Konrad-Zuse-Zentrum für Informationstechnik Berlin

Ralf Kornhuber

Monotone Multigrid Methods for Elliptic Variational Inequalities I

Preprint SC 93–18 (August 1993)

Monotone Multigrid Methods for Elliptic Variational Inequalities I

Ralf Kornhuber

Abstract. Extending well–known linear concepts of successive subspace correction, we arrive at extended relaxation methods for elliptic variational inequalities. Extended underrelaxations are called monotone multigrid methods, if they are quasioptimal in a certain sense. By construction, all monotone multigrid methods are globally convergent. We take a closer look at two natural variants, which are called symmetric and unsymmetric multigrid methods, respectively. While the asymptotic convergence rates of the symmetric method suffer from insufficient coarse–grid transport, it turns out in our numerical experiments that reasonable application of the unsymmetric multigrid method may lead to the same efficiency as in the linear, unconstrained case.

Key words: obstacle problems, adaptive finite element methods, multigrid methods

AMS (MOS) subject classifications: 65N30, 65N55, 35J85

Chapter 1

Introduction

Let Ω be a polygonal domain in the Euclidean space $\mathbb{R}^2$. We consider the optimization problem

$$u \in \mathcal{K}: \qquad \mathcal{J}(u) \leq \mathcal{J}(v), \;\; v \in \mathcal{K}, \tag{1.1}$$

on a closed, convex subset $\mathcal{K} \subset H_0^1(\Omega)$ of the form

$$\mathcal{K} = \{v \in H_0^1(\Omega) \mid v(x) \leq \varphi(x) \text{ a.e. in } \Omega\}, \tag{1.2}$$

with some obstacle function $\varphi \in H^1(\Omega) \cap C(\bar{\Omega})$, satisfying $\varphi(x) \geq 0$ on the boundary $\partial\Omega$. The quadratic functional $\mathcal{J}$,

$$\mathcal{J}(v) = \tfrac{1}{2} a(v,v) - \ell(v), \;\; v \in \mathcal{K}, \tag{1.3}$$

is induced by a continuous, symmetric and $H_0^1(\Omega)$–elliptic bilinear form

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \, \partial_i v \, \partial_j w \, dx$$

and a linear functional $\ell \in H^{-1}(\Omega)$. It is well–known (c.f. Glowinski [11]) that the obstacle problem (1.1) can be rewritten as the following elliptic variational inequality of the first kind

$$u \in \mathcal{K}: \qquad a(u, v-u) \geq \ell(v-u), \quad v \in \mathcal{K}, \tag{1.4}$$

and admits a unique solution $u \in \mathcal{K}$.

Obstacle problems play an important role in the mathematical modeling of a variety of free boundary problems, arising for instance in porous media flow, device simulation or nonlinear mechanics. We refer to Baiocchi and Capelo [1], Cottle et al. [7], Glowinski [11], Kinderlehrer and Stampaccia [19] or Rodrigues [27] for detailed information.

Let $\mathcal{T}_j$ be a given partition of Ω in triangles $t \in \mathcal{T}_j$ with minimal diameter of order 2^{-j}. We denote the set of interior nodes and edges by $\mathcal{N}_j$ and $\mathcal{E}_j$, respectively. Discretizing (1.1) by continuous, piecewise linear finite elements $\mathcal{S}_j$, we obtain the finite dimensional problem

$$u_j \in \mathcal{K}_j: \qquad \mathcal{J}(u_j) \leq \mathcal{J}(v), \;\; v \in \mathcal{K}_j. \tag{1.5}$$

Here the set $\mathcal{K} \subset H_0^1(\Omega)$ is replaced by its discrete analogue $\mathcal{K}_j \subset \mathcal{S}_j$,

$$\mathcal{K}_j = \{v \in \mathcal{S}_j \mid v(p) \leq \varphi_j(p), \; p \in \mathcal{N}_j\},$$

induced by the $\mathcal{S}_j$–interpolate φ_j of φ. Of course, (1.6) is uniquely solvable and can be reformulated as the variational inequality

$$u_j \in \mathcal{K}_j : \qquad a(u_j, v - u_j) \geq \ell(v - u_j), \;\; v \in \mathcal{K}_j. \tag{1.6}$$

It is well–known (c.f. Glowinski [11]) that u_j is converging to u in $H_0^1(\Omega)$, if the meshsize of $\mathcal{T}_j$ tends to zero and the interior angles of $t \in \mathcal{T}_j$ are uniformly bounded from below.

Let Λ_j denote the set of nodal basis functions of $\mathcal{S}_j$. The standard projected Gauss–Seidel method for the iterative solution of (1.5) is resulting from the successive optimization of $\mathcal{J}$ in the direction of the basis functions $\lambda \in \Lambda_j$. This single–grid relaxation typically suffers from rapidly deteriorating convergence rates when proceeding to more and more refined triangulations. This undesirable behavior stimulated the development of various multigrid methods based on a hierarchy of triangulations [6, 4, 14, 15, 16, 17, 25, 26, 30]. However, even if applied to simple model problems, all those methods suffer either from missing robustness or from unsatisfactory convergence rates. *Monotone multigrid methods* to be presented in this paper are globally convergent and exhibit asymptotic convergence rates of order $1-\mathcal{O}(j^{-2})$ without any regularity assumptions on the free boundary. Moreover, in our numerical experiments we observed the same efficiency as for related linear multigrid methods in the corresponding unconstrained case.

The paper is organized as follows. In Section 2, we present a general approach to construct iterative methods for the discrete problem (1.5) by extending the set of fine–grid search directions Λ_j. More precisely, for a given ν–th iterate u_j^ν, $\nu \geq 0$, we choose a suitable finite set $M_c^\nu \subset \mathcal{S}_j$ of additional search directions, to compute the next iterate $u_j^{\nu+1}$ by successive, constrained optimization of $\mathcal{J}$ in the direction of $\mu \in M^\nu = \Lambda_j \cup M_c^\nu$. This so–called *extended relaxation* can be regarded as a generalized multigrid method for (1.5). Indeed, in case of linear problems, we recover the standard multigrid V–cycle with incomplete Gauss–Seidel smoother by adding the set $M_c = \Lambda_c$ of all *new* coarse–grid nodal basis functions to the fine–grid nodal basis Λ_j. See the excellent overviews of Xu [31] and Yserentant [33] for further information. Of course, extended relaxations can be also interpreted as successive subspace corrections or nonlinear multiplicative Schwarz methods.

Usually, the local optimization problems corresponding to $\mu \in M_c^\nu$ cannot be evaluated efficiently, motivating the approximation by suitable inexact solvers. Replacing the fine–grid constraints by more restrictive *local obstacles*, we arrive at *extended underrelaxations* preserving non–increasing energy also by the approximate corrections. It is shown that extended underrelaxations are globally convergent.

In Section 3, we use this general approach to construct two families of monotone multigrid methods, which are introduced as extended underrelaxations based on sufficiently large sets M^ν, $\nu \geq 0$. Generalizing the standard V–cycle to variational inequalities, we first extend Λ_j by the set Λ_c to the multilevel nodal basis $\Lambda = \Lambda_j \cup \Lambda_c$, which then is used as a constant set $M^\nu = \Lambda$, $\nu \geq 0$, of search directions. In this way, the method of Mandel [25, 26] is recovered. For linear finite elements the local obstacles used by Mandel [26] can be easily improved.

Compared to the unconstrained case, this so–called *symmetric* multigrid approach usually still suffers from unsatisfactory asymptotic convergence rates, resulting from a possibly poor decomposition of the reduced subspace $\mathcal{S}_j^\circ$ by the corresponding subset $\Lambda^\circ \subset \Lambda$. Hence, we improve the coarse–grid transport by suitable *unsymmetric* truncation of the coarse–grid functions $\lambda \in \Lambda_c$ (c.f. Hoppe and Kornhuber [18]), providing variable search directions $M^\nu = \Lambda_j \cup \tilde{\Lambda}_c^\nu$, $\nu \geq 0$, which depend on the actual guess of the discrete free boundary. As a consequence, the possible inactivation of active points is performed only on the fine grid. We emphasize that both the symmetric and the unsymmetric approach are permanent extensions of linear multigrid methods, reducing to the standard V–cycle with incomplete Gauss–Seidel smoother, if u_j has no contact with the obstacle φ_j. Applying recent results on linear multigrid methods (c.f. Kornhuber and Yserentant [23]) to the reduced linear problems, we obtain $1 - \mathcal{O}(j^{-2})$ estimates of the asymptotic convergence rate without any regularity assumptions on the free boundary. Global estimates of the convergence rates will be a subject of future research.

In multigrid terminology, the unsymmetric truncation of coarse–grid basis functions amounts to well–known perturbations of the weighted restrictions and prolongations in the neighborhood of the free boundary. It seems that the good convergence properties of the multigrid method proposed by Brandt and Cryer [6] are an outcome of such modified restrictions and prolongations. However, the present lack of a convergence proof may be due to the slightly inconsistent use of these modifications.

The final Section 4 is devoted to some illustrative numerical experiments. Compared to the symmetric approach, we obtained tremendously improved asymptotic convergence rates of the unsymmetric multigrid method, while approximately the same number of iterations was needed to fix the exact active set. For the unsymmetric method, we also found a better convergence behavior than for the considered variant of Brandt and Cryer's method [6], which in turn was mostly superior to the symmetric scheme, but failed in some of our numerical experiments. Moreover, using the interpolated results from the previous level, we observed the same efficiency of the constrained unsymmetric multigrid method as in the corresponding unconstrained case.

The concept of extended underrelaxations as described herein is open to various generalizations. For example, the treatment of more general boundary conditions and lower or double obstacle problems is obvious. In view of recent work on multilevel methods in three space dimensions (c.f. Bornemann et al. [5], Erdmann et al. [9, 10]), the extension of the presented algorithms to the 3–D case is straightforward. Elliptic variational inequalities of the second kind occurring for instance in nonlinear mechanics or time–discretized two–phase Stefan problems, will be treated in the second part of this paper [20].

Chapter 2

Extended Relaxation Methods

We will describe one step of an extended relaxation method for the iterative solution of (1.5), as applied to a given ν–th iterate $u_j^\nu \in \mathcal{S}_j$, $\nu \geq 0$. For this reason, let $M^\nu = \Lambda_j \cup M_c^\nu$ be the union of the nodal basis functions $\Lambda_j = \{\lambda_p^{(j)} \mid p \in \mathcal{N}_j\}$ of $\mathcal{S}_j$ and a suitably chosen subset $M_c^\nu \subset \mathcal{S}_j$. To allow for a possible adaptation of the additional search directions to the unknown discrete free boundary, the set M_c^ν may change in each iteration step. In view of future applications, the elements of M_c^ν are called *coarse–grid functions* in contrast to the *fine–grid functions* contained in Λ_j. We select a suitable enumeration of $M^\nu = \{\mu_1^\nu \ldots, \mu_{m^\nu}^\nu\}$ and introduce the one–dimensional subspaces $V_l^\nu = \operatorname{span}\{\mu_l^\nu\}$, $l = 1, \ldots, m^\nu$. For notational convenience, the index ν will be suppressed whenever possible.

Starting with $w_0 = u_j^\nu$, the *extended relaxation method induced by* M^ν provides the next iterate $u_j^{\nu+1} = w_m$ by successive subspace correction, producing the intermediate iterates $w_l = w_{l-1} + v_l^*$, $l = 1, \ldots, m$. The corrections v_l^* are the unique solutions of the local subproblems

$$v_l^* \in \mathcal{D}_l^* : \qquad a(v_l^*, v - v_l^*) \geq \ell(v - v_l^*) - a(w_{l-1}, v - v_l^*), \quad v \in \mathcal{D}_l^*, \qquad (2.1)$$

where the closed, convex subsets $\mathcal{D}_l^* = \mathcal{D}_l^*(w_{l-1})$ are defined by

$$\mathcal{D}_l^*(w_{l-1}) = \{v \in V_l \mid w_{l-1}(p) + v(p) \leq \varphi_j(p),\ p \in \mathcal{N}_j \cap \text{ int supp } \mu_l\}.$$

Note that (2.1) is just the nonlinear multiplicative Schwarz method for (1.5) induced by the splitting

$$\mathcal{S}_j = \sum_{l=1}^{m} V_l, \qquad (2.2)$$

which may change in each iteration step. Of course, each of the local subproblems (2.1) provides a unique solution v_l^*. The scheme (2.1) is monotone in the sense that

$$\mathcal{J}(w_l) \leq \mathcal{J}(w_{l-1}), \qquad (2.3)$$

if $w_{l-1} \in \mathcal{K}_j$. For arbitrary $u_j^\nu = w_0 \in \mathcal{S}_j$, we have $w_l \in \mathcal{K}_{j,l} \supset \mathcal{K}_j$ with

$$\mathcal{K}_{j,l} = \{w \in \mathcal{S}_j \mid w(p) \leq \varphi_j(p), p \in \mathcal{N}_j \cap \bigcup_{i=1}^{l} \text{ int supp } \mu_i\},\ l = 0, \ldots, m,$$

providing $u_j^{\nu+1} \in \mathcal{K}_j$.

The efficient evaluation of (2.1) suffers from the fact that the values of a given $v \in V_l$ at all $p \in \mathcal{N}_j \cap \text{int supp } \mu_l$ are required, to check the constraints involved in the definition of $\mathcal{D}_l^*$. With multigrid methods in mind, this leads to an inadmissible number of additional interpolations.

Hence, the optimal corrections $v_l^* \in V_l$ are replaced by approximations $v_l \in V_l$ provided by approximate subproblems of the form

$$v_l \in \mathcal{D}_l : \quad a(v_l, v - v_l) \geq \ell(v - v_l) - a(w_{l-1}, v - v_l), \quad v \in \mathcal{D}_l, \tag{2.4}$$

with closed, convex subsets $\mathcal{D}_l = \mathcal{D}_l^\nu(w_0^\nu \ldots, w_{l-1}^\nu) \subset V_l^\nu$ given by

$$\mathcal{D}_l^\nu(w_0^\nu \ldots, w_{l-1}^\nu) = \{v \in V_l^\nu \mid v(p) \leq \psi_l^\nu(w_0^\nu \ldots, w_{l-1}^\nu)(p),\ p \in \mathcal{N}_j\},$$

where the *local obstacles* $\psi_l = \psi_l^\nu(w_0^\nu \ldots, w_{l-1}^\nu) \in V_l^\nu$ are understood to depend on ν and the preceding intermediate iterates $w_0^\nu, \ldots, w_{l-1}^\nu$.

As $\psi_l \in V_l$, we have to evaluate a given $v \in V_l$ at only one point $p \in \mathcal{N}_j \cap$ int supp μ_l, to see whether $v \in \mathcal{D}_l$ or not. In the multigrid methods to be considered in the following section, the local obstacles ψ_l are resulting from appropriate, recursive restrictions of the defect obstacle $\varphi_j - w_{l-1} \in \mathcal{S}_j$ to the coarse levels.

The local obstacle ψ_l is called *monotone* if the inclusion

$$\mathcal{D}_l(w_0, \ldots, w_{l-1}) \subset \mathcal{D}_l^*(w_{l-1}) \tag{2.5}$$

is valid for all $w_0 \in \mathcal{K}_{j,0}, \ldots, w_{l-1} \in \mathcal{K}_{j,l-1}$, where equality holds in case of fine–grid constraints corresponding to $\mu_l = \lambda_p^{(k)} \in \Lambda_j$ and if

$$0 \leq \psi_l(w_0, \ldots, w_{l-1}) \tag{2.6}$$

is satisfied for all $w_0, \ldots, w_{l-1} \in \mathcal{K}_j$. Obviously, the condition (2.5) is equivalent to

$$w_{l-1}(p) + \psi_l(w_0 \ldots, w_{l-1})(p) \leq \varphi_j(p), \quad p \in \mathcal{N}_j \cap \text{ int supp } \mu_l, \tag{2.7}$$

for all $w_0 \in \mathcal{K}_{j,0}, \ldots, w_{l-1} \in \mathcal{K}_{j,l-1}$ and the required equality for fine–grid constraints leads to the definition

$$\psi_l(w_{l-1}) = (\varphi_j(p) - w_{l-1}(p))\lambda_p^{(j)}, \quad \mu_l = \lambda_p^{(j)} \in \Lambda_j. \tag{2.8}$$

Lemma 2.1 *Assume that ψ_l is monotone and that $w_0, \ldots, w_{l-1} \in \mathcal{K}_j$. Then the corrections v_l^* and v_l computed from (2.1) and (2.4), respectively, are related by*

$$v_l^* = \omega_l v_l, \quad \omega_l \in [0, 1]. \tag{2.9}$$

Proof. If $v_l^* \le \psi_l$, we clearly have $v_l = v_l^*$ from (2.5). As $\psi_l, v_l^* \in V_l$ and V_l is one–dimensional we only have to consider the remaining case $v_l^* > \psi_l$. In this case, we obtain from the non–negativity (2.6) that $0 \le v_l = \psi_l < v_l^*$. This gives the assertion. ∎

In view of Lemma 2.1, the approximate version (2.4) induced by M^ν and a sequence of monotone constraints ψ_l^ν, $l = 1 \ldots, m^\nu$, is called *extended underrelaxation*. Exploiting the convexity of $\mathcal{J}$, it is easily seen that extended underrelaxations preserve the monotonicity (2.3). For any initial iterate $u_j^0 \in \mathcal{S}_j$, the subsequent iterates u_j^ν, $\nu \ge 1$, are contained in $\mathcal{K}_j$. Moreover, we have $w_l^\nu \in \mathcal{K}_j$, $\nu \ge 1$.

Note that both (2.1) and (2.4) may be regarded as perturbations of the classical projected Gauss–Seidel iteration, which in turn is resulting from the trivial choice $M_c = \emptyset$.

Theorem 2.1 *An extended underrelaxation is globally convergent.*

Proof. The sequence of iterates u_j^ν, $\nu = 0, 1, \ldots$, is bounded because the monotonicity of the iteration yields

$$\mathcal{J}(u_j^\nu) \le \mathcal{J}(u_j^1)\ , \quad \nu = 1, 2, \ldots\ ,$$

and we have $\mathcal{J}(v^\nu) \to \infty$ for any unbounded sequence $v^\nu \in \mathcal{S}_j$. As $\mathcal{S}_j$ has finite dimension, each subsequence of u_j^ν has a convergent subsequence. We will show that this leads to the convergence of the local corrections v_l^ν,

$$v_l^\nu \to 0, \quad \text{for } \nu \to \infty, \quad l = 1, \ldots, m. \tag{2.10}$$

From (2.10), the whole sequence u_j^ν must be convergent to some limit $u_j^* \in \mathcal{S}_j$. It is clear from (2.8) and (2.10) that u_j^* is also a fixed point of the single–grid relaxation, which is well–known to have the unique fixed point u_j (c.f. [11]). This gives the assertion.

We still have to show the convergence of the corrections (2.10). Recall that the optimal correction $v_l^* = v_l^*(w_{l-1})$, resulting from the exact evaluation of the next local subproblem, satisfies the variational inequality (2.1). As $w_{l-1}^\nu \in \mathcal{K}_j$ for $\nu \ge 1$, we can insert $v = 0 \in \mathcal{D}_l^*$ in (2.1), to obtain

$$\ell(v_l^*) - a(w_{l-1}, v_l^*) \ge a(v_l^*, v_l^*),$$

and some elementary calculations give

$$\mathcal{J}(w_{l-1}) - \mathcal{J}(w_{l-1} + v_l^*) \ge \tfrac{1}{2} a(v_l^*, v_l^*). \tag{2.11}$$

Now let v_l be the solution of the corresponding approximate local problem. Then we have from Lemma 2.1 that $v_l = \omega_l v_l^*$ with some $\omega_l \in [0,1]$. Exploiting (2.11) and the convexity of the functional $\mathcal{J}$ we get

$$\mathcal{J}(w_{l-1}) - \mathcal{J}(w_{l-1} + v_l) \geq \tfrac{1}{2} a(v_l, v_l). \tag{2.12}$$

Successive application of (2.12) gives

$$\begin{aligned}&\mathcal{J}(u_j^{\nu_1}) - \mathcal{J}(u_j^{\nu_2}) \geq \\ &\quad \geq \sum_{\nu=\nu_1}^{\nu_2-1} \sum_{l=1}^{m^\nu} \mathcal{J}(w_{l-1}^\nu) - \mathcal{J}(w_{l-1}^\nu + v_l^\nu) \geq \sum_{\nu=\nu_1}^{\nu_2-1} \sum_{l=1}^{m^\nu} \tfrac{1}{2} a(v_l^\nu, v_l^\nu) \geq 0.\end{aligned}$$

Now the convergence (2.10) follows from the continuity of $\mathcal{J}$. This completes the proof. ∎

In view of (2.10) we have even shown the convergence of the intermediate iterates w_l^ν, $l = 1, \ldots, m^\nu$,

$$w_l^\nu \to u_j, \quad l = 1, \ldots, m^\nu, \quad \nu \to \infty. \tag{2.13}$$

Theorem 2.1 allows for immediate extensions to other convex perturbations of the energy functional $\mathcal{J}$ allowing for a convergent single–grid relaxation. In particular, a related convergence result for elliptic variational inequalities of the second kind (i.e. Theorem 2.1 in [20]) can be shown almost literally in the same way. The convergence of single–grid relaxations applied to other elliptic variational inequalities is discussed to some extend by Glowinski [11].

The remainder of this section is devoted to the asymptotic behavior of extended underrelaxations. We define the active set $\mathcal{N}_j^\bullet(w)$ of some $w \in \mathcal{S}_j$ as the subset of nodes $p \in \mathcal{N}_j$ with the property $w(p) = \varphi_j(p)$. The remaining nodes $p \in \mathcal{N}_j^\circ(w) = \mathcal{N}_j \setminus \mathcal{N}_j^\bullet(w)$ are called inactive. The following Lemma states the convergence of the active sets $\mathcal{N}^\bullet(u_j^\nu)$, generalizing a related result of Mandel [26].

Lemma 2.2 *Assume that the functions $\mu_l^\nu \in M^\nu$ are uniformly bounded and positive in the sense that*

$$0 < c_1 \leq \mu_l^\nu(p) \leq c_2, \quad p \in \mathcal{N}_j \cap \text{ int supp } \mu_l^\nu, \tag{2.14}$$

holds for all $l = 1, \ldots, m^\nu$ and $\nu \geq 0$ with constants c_1, c_2 independent of ν. Assume further that the discrete problem satisfies the strict complementary condition

$$a(u_j, \lambda_p^{(j)}) < \ell(\lambda_p^{(j)}), \quad p \in \mathcal{N}_j^\bullet(u_j). \tag{2.15}$$

Then the active sets $\mathcal{N}_j^\bullet(w_l^\nu)$ of the intermediate iterates w_l^ν, resulting from the extended underrelaxation (2.4) induced by M^ν, converge to $\mathcal{N}^\bullet(u_j)$, i.e. there is a $\nu_0 \geq 0$ such that

$$\mathcal{N}^\bullet(w_l^\nu) = \mathcal{N}^\bullet(u_j), \quad l = 1, \ldots, m^\nu, \quad \nu \geq \nu_0. \tag{2.16}$$

Proof. As the intermediate iterates w_l^ν converge to u_j, we clearly have $\varphi_j(p) - w_l^\nu(p) > 0$, $p \in \mathcal{N}_j^\circ(u_j)$, if ν is greater than some suitable ν_1. This yields

$$\mathcal{N}^\circ(u_j) \subset \mathcal{N}_j^\circ(w_l^\nu), \;\; l = 1, \ldots, m^\nu, \;\; \nu \geq \nu_1. \tag{2.17}$$

Exploiting the uniform positivity (2.14), we can derive an extended strict complementary condition of the form

$$\ell(\mu_l^\nu) - a(u_j, \mu_l^\nu) \geq c > 0, \;\; \mathcal{N}_j^\bullet(u_j) \cap \text{ int supp } \mu_l^\nu \neq \emptyset,$$

for all $\mu_l \in M^\nu$, where c is independent of l and ν. As w_l^ν converges to u_j and the μ_l^ν are uniformly bounded according to (2.14), we can find a $\nu_2 \geq 0$ such that

$$\ell(\mu_l^\nu) - a(w_l^\nu, \mu_l^\nu) > 0, \;\; \mathcal{N}_j^\bullet(u_j) \cap \text{ int supp } \mu_l^\nu \neq \emptyset, \;\; \nu \geq \nu_2. \tag{2.18}$$

Using (2.18) and (2.4), it is easily checked that an active point $p \in \mathcal{N}_j^\bullet(u_j) \cap \mathcal{N}_j^\bullet(w_{l_0}^\nu)$ for some l_0 and $\nu \geq \nu_2$ is not inactivated in further iterations. Moreover, choosing some fixed $p \in \mathcal{N}_j^\bullet(u_j)$, we can apply (2.18) to the corresponding fine–grid basis function $\mu_{l_0}^\nu = \lambda_p^{(j)}$, $l_0 = l_0(p, j)$, to obtain $p \in \mathcal{N}_j^\bullet(w_{l_0}^\nu)$ for $\nu \geq \nu_2$. We have shown the inclusion

$$\mathcal{N}^\bullet(u_j) \subset \mathcal{N}_j^\bullet(w_l^\nu), \;\; l = 1, \ldots, m^\nu, \;\; \nu \geq \nu_2. \tag{2.19}$$

Now the assertion follows from (2.17) and (2.19). ∎

We say that M^ν is *regular*, if the functions $\mu_l^\nu \in M^\nu$ satisfy the condition (2.14) and if $\mathcal{N}_j^\bullet(w_l^\nu) = \mathcal{N}_j^\bullet(u_j)$, $l = 1, \ldots, m^\nu$, implies $M^\nu = M^* = \Lambda_j \cup M_c^*$ with a subset $M_c^* \subset \mathcal{S}_j$ not depending on ν.

Once the active set $\mathcal{N}_j^\bullet(u_j)$ of the exact solution u_j is known, the discrete variational inequality (1.6) is reducing to the variational equality

$$a(u_j, v) = \ell(v), \;\; v \in \mathcal{S}_j^\circ, \tag{2.20}$$

where the reduced subspace $\mathcal{S}_j^\circ \subset \mathcal{S}_j$ is defined by

$$\mathcal{S}_j^\circ = \{v \in \mathcal{S}_j \mid \text{ int supp } v \cap \mathcal{N}_j^\bullet(u_j) = \emptyset\}.$$

If M^ν is regular, the reduced set M°,

$$M^\circ = \{\mu \in M^* \mid \text{int supp } \mu \cap \mathcal{N}_j^\bullet(u_j) = \emptyset\} \subset M^*,$$

is inducing a linear extended relaxation method for the iterative solution of (2.20). The corrections $v_l \in V_l$ in the direction of $\mu_l \in M^\circ$ are computed from the local subproblems

$$v_l \in V_l : \qquad a(v_l, v) = \ell(v) - a(w_{l-1}, v), \;\; v \in V_l. \tag{2.21}$$

Assuming the strict complementarity (2.15), it is easily seen that the extended relaxation (2.1) induced by a regular sequence M^ν is asymptotically reducing to the linear scheme (2.21). We will prove a related result for extended underrelaxations.

A sequence of monotone local obstacles $\psi_l^\nu \in V_l^\nu$, $\nu \geq 0$, is called *quasioptimal*, if $M^\nu = M^*$ implies $\psi_l^\nu = \psi_l^*$, $l = 1, \ldots, m^\nu = m^*$, where the local obstacles $\psi_l^* = \psi_l^*(w_0, \ldots, w_{l-1})$ are continuous with respect to $w_0, \ldots, w_{l-1} \in \mathcal{K}_j$ and $\psi_l^*(u_j) = \psi_l^*(u_j, \ldots, u_j)$ satisfies the condition

$$\mu_l \in M^\circ \Rightarrow \psi_l^*(u_j)(p) > 0, \quad p \in \mathcal{N}_j \cap \text{ int supp } \mu_l, \tag{2.22}$$

for all $l = 1, \ldots, m^*$.

Now we are ready to state the main result of this section.

Theorem 2.2 *Assume that the strict complementary condition (2.15) holds. Then an extended underrelaxation induced by a regular sequence M^ν and quasioptimal local obstacles ψ_l^ν, $l = 1, \ldots, m^\nu$, $\nu \geq 0$, is globally convergent and is asymptotically reducing to the linear extended relaxation (2.21).*

Proof. The global convergence follows immediately from Theorem 2.1.

According to Lemma 2.2, there is a $\nu_0 \geq 0$ such that $\mathcal{N}_j^\bullet(w_l^\nu) = \mathcal{N}_j^\bullet(u_j)$, holds for $l = 1, \ldots, m^\nu$ and $\nu \geq \nu_0$. Hence, we have $M^\nu = M^*$ and $\psi_l^\nu = \psi_l^*$, $l = 1, \ldots, m^\nu = m^*$, for $\nu \geq \nu_0$.

Let us first consider the corrections v_l^ν in the direction of $\mu_l \in M^\circ$. By the continuity of $\psi_l^*(w_0^\nu, \ldots, w_{l-1}^\nu)$ with respect to $w_0^\nu, \ldots, w_{l-1}^\nu$, the local obstacles converge to $\psi_l^*(u_j)$ as $w_0^\nu, \ldots, w_{l-1}^\nu$ converge to u_j. Utilizing (2.22) and the convergence (2.10) of the corrections v_l^ν, we can find a $\nu_1 \geq \nu_0$ such that

$$v_l^\nu(p) < \psi_l^*(w_0^\nu, \ldots, w_{l-1}^\nu)(p), \quad p \in \mathcal{N}_j \cap \text{ int supp } \mu_l, \quad \nu \geq \nu_1. \tag{2.23}$$

In view of (2.23), it follows from the variational inequality (2.4) that v_l^ν solves the variational equality (2.21) for $\nu \geq \nu_1$.

In the remaining case $\mu_l \in M^* \setminus M^\circ$, the corresponding corrections v_l^ν satisfy $v_l^\nu = 0$, $\nu \geq \nu_0$, by the invariance of the active sets $\mathcal{N}_j^\bullet(w_l^\nu)$, $l = 1, \ldots, m^\nu$, for $\nu \geq \nu_0$. This completes the proof. ■

Roughly speaking, Theorem 2.2 states that the extended relaxation (2.1) and all extended underrelaxations induced by the same (regular) M^ν but different (quasioptimal) local obstacles ψ_l^ν have the same asymptotic behavior. The next section is devoted to the actual choice of regular sets $M_c^\nu \in \mathcal{S}_j$ and corresponding quasioptimal local obstacles ψ_l^ν, $l = 1, \ldots, m^\nu$.

Chapter 3

Monotone Multigrid Methods

Assume that $\mathcal{S}_j$ is resulting from several refinements of an intentionally coarse triangulation $\mathcal{T}_0$, producing a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ and a corresponding sequence of nested finite element spaces $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j$. Following Bank et al. [3], a triangle $t \in \mathcal{T}_k$ is refined either by subdividing it into four congruent subtriangles or by connecting one of its vertices to the midpoint of the opposite side. The first case is called *regular* (red) refinement while the second case is referred to as *irregular* (green) refinement. The refinement process has to obey further structural rules, which are meanwhile standard in the literature on multilevel methods [2, 3, 8, 18, 32]. We refer for example to Yserentant [32] for further information.

Let $\Lambda_k = \{\lambda_p^{(k)} |\ p \in \mathcal{N}_k\}$ denote the sets of nodal basis functions in $\mathcal{S}_k$, $k = 0, \ldots, j$. Collecting the $m_0 = n_0$ elements of Λ_0 and the m_k new basis functions on each level, we define the set Λ by

$$\Lambda = \Lambda_0 \cup \bigcup_{k=1}^{j} \Lambda_k \setminus \Lambda_{k-1}.$$

Note that $\Lambda_j \subset \Lambda$ holds by construction. In the following, we will use the canonical order of Λ, which is induced by the refinement levels,

$$\Lambda = \{\lambda_{p_1}^{(j)}, \lambda_{p_2}^{(j)}, \ldots, \lambda_{p_{m_j}}^{(j)}, \ldots, \lambda_{p_1}^{(0)}, \ldots, \lambda_{p_{m_0}}^{(0)}\}, \tag{3.1}$$

denoting $\lambda_l = \lambda_{p_i}^{(k)}$ with $l = l(p_i, k) = 1, \ldots, m$ for all $i = 0, \cdots, m_k$ and $k = 0, \cdots, j$. This enumeration is inherited by the corresponding reduced set Λ°,

$$\Lambda^\circ = \{\lambda \in \Lambda \mid \text{int supp } \lambda \cap \mathcal{N}_j^\bullet(u_j) = \emptyset\} \subset \Lambda.$$

An extended underrelaxation satisfying the conditions of Theorem 2.2 is called *monotone multigrid method*, if $\Lambda^\circ \subset M^\circ$. Roughly speaking, the reduced set of search directions has to be large enough.

Hence, it is natural to consider the fixed set $M^\nu = \Lambda = \Lambda_j \cup \Lambda_c$ of search directions using the coarse–grid functions $\Lambda_c = \Lambda \backslash \Lambda_j$ for all $\nu \geq 0$. Monotone multigrid methods induced by Λ are called *symmetric* because the elements of the reduced set Λ° may be regarded as symmetric truncations of $\lambda \in \Lambda$ (see a related notation in [18]). Of course, Λ is regular in the sense of the preceding section.

To complete the construction of a monotone multigrid method induced by Λ, we now derive a quasioptimal sequence ψ_l, $l = 1, \ldots, m$, of local obstacles.

This will be done by suitable successive restrictions of the defect obstacle $\varphi_j - w_l^\nu$. Here we will make use of the definition $v^{(k)} = v_{p_1}^{(k)} + \ldots, + v_{p_{m_k}}^{(k)} \in \mathcal{S}_k$ denoting the sum of all corrections $v_l = v_{p_i}^{(k)} \in \mathcal{S}_k$ corresponding to basis functions $\lambda_l = \lambda_{p_i}^{(k)}$ on level k. Recall that for a given iterate u_j^ν the corrections $v_l = v_l^\nu$ are solutions of the local problems (2.4).

Note that for each $p \in \mathcal{N}_k$ the corresponding nodal basis function $\lambda_p^{(k)} \in \mathcal{S}_k$ is contained either in Λ_j or in Λ_c.

Lemma 3.1 *Assume that the mappings $R_{k+1}^k : \mathcal{S}_{k+1} \to \mathcal{S}_k$, $k = j-1, \ldots, 0$, are continuous and have the properties*

$$R_{k+1}^k v(p) \leq v(p), \quad p \in \mathcal{N}_{k+1}, \tag{3.2}$$

and

$$\min\{v(q) \mid q \in \mathcal{N}_{k+1} \cap \text{ int supp } \lambda_p^{(k)}\} \leq R_{k+1}^k v(p), \quad p \in \mathcal{N}_k, \tag{3.3}$$

for all $v \in \mathcal{S}_{k+1}$. Then, for a given iterate u_j^ν, the recursive restriction

$$\psi^{(k)} = R_{k+1}^k(\psi^{(k+1)} - v^{(k+1)}), \quad k = j-1, \ldots, 0, \tag{3.4}$$

of the defect obstacle $\psi^{(j)} = \varphi_j - u_j^\nu$ generates the quasioptimal local obstacles $\psi_l \in V_l$ by the definition

$$\psi_l = \psi^{(k)}(p)\lambda_p^{(k)}, \quad l = l(p,k) = 1, \ldots, m. \tag{3.5}$$

Proof. We have to show that the local obstacles ψ_l defined by (3.5) are continuous with respect to the intermediate iterates $w_0, \ldots, w_{l-1}$ and satisfy the conditions (2.6), (2.7), (2.8) and (2.22). By straightforward induction, these assertions can be traced back to the continuity of the restrictions R_{k+1}^k and the assumptions (3.2) and (3.3), exploiting the enumeration (3.1) of Λ. To give an example, let us sketch the proof of condition (2.8).

Assume that $\lambda_l = \lambda_p^{(k_0)} \in \Lambda_j$. In this case it follows from (3.2) and (3.3) that

$$R_{k+1}^k v(p) = v(p), \quad v \in \mathcal{S}_{k+1}. \tag{3.6}$$

We further have $\lambda_p^{(k_0)} = \lambda_p^{(k)}$ and $v^{(k)}(p) = 0$ for all $k = k_0 + 1, \ldots, j$ giving $w_{l-1}(p) = u_j^\nu(p)$. Hence, the equality (3.6) inductively leads to

$$\psi_l = \psi^{(k_0)}(p)\lambda_p^{(k_0)} = (\varphi_j(p) - u_j^\nu(p))\lambda_p^{(j)},$$

which proves (2.8). ∎

We are left with the problem to construct *quasioptimal restriction operators* $R_{k+1}^k : \mathcal{S}_{k+1} \to \mathcal{S}_k$ satisfying the assumptions of Lemma 3.1.

It is easily seen that the restrictions $r_{k+1}^k : \mathcal{S}_{k+1} \to \mathcal{S}_k$, $k = 0, \dots, j-1$,

$$r_{k+1}^k v(p) = \min\{v(q) \mid q \in \mathcal{N}_{k+1} \cap \text{ int supp } \lambda_p^{(k)}\}, \quad p \in \mathcal{N}_k, \tag{3.7}$$

proposed by Mandel [26], are quasioptimal in this sense. Though definition (3.7) looks quite natural in view of condition (3.3), it does not take advantage of the fact that the arguments $v \in \mathcal{S}_{k+1}$ of r_{k+1}^k are piecewise linear on $\mathcal{T}_k$. As a consequence, the resulting local constraints are too pessimistic compared with the quasioptimal restrictions R_{k+1}^k, $k = 0, \dots, j-1$, derived in the sequel.

For some fixed k, $0 \le k \le j-1$, let $\mathcal{E}_k' \subset \mathcal{E}_k$ denote the set of bisected edges in $\mathcal{E}_k$ with midpoints p_e, $e \in \mathcal{N}_{k+1}$. Selecting a certain order $\mathcal{E}_k' = \{e_1, \dots, e_s\}$ of $\mathcal{E}_k'$, we define the restriction operator $R_{k+1}^k : \mathcal{S}_{k+1} \to \mathcal{S}_k$ according to

$$R_{k+1}^k v = I_{\mathcal{S}_k} \circ R_{e_s} \circ \dots \circ R_{e_1} v, \quad v \in \mathcal{S}_{k+1}, \tag{3.8}$$

where $I_{\mathcal{S}_k}$ denotes the $\mathcal{S}_k$–interpolation and the operators $R_e : \mathcal{S}_{k+1} \to \mathcal{S}_{k+1}$, $e \in \mathcal{E}_k'$, are of the form

$$R_e v = v + v_1 \lambda_{p_1}^{(k+1)} + v_2 \lambda_{p_2}^{(k+1)}, \quad v \in \mathcal{S}_{k+1} \tag{3.9}$$

with $p_1, p_2 \in \mathcal{N}_k$ denoting the vertices of $e = (p_1, p_2) \in \mathcal{E}_k'$. In (3.9) the scalars $v_1, v_2 \in \mathbb{R}$ are chosen such that

$$R_e v(p) \le v(p), \quad p = p_1, p_e, p_2.$$

In particular, we set $v_1 = 0$ if $v(p_1) \le v(p_e)$ or $v(p_1) + v(p_2) \le 2v(p_e)$. In the remaining case, v_1 is determined by

$$v_1 = \begin{cases} 2v(p_e) - v(p_1) - v(p_2), & \text{if} \quad v(p_2) \le v(p_e) \le v(p_1), \\ v(p_e) - v(p_1), & \text{if} \quad v(p_e) \le v(p), \ p = p_1, p_2. \end{cases}$$

The value of v_2 is obtained in a symmetrical way.

It can be checked by elementary considerations that for each enumeration of $\mathcal{E}_k'$ the definition (3.8) provides a quasioptimal restriction operator R_{k+1}^k. In particular, R_{k+1}^k is less restrictive than r_{k+1}^k in the sense that

$$r_{k+1}^k v \le R_{k+1}^k v, \quad v \in \mathcal{S}_{k+1}. \tag{3.10}$$

Hence, using R_{k+1}^k instead of r_{k+1}^k, we can expect less damping of the coarse–grid corrections, providing faster convergence of the corresponding algorithm. This heuristic reasoning is strengthened by the numerical experiments reported below. On the other hand, it is known from Theorem 2.2 that for a large class of discrete problems (1.5) the asymptotic behavior of both methods based on r_{k+1}^k and R_{k+1}^k is the same.

The main properties of the monotone multigrid method induced by Λ are summarized in the following Theorem.

Theorem 3.1 *Assume that the local obstacles ψ_l, $l = 1, \ldots, m$, defined in (3.5) are based on the restriction operators R^k_{k+1}, $k = 0, \ldots, j-1$, given by (3.8). Then the monotone multigrid method induced by Λ and ψ_l, $l = 1, \ldots, m$, is globally convergent.*

If additionally the discrete problem (1.5) satisfies the strict complementary condition (2.15), then the a posteriori error estimate

$$\|u_j^\nu - u_j^{\nu+1}\| \leq (1 - c(j+1)^{-2})\|u_j^{\nu-1} - u_j^\nu\| \qquad (3.11)$$

holds for $\nu \geq \nu_0$ and suitable $\nu_0 \geq 0$. Here $\|\cdot\|^2 = a(\cdot,\cdot)$ denotes the energy norm and the positive constant $c < 1$ only depends on the ellipticity of $a(\cdot,\cdot)$ and the shape regularity of $\mathcal{T}_0$.

Proof. From the assumptions and Lemma 3.1, it is clear that the local obstacles ψ_l, $l = 1, \ldots, m$, are quasioptimal. It is easily checked that Λ is regular in the sense of the preceding section.

Hence, it follows from Theorem 2.2 that the induced extended underrelaxation is globally convergent. Moreover, Theorem 2.2 states that the iteration (2.4) asymptotically reduces to the linear scheme (2.21) induced by $M^\circ = \Lambda^\circ$ for the iterative solution of the reduced linear problem (2.20). The convergence of this linear method is investigated in a recent paper of Kornhuber and Yserentant [23], from which the error estimate (3.11) is taken. ■

Note that the proof of the asymptotic error estimate (3.11) is restricted to two space dimensions, referring to Kornhuber and Yserentant [23] for details. We emphasize that no regularity assumptions on the free boundary have entered into our considerations.

Excluding contributions of the coarse–grid functions $\lambda_l \in \Lambda \setminus \Lambda^\circ$ may considerably deteriorate the coarse–grid transport of the linear scheme (2.21), at least compared to standard linear multigrid methods. This explains, why the symmetric scheme induced by Λ usually suffers from unsatisfying asymptotic convergence rates (c.f. Mandel [26]). To improve the asymptotic convergence by improved coarse–grid transport, we will extend the set Λ° by suitable *unsymmetric* truncations of the multilevel nodal basis functions $\lambda \in \Lambda$.

More precisely, we use the variable search directions $\tilde{\Lambda}^\nu = \Lambda_j \cup \tilde{\Lambda}^\nu_c$ with $\tilde{\Lambda}^\nu_c$ given by

$$\tilde{\Lambda}^\nu_c = \{\tilde{\lambda} \mid \tilde{\lambda} = I_{\mathcal{S}^\nu_j}\lambda^{(k)}_p,\ \lambda^{(k)}_p \in \Lambda_c,\ p \in \mathcal{N}_j \setminus \mathcal{N}^\nu_j\}, \qquad (3.12)$$

using the $\mathcal{S}^\nu_j$–interpolate $I_{\mathcal{S}^\nu_j}$. The space $\mathcal{S}^\nu_j$,

$$\mathcal{S}^\nu_j = \{v \in \mathcal{S}_j \mid \text{ int supp } v \cap \mathcal{N}^\nu_j = \emptyset\}, \qquad (3.13)$$

is the reduced subspace of $\mathcal{S}_j$ with respect to the subset $\mathcal{N}^\nu_j \subset \mathcal{N}_j$,

$$\mathcal{N}^\nu_k = \{p \in \mathcal{N}_j \mid w^\nu_{l_0}(p) = \varphi_j(p),\ l_0 = l_0(p,j)\}, \qquad (3.14)$$

which is fixed by the intermediate iterates w_l^ν resulting from fine–grid corrections. To make sure that $\tilde{\Lambda}^\nu$ is well–defined, the m^ν elements $\tilde{\lambda}_l = I_{\mathcal{S}_j^\nu}\lambda_p^{(k)}$ are ordered according to the refinement levels k, as indicated in (3.1). Indeed, to evaluate the interpolation $I_{\mathcal{S}_j^\nu}\lambda_p^{(k)}$, $\lambda_l \in \Lambda$, only the subset $\mathcal{N}_j^\nu \cap$ int supp λ_l is required. But this subset is known from the preceding fine–grid corrections corresponding to $p \in$ int supp λ_l. Note that the inactivation of an active point $p \in \mathcal{N}_j^\bullet(u_j^\nu)$ of u_j^ν can be caused only by the corresponding fine–grid correction.

It is easily checked that the truncated functions $\tilde{\lambda}_l \in \tilde{\Lambda}^\nu$ still satisfy condition (2.14) with constants c_1, c_2 not depending on ν. We clearly have $\mathcal{N}_j^\nu \subset \mathcal{N}_j^\bullet(u_j^{\nu+1})$ for all $\nu \geq 0$. Moreover, it immediately follows that

$$\mathcal{N}_j^\nu = \mathcal{N}_j^\bullet(u_j), \quad \mathcal{S}_j^\nu = \mathcal{S}_j^\circ, \quad \nu \geq \nu_0, \tag{3.15}$$

holds, if the active sets satisfy $\mathcal{N}^\bullet(w_l^\nu) = \mathcal{N}_j^\bullet(u_j)$, for $l = 1, \ldots, m^\nu$ and $\nu \geq \nu_0$. Hence, $\tilde{\Lambda}^\nu$ is regular for any sequence $\mathcal{N}_j^\nu$ defined in (3.14). Of course, we have

$$\Lambda^\circ \subset \tilde{\Lambda}^\circ = \{\tilde{\lambda} \mid \tilde{\lambda} = I_{\mathcal{S}_j^\circ}\lambda_p^{(k)}, \ \lambda_p^{(k)} \in \Lambda, \ p \in \mathcal{N}_j^\circ\}$$

so that we can expect an improved asymptotic convergence of *unsymmetric* multigrid methods induced by $\tilde{\Lambda}^\nu$.

Utilizing a related version of Lemma 3.1, we can derive a sequence of quasioptimal local obstacles $\tilde{\psi}_l^\nu \in \tilde{V}_l = \text{span}\{\tilde{\lambda}_l\}$ by suitable recursive restriction

$$\tilde{\psi}^{(k)} = \tilde{R}_{k+1}^k(\tilde{\psi}^{(k+1)} - v^{(k+1)}), \quad k = j-1, \ldots, 0, \tag{3.16}$$

of the defect obstacle $\tilde{\psi}^{(0)} = \varphi_j - u_j^\nu$ and the definition

$$\tilde{\psi}_l^\nu = \tilde{\psi}^{(k)}(p)\tilde{\lambda}_p^{(k)}, \quad l = l(p,k) = 1, \ldots, m^\nu. \tag{3.17}$$

Appropriate restriction operators $\tilde{R}_{k+1}^k$ are obtained by a slight modification of the restrictions R_{k+1}^k defined in (3.8). More precisely, for given $v \in \mathcal{S}_{k+1}$, we formally set $v(p_e) = \infty$, if $p_e \in \mathcal{N}_{k+1} \cap \mathcal{N}_j^\nu$ and then compute the coefficients v_p and v_q appearing in (3.9) as described above. In this way, we obtain modified local operators $\tilde{R}_e : \mathcal{S}_{k+1} \to \mathcal{S}_{k+1}$, $e \in \mathcal{E}_k'$, generating the restrictions $\tilde{R}_{k+1}^k$, $k = j-1, \ldots, 0$, by the definition

$$\tilde{R}_{k+1}^k v = I_{\mathcal{S}_k} \circ \tilde{R}_{e_s} \circ \ldots \circ \tilde{R}_{e_1} v, \quad v \in \mathcal{S}_{k+1}. \tag{3.18}$$

Theorem 3.2 *Assume that the local obstacles $\tilde{\psi}_l^\nu$, $l = 1, \ldots, m^\nu$, defined in (3.17) are based on the restriction operators $\tilde{R}_{k+1}^k$, $k = 0, \ldots, j-1$, given by (3.18). Then the monotone multigrid method induced by $\tilde{\Lambda}^\nu$ and $\tilde{\psi}_l^\nu$, $l = 1, \ldots, m^\nu$, is globally convergent.*

If additionally the discrete problem (1.5) satisfies the strict complementary condition (2.15), then the iterates u_j^ν satisfy an a posteriori error estimate of the form (3.11) for all $\nu \geq \nu_0$ and suitable $\nu_0 \geq 0$.

Proof. Following the proof of Theorem 3.1 the assertions are immediately deduced from Theorem 2.2 and the results in [23]. ■

We emphasize that both the symmetric and the unsymmetric monotone multigrid method can be implemented as a usual multigrid V–cycle.

Indeed, the symmetric multigrid method amounts to a sequence of projected Gauss–Seidel smoothings on the refinement levels $k = j, \ldots, 0$. Starting with $\psi^{(j)} = \varphi_j - u_j^\nu$, the obstacle $\psi^{(k)} \in \mathcal{S}_k$ on level k is resulting from the restriction (3.4) of the corrected obstacle $\psi^{(k+1)} - v^{(k+1)}$ on the preceding level. On coarse grids the stiffness matrix and the residual are obtained by usual weighted restriction. Finally, the corrections are interpolated to the finest level by canonical prolongation.

The unsymmetric multigrid method can be arranged as a modification of the symmetric scheme, making sure that the points $p \in \mathcal{N}_j^\nu$ do not enter further calculations. In particular, these points must not contribute to the weighted restriction of the residual *and* of the stiffness matrix and to the prolongated corrections. To avoid unnecessary computational work, the possibly varying restriction of the stiffness matrix has to be implemented carefully.

We have recovered the well–known local modifications of the canonical restrictions and prolongations in the neighborhood of the free boundary (c.f. Brandt and Cryer [6]). Recall that the way we are using these modifications is justified by the convergence results mentioned above. It turns out that this advantage pays off in actual computations as reported in the next section.

Chapter 4

Numerical Experiments

In this section, we consider the numerical solution of the obstacle problem (1.1) with the quadratic form $a(\cdot,\cdot)$, the right hand side $\ell(\cdot)$ and the constraints $\mathcal{K} \subset H_0^1(\Omega)$ given by

$$a(v,w) = \int_\Omega \partial_1 v\ \partial_1 w + \partial_2 v\ \partial_2 w\ dx, \quad \ell(v) = 2C \int_\Omega v\ dx$$

and

$$\mathcal{K} = \{v \in H_0^1(\Omega)|\ v(x) \leq \mathrm{dist}(x, \partial\Omega),\ \text{a.e. in } \Omega\},$$

respectively. For simplicity, we use the unit square $\Omega = (0,1) \times (0,1)$. The resulting optimization problem is modeling the elasto–plastic torsion of a cylindrical bar with cross–section Ω, where the active points characterize the plastic region, while the material is considered elastic in nonactive points. The solution u represents the stress potential and the applied twist angle is expressed by the parameter C. We refer for example to Rodrigues [27] for further information.

The elastic region is located along the diagonals of Ω and becomes arbitraryly small with increasing C, rendering a challenging test example for various numerical methods (see [9, 12, 15, 18]).

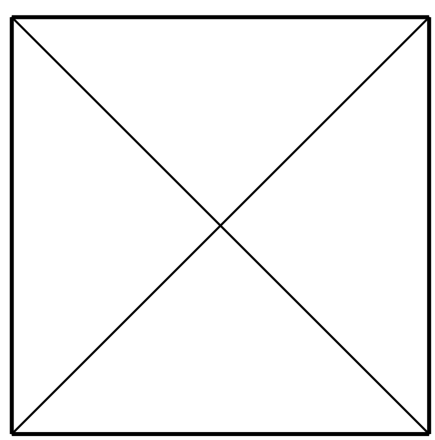

Figure 4.1: Initial Triangulation $\mathcal{T}_0$

The continuous problem (1.1) is discretized with respect to various triangulations $\mathcal{T}_j$, $j = 0, \ldots, 8$, resulting from j uniform refinement steps applied to

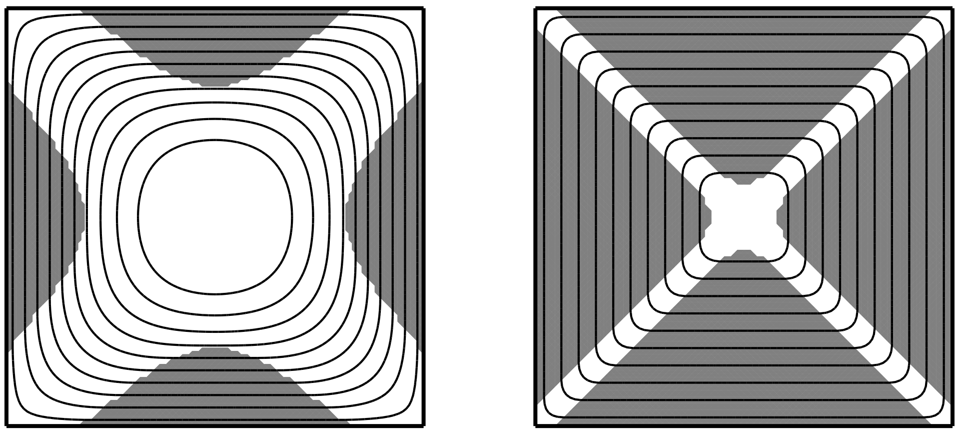

Figure 4.2: Level Curves and Active Sets for $C = 2.5$ and $C = 10$.

the initial triangulation $\mathcal{T}_0$, which is shown in Figure 4.1. The combination with adaptive techniques as applied successfully in [10, 18, 22, 21] will be treated in a forthcoming paper. The level curves and (shaded) plastic regions of the approximate solutions on refinement level $j = 6$ corresponding to the parameters $C = 2.5$ and $C = 10.$ are depicted in the left and the right picture of Figure 4.2, respectively.

The resulting discrete problems of the form (1.5) are solved iteratively comparing the four different multigrid methods described as follows.

SYMMA:	The symmetric multigrid method proposed by Mandel [25, 26].
SYMKH:	The symmetric multigrid method with new restriction of the defect obstacle (c.f. Theorem 3.1).
BRCR:	A variant of the multigrid methods proposed by Brandt and Cryer [6].
UNSYMKH:	The unsymmetric multigrid method with modified new restrictions (c.f. Theorem 3.2).

For all four methods we only consider the V–cycle with one pre–smoothing step. The multigrid method BRCR needs some further explanation. The method is based on the full approximation scheme (FAS), using pointwise restrictions of the unknowns and the obstacle function. The stiffness matrix on the coarse grids is induced by the quadratic form $a(\cdot, \cdot)$. The weighted restriction of the residuals and the weighted interpolation of the corrections are modified such that active points do not contribute to the restriction and such that coarse–grid corrections do not cause the inactivation of active points. Note that BRCR seems to be the most prominent of several other variants proposed in [6]. Though there is a lot of similarity to the unsymmetric method, BRCR is not monotone.

The implementation was carried out in the framework of the finite element code KASKADE (c.f. Roitzsch [28, 29]) and we used a SPARC IPX workstation for the actual computation.

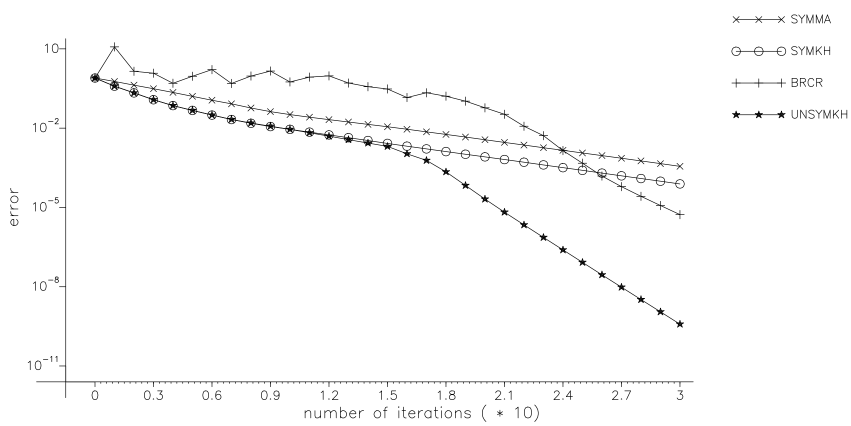

Figure 4.3: Comparison of the Iterative Errors for $C = 2.5$ and $j = 6$

In our first experiment we investigate the convergence behavior of the four multigrid methods for the fixed initial iterate $u_j^0 = 0$ and fixed refinement level $j = 6$. Figure 4.3 gives an overview on the iterative errors occurring for the parameter $C = 2.5$. The overall convergence of all methods can be divided into a transient phase, dominated by the search for the exact active set, and an asymptotic phase, corresponding to the iterative solution of the reduced linear scheme. This observation supports the analysis contained in the second section. Moreover, it indicates that the common description of the global convergence behavior by just one averaged convergence rate may be misleading.

As expected, the new restrictions (3.8) applied in SYMKH provide slightly better transient convergence rates than Mandel's restrictions (3.7) used in SYMMA. The asymptotic behavior of both symmetric methods, relying on the same reduced splitting, clearly remains the same. Compared to SYMKH, the transient convergence behavior of UNSYM scarcely suffers from the fact that coarse–grid corrections must not cause any inactivation. On the other hand, we have a tremendous improvement of the asymptotic convergence rate, becoming approximately the same as in the unconstrained case. Though the transient behavior of BRCR exhibits the intrinsic lack of monotonicity, both symmetric schemes SYMMA and SYMKH are clearly outperformed once the asymptotic phase is reached.

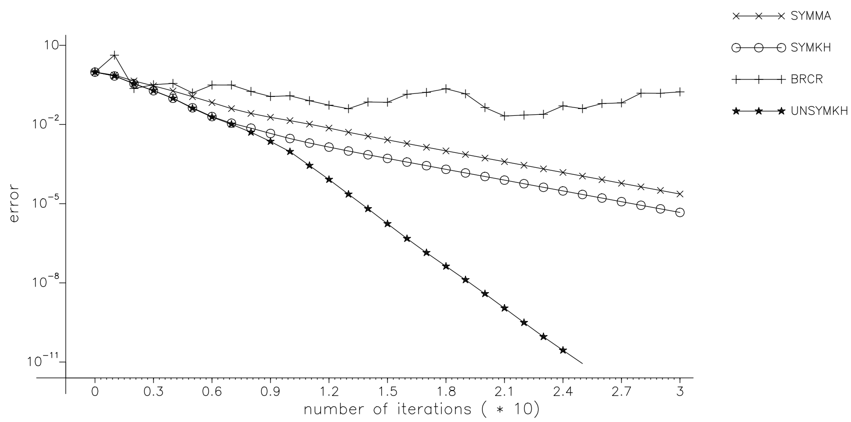

Figure 4.4: Comparison of the Iterative Errors for $C = 10.$ and $j = 6$

To check the robustness of the four multigrid methods, the twist angle C is now switched to $C = 10.$. The corresponding iteration history is shown in Figure 4.4. While the convergence behavior of the three monotone multigrid methods remains basically unchanged, we did not observe convergence of BRCR within the first 400 iteration steps. Note that BRCR entered an infinite loop if applied to the same problem on the lower refinement level $j = 4$, while (slow) convergence occurred for $j = 5$. This lack of robustness of Brandt and Cryer's method is well–known for quite a while (c.f. Bollrath [4], p. 29). However, such cases as resulting from the choice of $C = 10.$ were considered as artificial compared to moderate situations as provided by $C = 2.5$.

Using the moderate value $C = 2.5$, we now concentrate on the variation of the convergence behavior with varying refinement level j. Starting with $u_j^0 = 0$ on the levels $j = 0, \ldots, 8$, we found that the transient and the asymptotic convergence rates of UNSYM seem to be uniformly bounded by about 0.8 and 0.4, respectively. However, the number of iterations, which is needed to reach the asymptotic phase, grows exponentially with the refinement levels j. Related results were obtained for the other three methods.

To provide a more realistic situation, the artificial initial iterate $u_j = 0$ is now replaced by the interpolated solution from the previous level. It turns out that in this way the transient phase is almost eliminated from the convergence history. As the iteration immediately enters the asymptotic phase, it now makes sense to consider the usual averaged convergence rates ρ_j given by

$$\rho_j = \sqrt[\nu_0]{\varepsilon_j^{\nu_0} / \varepsilon_j^0}, \quad j = 0, \ldots, 8,$$

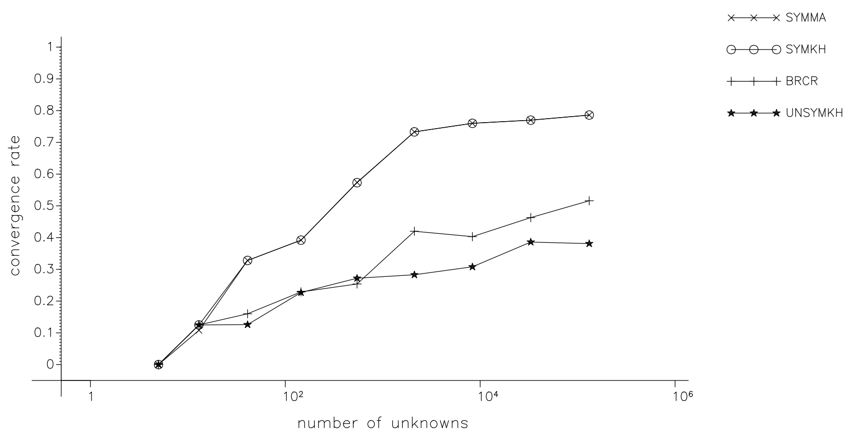

Figure 4.5: Comparison of the Convergence Rates for $C = 2.5$

where ε_j^ν denotes the iterative error after ν iteration steps and ν_0 is chosen such that $\varepsilon_j^{\nu_0} < 10.^{-12}$. For each of the four multigrid methods in question, Figure 4.5 shows the dependency of ρ_j on the refinement level j. As expected from the same asymptotic behavior of SYMMA and SYMKH, we obtain almost the same results of both symmetric methods. The convergence rates seem to saturate at about 0.8. Due to the good initial iterates, the excellent asymptotic convergence rates of BRCR and UNSYM now preserve during the whole iteration process.

In practical calculations, the discrete problem (1.5) should be solved only up to discretization accuracy. If applied to the present example, this strategy requires one or two iteration steps of UNSYM on each level, to reduce the iterative error to one tenth of the discretization error. This is the same efficiency as observed in the corresponding unconstrained case.

Acknowledgements. The work on the present paper was stimulated by frequent discussions with F. Bornemann, R.H.W. Hoppe, H. Yserentant and my students. Special thanks to P.Deuflhard for various important remarks and suggestions an to R. Roitzsch for providing a comfortable software environment for the implementation.

Bibliography

[1] C. Baiocchi, A. Capelo: *Variational and Quasivariational Inequalities.* Wiley, Chichester (1984)

[2] R.E. Bank, T. Dupont, H. Yserentant: *The Hierarchical Basis Multigrid Method.* Numer. Math. **52**, p. 387–404 (1988)

[3] R.E. Bank, A.H. Sherman, H. Weiser: *Refinement Algorithm and Data Structures for Regular Local Mesh Refinement.* Scientific Computing, (R. Stepleman et al., eds.), Amsterdam: IMACS North–Holland, p. 3–17 (1983)

[4] C. Bollrath: *Zwei Mehrgitterverfahren zur numerischen Berechnung von stationären Strömungen durch poröse Medien mit freiem Rand.* Dissertation, Universität Bochum (1985)

[5] F. Bornemann, B. Erdmann, R. Kornhuber: *Adaptive Multilevel Solution of 3–D Elliptic Boundary Value Problems.* To appear in: Int. J. Numer. Meth. Eng. (1993)

[6] A. Brandt, C.W. Cryer: *Multigrid Algorithms for the Solution of Linear Complementarity Problems Arising from Free Boundary Problems.* SIAM J. Sci. Stat. Comput., **4**, p. 655–684 (1983)

[7] R.W. Cottle, J.S Pang, R.E. Stone: *The Linear Complementary Problem.* Academic Press, Boston (1992)

[8] P. Deuflhard, P. Leinen, H. Yserentant: *Concepts of an Adaptive Hierarchical Finite Element Code.* IMPACT Comput. Sci. Engrg. **1**, p. 3–35 (1989)

[9] B. Erdmann, R.H.W. Hoppe, R. Kornhuber: *Adaptive Multilevel – Methods for Obstacle Problems in Three Space Dimensions.* To appear in: Proceedings of the 9. GAMM–Seminar Kiel (W. Hackbusch, G. Wittum, eds.), Vieweg, Braunschweig (1994)

[10] B. Erdmann, M. Frei, R.H.W. Hoppe, R. Kornhuber, U. Wiest: *Adaptive Finite Element Methods for Variational Inequalities.* To appear in: East–Western Journal on Numerical Mathematics (1993)

[11] R. Glowinski: *Numerical Methods for Nonlinear Variational Problems.* Springer Verlag, New York (1984)

[12] R. Glowinski, J.L. Lions, R. Trémolières: *Numerical Analysis of Variational Inequalities.* North–Holland, Amsterdam (1981)

[13] W. Hackbusch: *Multi–Grid Methods and Applications.* Springer Verlag, Berlin (1985)

[14] W. Hackbusch, H.D. Mittelmann: *On Multigrid Methods for Variational Inequalities.* Numer. Math., **42**, p. 65–76 (1983)

[15] R.H.W. Hoppe: *Multigrid Algorithms for Variational Inequalities.* SIAM J. Numer. Anal., **24**, p. 1046–1065 (1987)

[16] R.H.W. Hoppe: *Two–sided Approximations for Unilateral Variational Inequalities by Multigrid Methods.* Optimization, **18**, p. 867–881 (1987)

[17] R.H.W. Hoppe: *Une méthode multigrille pour la solution des problèmes d'obstacle.* M^2 AN, **24**, p. 711–736 (1990)

[18] R.H.W. Hoppe, R. Kornhuber: *Adaptive Multilevel–Methods for Obstacle Problems.* To appear in: SIAM J. Numer. Anal. (1993)

[19] D. Kinderlehrer, G. Stampacchia: *An Introduction to Variational Inequalities and Their Applications.* Academic Press, New York (1980)

[20] R. Kornhuber: *Monotone Multigrid Methods for Elliptic Variational Inequalities II.* Preprint SC 93–19, Konrad–Zuse–Zentrum Berlin (ZIB) (1993)

[21] R. Kornhuber, R. Roitzsch: *Self Adaptive Computation of the Breakdown Voltage of Planar pn–Junctions with Multistep Field Plates.* In: Proceedings of the 4th International Conference of Simulation of Semiconductor Devices and Processes. (Fichtner et al., eds.), Zurich, p. 535–543 (1991)

[22] R. Kornhuber, R. Roitzsch: *Self Adaptive Finite Element Simulation of Reverse Biased pn–Junctions.* Comm. Numer. Meth. Eng. **9**, p. 243–250 (1993)

[23] R. Kornhuber, H. Yserentant: *Multilevel Methods for Elliptic Problems on Roughly Bounded Domains.* Preprint SC 93–21, Konrad–Zuse–Zentrum Berlin (ZIB) (1993)

[24] P.L. Lions: *On the Schwarz Alternating Method II.* In: Domain Decomposition Methods for Partial Differential Equations. (T.F. Chan et al., eds.), SIAM, Philadelphia, p. 47–70 (1989)

[25] J. Mandel: *Etude algébrique d'une méthode multigrille pour quelques problèmes de frontière libre.* C.R. Acad. Sci. Paris, Ser. I, **298**, p. 469–472 (1984)

[26] J. Mandel: *A Multilevel Iterative Method for Symmetric, Positive Definite Linear Complementarity Problems.* Appl. Math. Optimization, **11**, p. 77–95 (1984)

[27] J.–F. Rodrigues: *Obstacle Problems in Mathematical Physics.* North–Holland Mathematical Studies 134, Amsterdam (1987)

[28] R. Roitzsch: KASKADE *User's Manual.* Technical Report TR 89–4 Konrad–Zuse–Zentrum Berlin (ZIB) (1989)

[29] R. Roitzsch: KASKADE *Programmer's Manual.* Technical Report TR 89–5 Konrad–Zuse–Zentrum Berlin (ZIB) (1989)

[30] R. Schwenkert: *Lösung linearer Komplementaritätsprobleme mittels hierarchischer Basen.*, Report 108, Institut für Angewandte Mathematik und Statistik der Universität Würzburg (1988)

[31] J. Xu: *Iterative Methods by Space Decomposition and Subspace Correction.* SIAM Review **34**, p. 581–613 (1992)

[32] H. Yserentant: *Two Preconditioners Based on the Multilevel Splitting of Finite Element Spaces.* Numer. Math., **58**, p. 163–184 (1990)

[33] H. Yserentant: *Old and New Convergence Proofs for Multigrid Methods.* Acta Numerica, p. 285–326 (1993)