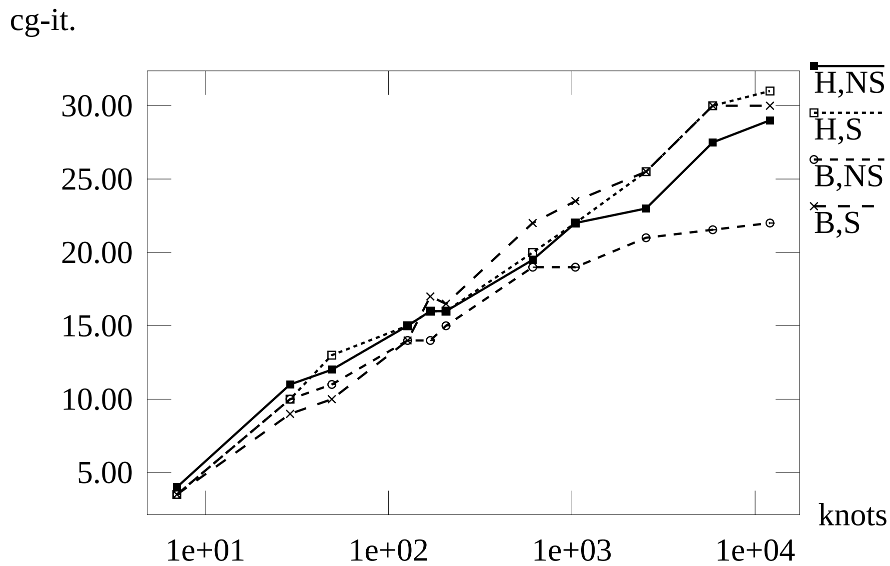

Fig. 5.8 Comparison of the preconditioners

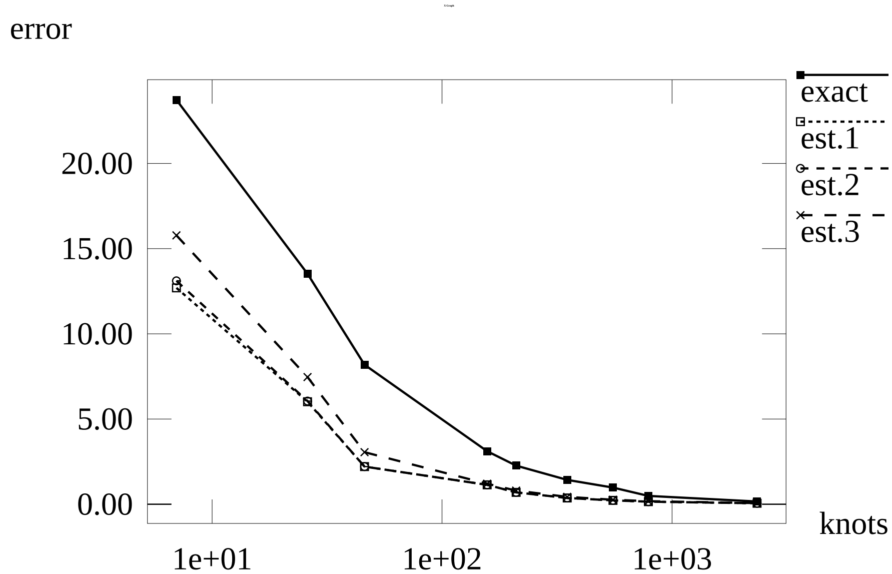

Fig. 5.9 Comparison of the error estimators

Table 4: History of the iterative process

Level	Depth	Nodes	Iterations	
			Solution	Error Est.
0	0	1	1/0.0	1/3.0
1	1	7	2/1.5	1/3.0
2	2	26	2/1.5	1/3.0
3	3	46	1/3.0	1/1.0
4	4	157	2/3.0	1/1.0
5	4	210	2/2.0	1/1.0
6	5	350	2/2.0	1/1.0
7	5	554	1/3.0	1/1.0
8	5	789	2/2.0	1/1.0
9	6	2230	2/2.5	1/1.0
10	7	3491	2/2.0	1/1.0
11	7	8317	2/2.5	1/0.0

5.3 Seepage flow through a three-dimensional dam

We consider a porous dam occupying a 3-D domain $Q := \Omega \times (0, H)$ with an L-shaped cross section $\Omega := (0, 2a) \times (0, 2b) \setminus [a, 2a) \times (0, b]$ which separates two water reservoirs at constant height so that the inlet face ∂Q_{in} and the outlet face ∂Q_{out} are given by $\partial Q_{\text{in}} := \Gamma_{\text{in}} \times [0, H]$, $\Gamma_{\text{in}} := \{0\} \times [0, 2b]$ and $\partial Q_{\text{out}} := \Gamma_{\text{out}} \times [0, h]$, $\Gamma_{\text{out}} := (\{a\} \cup \{2a\}) \times [0, b]$. We assume that the dam consists of a homogeneous, isotropic material and has an impervious bottom $\partial Q_b := \Omega \times \{0\}$. We denote by Q_w the wet part of the dam and by $u(x_1, x_2, x_3) = p(x_1, x_2, x_3) + x_3$ the piezometric head where p stands for the inner pressure of the water in Q_w. Then extending u into $Q \backslash Q_w$ by $u(x_1, x_2, x_3) = x_3$ and using the Baiocchi transformation (cf. e.g. Baiocchi et al. (1973))

$$w(x_1, x_2, x_3) = \int_{x_3}^{H} (u(x_1, x_2, s) - s)\, ds,$$

it can be shown (cf. e.g. Caffrey and Bruch (1979)) that w satisfies the variational inequality:

Find $w \in K := \{v \in H^1(Q) |\ v|_{\partial Q_b} = w_b,\ v|_{\partial Q_D} = g,\ v \geq 0 \text{ a.e. in } Q\}$ such that

$$\int_Q \nabla w \cdot \nabla (v - w)\, dx \geq - \int_Q (v - w)\, dx, \quad v \in K \tag{5.5}$$

where the Dirichlet data g on $\partial Q_D := \partial Q \setminus (\partial Q_N \cup \partial Q_b)$, $\partial Q_N := (0, a) \times \{0\} \times (0, H) \cup (0, 2a) \times \{2b\} \times (0, H)$ are given by

$$g(x_1, x_2, x_3) = \begin{cases} \frac{1}{2}(H - x_3)^2 & , \quad (x_1, x_2, x_3) \in \partial Q_{\text{in}} \\ \frac{1}{2}(h - x_3)^2 & , \quad (x_1, x_2, x_3) \in \partial Q_{\text{out}} \\ 0 & , \quad (x_1, x_2, x_3) \in \Gamma_{\text{out}} \times [h, H] \\ 0 & , \quad (x_1, x_2, x_3) \in \bar{\Omega} \times \{H\} \end{cases}$$

and where w_b is harmonic in ∂Q_b fulfilling the boundary conditions

$$w_b = \begin{cases} \frac{1}{2}H^2 & \text{on } \Gamma_{\text{in}} \\ \frac{1}{2}h^2 & \text{on } \Gamma_{\text{out}} \end{cases}$$

$$\partial w_b/\partial x_2 = 0 \text{ on } (0,a)\times\{0\},\ (a,2a)\times\{b\} \text{and } (0,2a)\times\{2b\}.$$

In particular, the computation of the Dirichlet data on the bottom ∂Q_b of the dam requires the solution of the 2-D Laplace equation with boundary data as given above.

We have applied the adaptive 3-D algorithm to a dam with the data $a = b = 10.0$, $H = 10.0$ and $h = 2.0$. Figures 5.10a,b represent the surface of the initial and of the final triangulation, respectively, while Figure 5.10c shows the clipping plane $\partial Q_1 := [0,20]\times\{10\}\times[0,10]$ of the final triangulation. Moreover, Figure 5.11a illustrates the level curves of the pressure in the wet part of the dam. Note, that the dry part is shown as the shaded region. To demonstrate the corner singularity of the solution, Figure 5.11b contains the level curves on the clipping plane ∂Q_1. As can be clearly seen from Figure 5.10c most refinement has been done in a neighborhood of the reentrant corner. Finally, Table 5.3 summarizes the history of the adaptive refinement process.

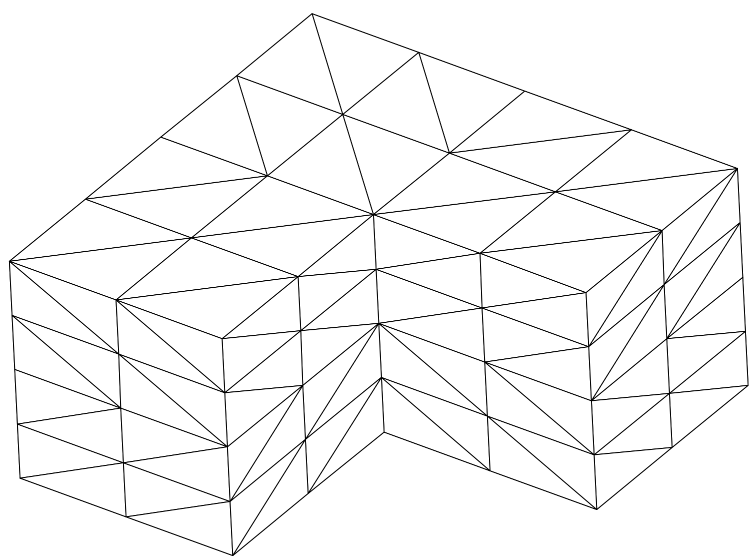

Fig. 5.10a Surface of the initial triangulation

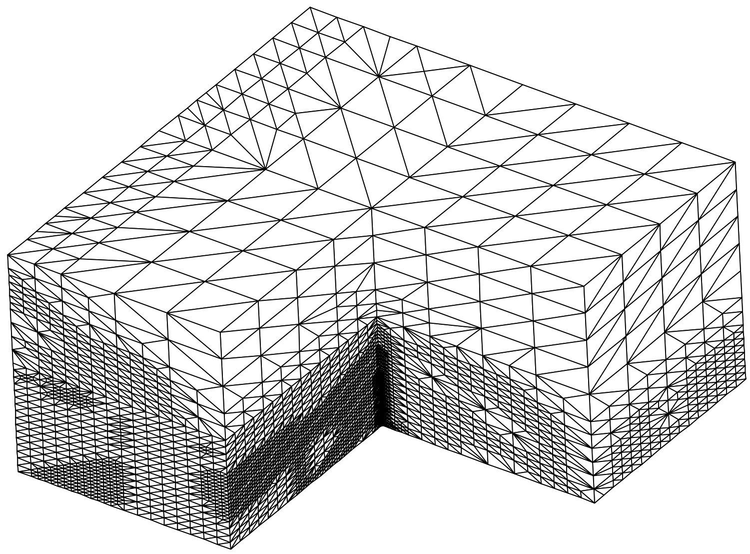

Fig. 5.10b Surface of the final triangulation

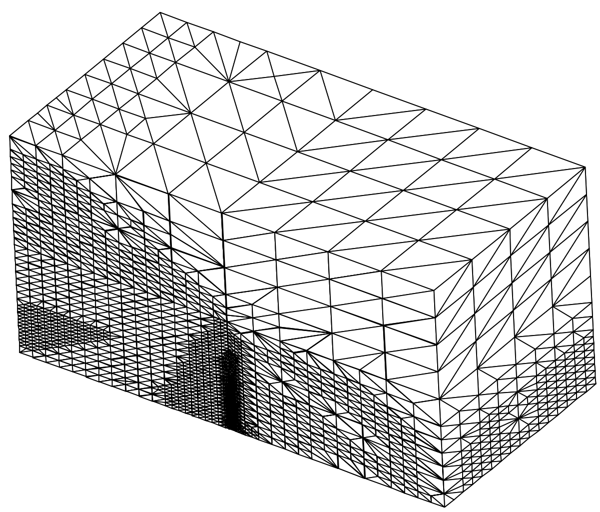

Fig. 5.10c Clipping plane ∂Q_1

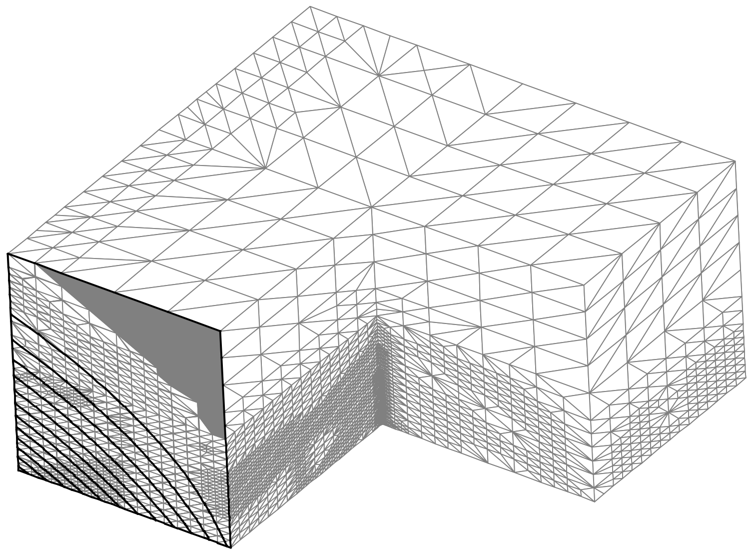

Fig. 5.11a Level curves of the final solution

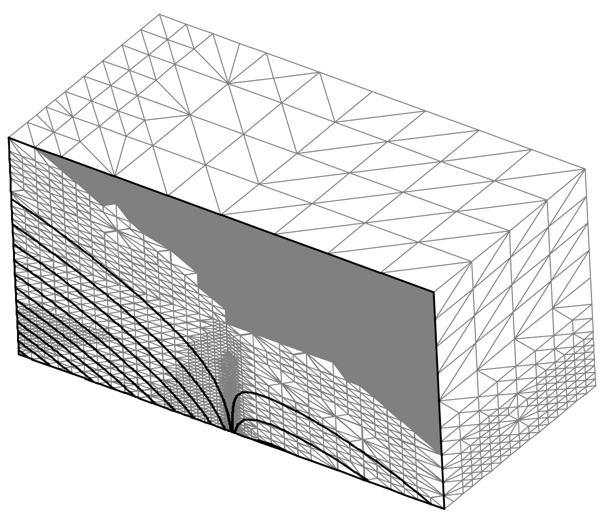

Fig. 5.11b Level curves on the clipping plane ∂Q_1

Table 5: History of the iterative process

Level	Depth	Nodes	Iterations: Solution	Iterations: Error Est.
0	0	105	4/3.8	2/0.0
1	1	585	5/6.4	2/0.0
2	2	2184	4/9.8	2/0.0
3	3	4123	4/9.5	2/0.0
4	4	10192	5/8.4	2/0.0
5	5	15881	5/8.2	2/0.0
6	6	34280	5/8.6	2/0.0

5.4 The axialsymmetric water cone problem in oil reservoir simulation

Another free boundary problem arising from stationary flow in porous media is the following: We consider an oil reservoir which is bounded from below by a layer of sand saturated with water and from above by an impermeable sediment. If we suppose that oil is produced at the production well, then due to the gradient of the inner pressure of the oil in the oilbearing layer a water cone forms below the well. At constant production rate the flow becomes stationary and so does the free boundary separating the oil from the water (cf. Figure 5.12).

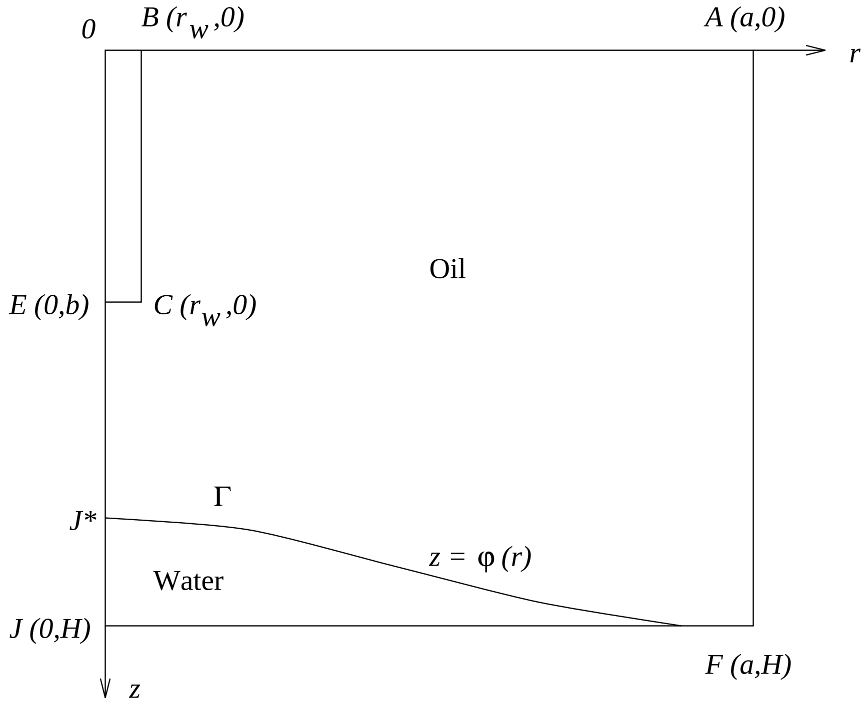

Fig. 5.12 Axialsymmetric water cone problem

For symmetry reasons the problem can be treated in 2-D with the computational domain Ω given in terms of cylindrical coordinates by $\Omega := \{(r,z)|\ 0 < r < a,\ 0 < z < H\} \setminus \{(r,z)|\ 0 < r < r_w,\ 0 < z \leq b\}$. Using again the piezometric head as unknown and performing the Baiocchi transformation, the problem can be stated as the following variational inequality in the pressure-like quantity w and the unknown constant production rate q^* (cf. e.g. Brakhagen (1989)):

Find $q^* \in (0, q_{\max})$, $q_{\max} := (H^2 + b^2 - 2\alpha b)/(2\ln(a/r_w))$, and $w \in K_{q^*} := \{v \in H^1(\Omega)|\ v|_{\Gamma_D} = g_D(q^*),\ v \geq 0$ a.e. in $\Omega\}$ such that

$$a(w, v-w) \geq l(v-w), \quad v \in K_{q^*} \tag{5.6}$$

where

$$\begin{aligned} a(w,v) &:= \int_\Omega (\frac{\partial w}{\partial r}\frac{\partial v}{\partial r} + \frac{\partial w}{\partial z}\frac{\partial v}{\partial z}) r\,dr\,dz, \\ l(v) &:= -\int_\Omega v r\,dr\,dz + \int_{\Gamma_N} g_N v r\,dr\,d\sigma \end{aligned}$$

and the Neumann data g_N on $\Gamma_N := [CE] \cup [EJ]$ and the Dirichlet data $g_D(q)$ on $\Gamma_D := \partial\Omega \setminus \Gamma_N$ are given by

$$g_N := \begin{cases} \alpha - b & \text{on} \quad [CE] \\ & \text{on} \quad [EJ] \end{cases}$$

$$g_D(q) := \begin{cases} \frac{1}{2}(H - z)^2 & \text{on} \quad [AF] \\ \frac{1}{2}H^2 + q\ln(r/a) & \text{on} \quad [AB] \\ \frac{1}{2}H^2 + q\ln(r_w/a) + \frac{1}{2}z^2 - \alpha z & \text{on} \quad [BC] \\ 0 & \text{on} \quad [FJ] \end{cases} .$$

Note that the unknown production rate q^* can be calculated in advance as the solution of the nonlinear problem

$$F(q) := \limsup_{z \to b+} \left[\frac{w_q(r_w, z) - w_q(r_w, b)}{z - b} - (b - z) \right] = 0 \qquad (5.7)$$

where w_q is the unique solution of the variational inequality:

Find $w_q \in K_q$ such that

$$a(w_q, v - w_q) \geq l(v - w_q), \quad v \in K_q. \qquad (5.8)$$

We have applied our adaptive algorithms for the data $a = H = 1$, $\alpha = b = 0.4$ and $r_w = 5 \cdot 10^{-4}$ yielding $q_{\max} = 5.53 \cdot 10^{-2}$. In view of the geometry of the computational domain we have provided a suitable initial triangulation by the grid generator BOXES (cf. Roitzsch and Kornhuber (1990)). The constant production rate q^* has been computed by the secant method applied to (5.7) solving (5.8) by the adaptive algorithm. Using 10^{-4} as termination criterion, after seven iterations we ended up with $q^* = 3.75 \cdot 10^{-2}$.

Figures 5.13a-c show the initial triangulation and some selected levels of the refinement process while Figure 5.13d represents level curves of the pressure-like quantity w on level 11. Note that the radius r_w of the production well is so small compared to the dimension of the reservoir that the well is hardly visible in the figures. As can be seen in Figure 5.13d, we have high pressure gradients at the production well but almost vanishing gradients at the oil-water interface: This explains the fact that the refinement is concentrated around the well but less pronounced at the interface causing difficulties in the resolution of the free boundary.

Figure 5.14 illustrates the performance of the preconditioners. While the symmetric and nonsymmetric versions of the BPX preconditioner behave similarly, the symmetric hierarchical preconditioner clearly outperforms its nonsymmetric counterpart which is in accordance with the theoretical reasoning. Finally, Table 5.4 represents the history of the adaptive refinement process.

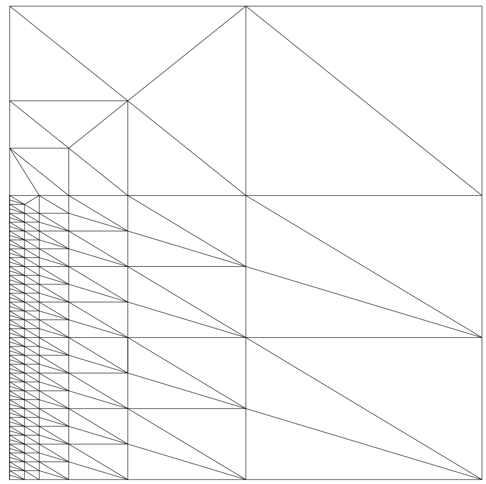

Fig. 5.13a Initial triangulation

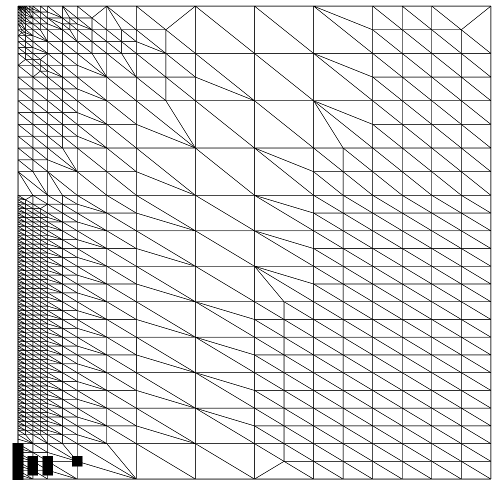

Fig. 5.13b Level 10

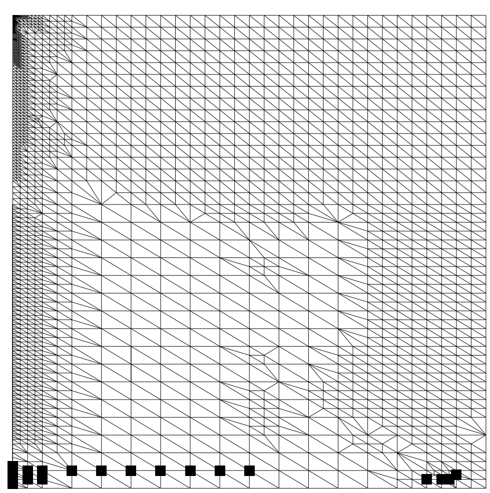

Fig. 5.13c Level 11

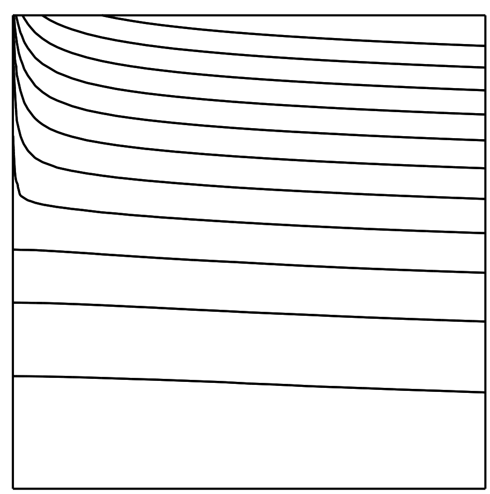

Fig. 5.13d Level curves (pressure)

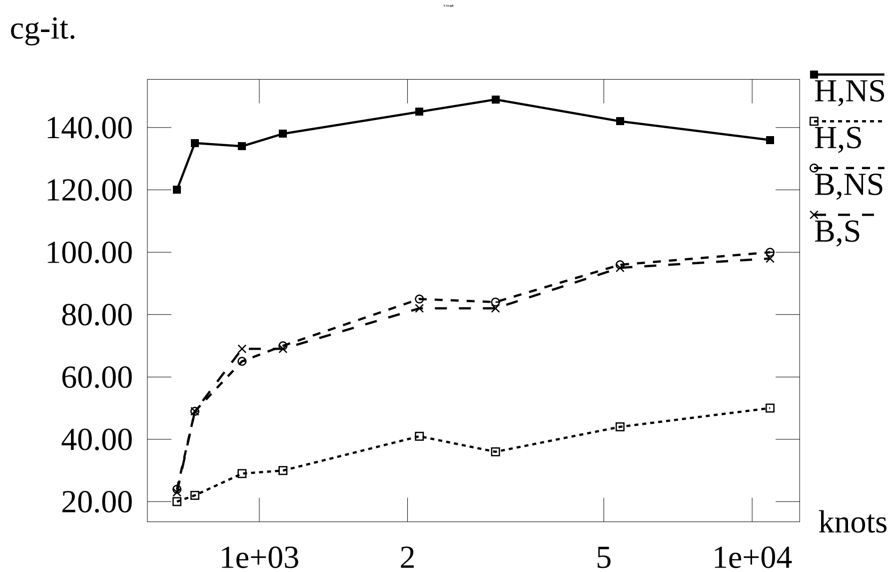

Fig. 5.14 Comparison of the preconditioners

Table 6: History of the iterative process

Level	Depth	Nodes	Iterations: Solution	Iterations: Error Est.
0	0	219	2/0.0	1/1.0
1	1	681	3/1.6	1/1.0
7	7	752	2/4.0	1/1.0
9	9	950	2/5.0	1/1.0
10	10	1102	2/5.0	1/1.0
11	11	2033	2/6.0	1/1.0
12	12	3303	2/6.5	1/1.0
13	13	5836	2/6.5	1/0.0

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