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Adaptive Multilevel – Methods for Obstacle Problems

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Abstract. We consider the discretization of obstacle problems for second order elliptic differential operators by piecewise linear finite elements. Assuming that the discrete problems are reduced to a sequence of linear problems by suitable active set strategies, the linear problems are solved iteratively by preconditioned cg–iterations. The proposed preconditioners are treated theoretically as abstract additive Schwarz methods and are implemented as truncated hierarchical basis preconditioners. To allow for local mesh refinement we derive semi–local and local a posteriori error estimates, providing lower and upper estimates for the discretization error. The theoretical results are illustrated by numerical computations.

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Contents

1. Introduction

Given a closed subspace $V \subset H^1(\Omega)$, Ω being a bounded polygonal domain in the Euclidean space $\mathbb{R}^2$, we consider obstacle problems of the form

$$\text{Find } u \in K \text{ such that } \quad \mathcal{J}(u) \leq \mathcal{J}(v) \ , \ \ v \in K, \tag{1.1}$$

for the energy functional $\mathcal{J}$,

$$\mathcal{J}(v) = \frac{1}{2} a(v,v) - \ell(v) \ , \ \ v \in V,$$

and a closed, convex set $K \subset V$,

$$K = \{ v \in V | \ v(x) \leq \varphi(x) \text{ a.e. in } \Omega \}.$$

Assuming that $\mathcal{J}$ is induced by a symmetric V–elliptic bilinear form $a(\cdot,\cdot)$,

$$a(v,w) = \int_\Omega \sum_{i,j=1}^{2} a_{ij} \ \partial_i v \ \partial_j w \ dx,$$

and some functional $\ell \in V'$, it is well–known that (1.1) is equivalent to the variational inequality

$$\text{Find } u \in K \text{ such that } \quad a(u, u-v) \leq \ell(u-v) \ , \quad v \in K. \tag{1.2}$$

For simplicity we restrict our considerations to the case $V = H_0^1(\Omega)$. To ensure existence and uniqueness of the solution u of (1.1) and (1.2), respectively, we assume $\varphi \in L^\infty(\Omega)$, $\varphi \geq 0$ a.e. on $\Gamma = \partial\Omega$, and $a_{ij} \in L^\infty(\Omega)$ satisfying

$$\begin{array}{lrcl} \text{a)} & a_{ij}(x) & = & a_{ji}(x) \ , \ \ 1 \leq i, \ j \leq 2 \ , \\ \text{b)} & \alpha_0 |\xi|^2 & \leq & \displaystyle\sum_{i,j=1}^{2} a_{ij}(x) \xi_i \xi_j \leq \alpha_1 |\xi|^2, \ \xi \in R^2, \ 0 < \alpha_0 \leq \alpha_1 \ . \end{array} \tag{1.3}$$

for almost all $x \in \Omega$.

Discretizing (1.2) in space by continuous, piecewise linear finite elements with respect to a triangulation of Ω, standard numerical schemes for the solution of the resulting finite dimensional variational inequality are projected relaxation methods (e.g. [15]). These iterative methods typically suffer from rapidly deteriorating convergence rates when proceeding to more and more refined triangulations which renders them inefficient from a numerical point of view. However, this drawback can be overcome by using multilevel techniques with respect to a hierarchy of triangulations. Multigrid approaches to obstacle problems have been developed by various authors ([10, 16, 17, 18, 19, 28, 29]). For obstacle type problems an alternative to projected relaxation is to use some sort of linearization techniques based on active set strategies (e.g. [16, 17, 18]). This is an iterative scheme where in each iteration step a set of active constraints is prespecified and then a linear subproblem has to be solved for the computation of the new iterate. Note that the multigrid techniques used in [16, 17, 18] consist of outer and inner iterations where the outer iteration is an active set strategy and the inner iterations are multigrid iterations for the approximate solution of the auxiliary problems.

Since for the obstacle problems under consideration the coefficient matrices of the auxiliary systems are symmetric positive definite, an alternative choice for the inner iterations are preconditioned conjugate gradient (pcg) methods, especially those based on multilevel preconditioners such as Yserentant's hierarchical basis preconditioner [36] or the BPX–preconditioner [9]. A related approach has been proposed in [34] where relaxation methods have been applied with respect to hierarchical bases.

For the adaptive construction of a suitable hierarchy of triangulations efficient and reliable a–posteriori error estimates are required. While a variety of well–established results are available in the case of linear elliptic problems (see [3, 13, 21, 35] for further references) the situation is less clear in the case of obstacle problems. Recently, the concepts introduced in [13] have been extended and applied successfully to a special obstacle problem arising in semi–conductor device simulation [24]. A more detailed investigation of this approach will be a subject of this paper. A–posteriori error estimates for the penalty method together with strategies for the adaptive choice of a space dependent penalty parameter and the mesh size have been given in [22].

The paper is organized as follows. After a brief discussion of the active–set strategy proposed in [17], we will focus on the construction and analysis of multilevel preconditioners providing the efficient solution of the arising linear subproblems. In particular, we will derive two variants of hierarchical basis type by suitable modifications of the standard hierarchical basis preconditioner. It will turn out that both variants are performing asymptotically as in the unconstrained case but that only one of them is robust with respect to the regularity of the free boundary. Inspired by a paper of Dryja and Widlund [14], the preconditioners will be regarded as multilevel additive Schwarz (MAS) methods. This abstract framework allows for obvious extensions to other variants of the MAS method, in particular to the BPX–preconditioner. Comparing the actual approximation with another approximation of higher accuracy we will derive semi–local and local a–posteriori error estimates, followed by a detailed analysis of their efficiency and reliability. The final chapter is devoted to some numerical experiments supporting the theoretical findings.

2. Outer–Inner Iterations

Let $\mathcal{T}$ denote a triangulation of the computational domain $\Omega \subset \mathbb{R}^2$. We assume that $\mathcal{T}$ is regular in the sense that the intersection of two triangles $t, t' \in \mathcal{T}$ is containing a common edge, a common vertex or is empty. The sets of vertices p and edges e which are not part of the boundary $\partial\Omega$ are called $\mathcal{P}$ and $\mathcal{E}$, respectively. We approximate V by the subspace $\mathcal{S}$ of continuous, piecewise linear finite elements vanishing on the boundary $\partial\Omega$ with the associated nodal basis λ_p, $p \in \mathcal{P}$, of $\mathcal{S}$ defined by $\lambda_p(q) = \delta_{pq}$, $p, q \in \mathcal{P}$, (Kronecker delta).

Further, let $\varphi_{\mathcal{T}} \in \mathcal{S}$ be a discrete obstacle approximating the given obstacle φ in an appropriate sense. For example, $\varphi_{\mathcal{T}}$ may be chosen as the L^2–projection of φ onto $\mathcal{S}$ or, if $\varphi \in C(\bar{\Omega})$, as the $\mathcal{S}$–interpolate. Correspondingly, we denote by $K_{\mathcal{T}} = \{v \in \mathcal{S} | v \le \varphi_{\mathcal{T}}\}$ the sets of discrete constraints. Then the finite element approximation of (1.1) amounts to the computation of an element $u_{\mathcal{T}} \in K_{\mathcal{T}}$ satisfying

$$a(u_{\mathcal{T}}, u_{\mathcal{T}} - v) \le \ell(u_{\mathcal{T}} - v), \; v \in K_{\mathcal{T}} \; . \tag{2.1}$$

It is easy to see that the finite dimensional variational inequality (2.1) is equivalent to a linear complementarity problem.

Lemma 2.1 *An element $u_{\mathcal{T}} \in K_{\mathcal{T}}$ is a solution to* (2.1) *if and only if the vector $\underline{u} \in \mathbb{R}^N$, $N := |\mathcal{P}|$ with components $u_p = u_{\mathcal{T}}(p)$, $p \in \mathcal{P}$, satisfies*

$$\max(A\underline{u} - \underline{b}, \; \underline{u} - \underline{\varphi}) = 0 \tag{2.2}$$

where A is the $N \times N$ stiffness matrix with entries $a_{pq} = a(\lambda_q, \lambda_p)$, $p, q \in \mathcal{P}$, and $\underline{b} \in \mathbb{R}^N$ and $\underline{\varphi} \in \mathbb{R}^N$ are the vectors with components $b_p = \ell(\lambda_p)$ and $\varphi_p = \varphi_{\mathcal{T}}(p)$, $p \in \mathcal{P}$. Note that (2.2) *has to be understood componentwise.*

Proof. Let $u_{\mathcal{T}} \in K_{\mathcal{T}}$ be the solution of (2.1). Then $A\underline{u} \le \underline{b}$ which can be deduced by choosing $v = u_{\mathcal{T}} - z$ in (2.1) with arbitrarily given $z \in \mathcal{S}$, $z \ge 0$. Since $\underline{u} \le \underline{\varphi}$, we thus have $(\underline{u} - \underline{\varphi})^T(A\underline{u} - \underline{b}) \ge 0$. But $v = \varphi_{\mathcal{T}}$ in (2.1) gives $(\underline{u} - \underline{\varphi})^T(A\underline{u} - \underline{b}) \le 0$ where $(\underline{u} - \underline{\varphi})^T(A\underline{u} - \underline{b}) = 0$ proving (2.2). The converse statement is obvious. ■

In the following we will consider an outer–inner iteration technique for the numerical solution of the complementarity problem (2.2). The outer iterations are governed by an active set strategy as presented in [17, 18]:

Outer iteration (active set strategy):

Step 1: Chose a startvector $\underline{u}^{(0)} \in \mathbb{R}^N$.

Step 2: Given $\underline{u}^{(\nu)} \in \mathrm{IR}^N$, $\nu \geq 0$, determine $\mathcal{P}^\bullet \subset \mathcal{P}$ as the set of points $p \in \mathcal{P}$ such that $(A\underline{u}^{(\nu)} - \underline{b})_p > (\underline{u}^{(\nu)} - \underline{\varphi})_p$ and set $\mathcal{P}^\circ := \mathcal{P} \setminus \mathcal{P}^\bullet$. Then compute $\underline{u}^{(\nu+1)} \in \mathrm{IR}^N$ from the splitting

$$\underline{u}^{(\nu+1)} = \underline{u}^\bullet + \underline{u}^\circ \tag{2.3}$$

where

$$u_p^\bullet = \varphi_p,\ p \in \mathcal{P}^\bullet\ ,\ \ u_p^\bullet = 0,\ p \in \mathcal{P}^\circ \tag{2.4}$$

and $\underline{u}^\circ$ satisfies

$$u_p^\circ = 0,\ p \in \mathcal{P}^\bullet \tag{2.5}$$

and

$$A\underline{u}^\circ = \underline{b} - A\underline{u}^\bullet . \tag{2.6}$$

It is obvious that the computation of the iterate $\underline{u}^{(\nu+1)}$ according to (2.6) actually requires the solution of a “reduced”, i.e., lower dimensional linear system.

The set $\mathcal{P}^\bullet$ is called active, since in view of $u_p^{(\nu+1)} = \varphi_p$, $p \in \mathcal{P}^\bullet$, it contains the nodal points where the obstacle is active. Correspondingly, $\mathcal{P}^\circ$ is said to be the inactive set. Introducing a corresponding splitting of the finite element space $\mathcal{S} = \mathcal{S}^\circ \oplus \mathcal{S}^\bullet$ in linear subspaces $\mathcal{S}^\circ$, $\mathcal{S}^\bullet \subset \mathcal{S}$ defined by

$$\mathcal{S}^\circ = \{v \in \mathcal{S} |\ v(p) = 0, p \in \mathcal{P}^\bullet\}\ ,\ \ \mathcal{S}^\bullet = \{v \in \mathcal{S} |\ v(p) = 0, p \in \mathcal{P}^\circ\} \tag{2.7}$$

the reduced system (2.6) can be rewritten as the variational equality

$$\text{Find } u^\circ \in \mathcal{S}^\circ \text{ such that } a(u^\circ, v) = \ell(v) - a(u^\bullet, v)\ ,\ \ v \in \mathcal{S}^\circ \tag{2.8}$$

with solution $u^\circ \in \mathcal{S}^\circ$ and $u^\bullet \in \mathcal{S}^\bullet$ defined by $u^\bullet(p) = u_p^\bullet$.

Remark 2.1 If (2.6) respectively (2.8) is solved exactly, it can be shown that for arbitrarily given initial iterate $\underline{u}^{(0)}$ the sequence $\underline{u}^{(\nu)}$, $\nu \geq 0$, of iterates is a monotonically decreasing sequence converging to the unique solution $\underline{u}$ of (2.2) (see e.g. [17, 18]). Actually, we do not want to solve (2.8) exactly but compute an approximation up to a certain accuracy κ_0 by means of an efficient iterative solver. In this inexact case, the convergence of a related most constrained strategy has been proved in [16] providing a stopping criterion for the inner iteration. However, this strategy turns out to be much too pessimistic in actual computations leading to a prohibitive large number of outer iteration steps.

In contrast to [17, 18] where multigrid techniques have been used, in this paper we will focus our interest on multilevel preconditioned *cg*–iterations which for well–known reasons are more suited to be used within an adaptive FEM code. For an introduction to the preconditioned *cg*–method we refer to [1] while the construction of appropriate multilevel preconditioners will be subject of the next chapter.

3. Additive Schwarz Methods and Hierarchical Bases

Let $\mathcal{T}_0$ be an intentionally coarse regular triangulation of Ω.

The triangulation $\mathcal{T}_0$ is refined several times providing a sequence of triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ and a corresponding sequence of nested finite element spaces $\mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_j$. The underlying refinement process described in the sequel, is meanwhile standard in the literature on multilevel preconditioning [3, 4, 5, 6, 8, 13, 37]. Note that this refinement in general does not coincide with the actual refinement process performed by some finite element code. Nevertheless, the triangulations $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ are available without any computational effort, if the underlying data structures are chosen properly [3, 26, 31, 32].

A triangle $t \in \mathcal{T}_k$ is refined either by subdividing it into four congruent subtriangles or by connecting one of its vertices with the midpoint of the opposite side. The first case is called regular (red) refinement and the resulting triangles are regular as well as the triangles of the initial triangulation $\mathcal{T}_0$. The second case is called irregular (green) refinement and results in two irregular triangles. As we do not want that new points are generated by green refinement we introduce the rule

(T1) Each vertex of $\mathcal{T}_{k+1}$ which does not belong to $\mathcal{T}_k$ is a vertex of a regular triangle.

Note that irregular refinement is potentially dangerous, because the interior angles are reduced. Hence we add the rule

(T2) Irregular triangles must not be further refined.

We say that a refined triangle is the father of the resulting triangles which in turn are called sons. We define the depth of a given triangle $t \in \bigcup_{k=0}^{j} \mathcal{T}_k$ as the number of ancestors of t. Of course, the depth of all triangles $t \in \mathcal{T}_k$ is bounded by k. Due to the final rule

(T3) Only triangles $t \in \mathcal{T}_k$ of depth k may be refined for the construction of $\mathcal{T}_{k+1}$, $0 \leq k \leq j$.

the whole sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ can be uniquely reconstructed from the initial triangulation $\mathcal{T}_0$ and the final triangulation $\mathcal{T}_j$ alone, neglecting the preceding dynamic refinement process. Recall that in actual computations we may chose the data structures representing the triangulations cleverly so that the sequence $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ is explicitly given. Note that the subscript j does in general *not* coincide with the number $l \geq j$ of refinement steps which have been necessary to create $\mathcal{T}_j$ from $\mathcal{T}_0$ by the actual finite element code. In practical calculations the difference $l - j$ of the refinement level l and the maximal depth j can be used to judge the quality of the implemented refinement strategy.

Of course, adaptive refinement should be based on reliable a–posteriori error estimates which will be considered in the following chapter. For the moment let us assume that a hierarchy $\mathcal{T}_0, \mathcal{T}_1, \ldots, \mathcal{T}_j$ with the property (T1 - T3) is available. We further assume that we have a disjoint splitting $\mathcal{P}_j = \mathcal{P}_j^\bullet \cup \mathcal{P}_j^\circ$ which may result from an active set strategy applied to (2.1) with respect to the triangulation $\mathcal{T} = \mathcal{T}_j$. Recall that this splitting is supposed to change in each outer iteration step. In the sequel we will deal with the construction of two multilevel preconditioners of hierarchical basis type to provide an efficient iterative solution of the corresponding reduced system

$$\text{Find } u_j^\circ \in \mathcal{S}_j^\circ \text{ such that } a(u_j^\circ, v) = \ell(v) - a(u_j^\bullet, v)\ , \quad v \in \mathcal{S}_j^\circ. \tag{3.1}$$

For this purpose we provide a decomposition $\mathcal{P}_k = \mathcal{P}_k^\bullet \cup \mathcal{P}_k^\circ$ of the sets $\mathcal{P}_k$ of the nodal points on the lower levels $0 \le k < j$ by means of the definition

$$\mathcal{P}_k^\bullet = \mathcal{P}_k \cap \mathcal{P}_j^\bullet\ , \quad \mathcal{P}_k^\circ = \mathcal{P}_k \setminus \mathcal{P}_k^\bullet\ , \quad 0 \le k \le j-1. \tag{3.2}$$

For $0 \le k \le j$ and $p \in \mathcal{P}_k$ we refer to $\lambda_p^{(k)} \in \mathcal{S}_k$ as the level k nodal basis function having p as its supporting point, i.e., $\lambda_p^{(k)}(p) = 1$. According to (2.7) the splitting (3.2) induces the subspaces $\mathcal{S}_k^\circ = \operatorname{span}\{\lambda_p^{(k)} |\ p \in \mathcal{P}_k^\circ\} \subset \mathcal{S}_k$, $0 \le k \le j$. Collecting the hierarchical basis functions with inactive supporting points according to

$$\hat\Lambda_0 := \{\lambda_p^{(0)} |\ p \in \mathcal{P}_0^\circ\}\ , \quad \hat\Lambda_k := \{\lambda_p^{(k)} |\ p \in \mathcal{P}_k^\circ \setminus \mathcal{P}_{k-1}^\circ\},\ 1 \le k \le j\ , \tag{3.3}$$

we denote by $\hat\Lambda_H = \bigcup_{k=1}^j \hat\Lambda_k$ the set of all hierarchical basis functions on the levels $k \ge 1$. Furthermore, we will utilize the subspaces $\hat V_0 = \operatorname{span} \hat\Lambda_0$ and $\hat V_\lambda = \operatorname{span}\{\lambda\}$, $\lambda \in \hat\Lambda_H$. However, the hierarchical decomposition of functions $v \in \mathcal{S}_j^\circ$ cannot be given in the standard way, since the subsets $\hat\Lambda_0$ and $\hat\Lambda_k$ of $\mathcal{S}_j$ in general are not contained in $\mathcal{S}_j^\circ$. This is due to the fact that functions $v \in \mathcal{S}_{k-1}^\circ$, $1 \le k \le j$, in general do not vanish in active nodal points $p \in \mathcal{P}_k^\bullet \backslash \mathcal{P}_{k-1}^\bullet$ appearing on the subsequent level k. We will modify such functions by means of suitable truncation operators $T_k : \mathcal{S}_l \to \mathcal{S}_k^\circ$, $0 \le l \le k \le j$, defined by

$$T_k v = \sum_{p \in \mathcal{P}_k^\circ} v(p) \lambda_p^{(k)}\ , \tag{3.4}$$

Note that $T_k v = v$, $v \in \mathcal{S}_k^\circ$. Now a feasible multilevel splitting of $\mathcal{S}_j^\circ$ is defined by simple truncation of the standard hierarchical basis

$$\Lambda_k^{(1)} = T_j \hat\Lambda_k\ , \quad 0 \le k \le j\ . \tag{3.5}$$

We will consider a second multilevel splitting which is based on a more restrictive choice of coarse grid functions. For this reason we define

$$\mathcal{P}_k^{\circ,\mathrm{reg}} = \{p \in \mathcal{P}_k^\circ |\ T_j \lambda_p^{(k)} = \lambda_p^{(k)}\}\ ,\quad 0 \le k \le j \tag{3.6}$$

Obviously (3.6) can be regarded as a weighted modification of the pointwise restriction (3.2) of the active set $\mathcal{P}_j^\bullet$ to the lower levels. Note that we have $\mathcal{P}_k^{\circ,\mathrm{reg}} \subset \mathcal{P}_k^\circ$ and $\mathcal{P}_j^{\circ,\mathrm{reg}} = \mathcal{P}_j^\circ$. Now the standard hierarchical splitting with respect to $\mathcal{P}_k^{\circ,\mathrm{reg}}$, $0 \le k \le j$ is given by

$$\Lambda_0^{(2)} = \{\lambda_p^{(0)} |\ p \in \mathcal{P}_0^{\circ,\mathrm{reg}}\}\ ,\quad \Lambda_k^{(2)} = \{\lambda_p^{(k)} |\ p \in \mathcal{P}_k^{\circ,\mathrm{reg}} \setminus \mathcal{P}_{k-1}^{\circ,\mathrm{reg}}\},\ 1 \le k \le j\ . \tag{3.7}$$

Note that a restriction of the active set which is similar to (3.6) has been used in [17]. In the context of hierarchical bases (3.6) has been proposed by Yserentant [38].

Remark 3.1 The difference between $\Lambda_k^{(1)}$ and $\Lambda_k^{(2)}$ is illustrated in Figure 3.1 where for ease of exposition we have considered the 1–D case.

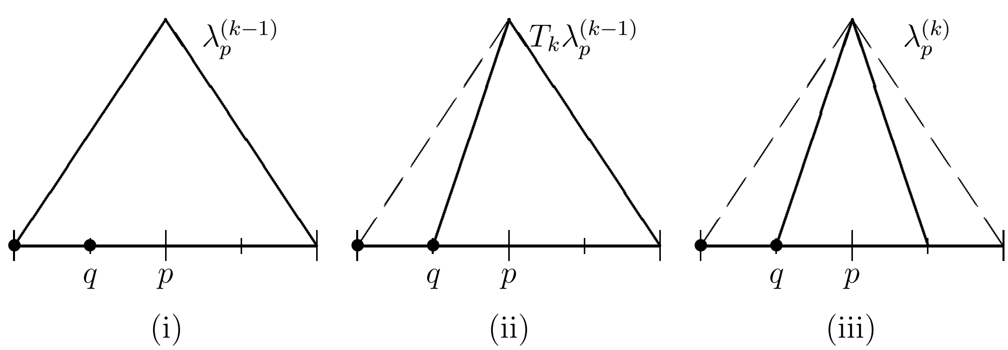

Figure 3.1

In particular, Figure 3.1 (i) represents a level $k-1$ basis function $\lambda_p^{(k-1)}$ with supporting point $p \in \mathcal{P}_{k-1}^\circ \setminus \mathcal{P}_{k-1}^{\circ,\mathrm{reg}}$ having a level k active neighbour $q \in \mathcal{P}_k^\bullet$ on the left. Figures 3.1 (ii) and (iii) display the basis functions $T_k\lambda_p^{(k-1)}$ and $\lambda_p^{(k)}$ selected in (3.5) and (3.7), respectively. Note that $T_k\lambda_p^{(k-1)}$ generally results in a "nonsymmetric" truncation while the choice of the higher level basis function $\lambda_p^{(k)}$ may be regarded as a "symmetric" cut.

As proposed in [14] the hierarchical basis preconditioners obtained from (3.5) and (3.7) will be treated in the framework of additive Schwarz methods. For recent results on the BPX preconditioner as an additive Schwarz method we refer to Bornemann [6] and Zhang [39]. As far as the following definitions and assertions do not differ for $\Lambda_k^{(\mu)}$, $\mu = 1, 2$ the index μ is skipped for notational convenience.

The direct subspace decomposition

$$S_j^\circ = V_0 \oplus \bigoplus_{\lambda \in \Lambda_H} V_\lambda \ , \tag{3.8}$$

of S_j° where $V_0 = \operatorname{span} \Lambda_0$ and $V_\lambda = \operatorname{span}\{\lambda\}$, $\lambda \in \Lambda_H = \bigcup_{k=1}^{j} \Lambda_k$, gives rise to an additive Schwarz method providing a reformulation

$$P u_j^\circ = \ell' \ ,$$

of the original problem (3.1) where

$$P = P_0 + \sum_{\lambda \in \Lambda_H} P_\lambda$$

is the sum of the Ritz projections $P_0 : \mathcal{S}_j^\circ \to V_0$, $P_\lambda : \mathcal{S}_j^\circ \to V_\lambda$, $\lambda \in \Lambda_H$, defined by

$$a(P_\nu w, v) = a(w, v) \ , \quad v \in V_\nu \ , \quad \nu = 0, \lambda \ ,$$

for each $w \in \mathcal{S}_j^\circ$ and $\ell' \in (\mathcal{S}_j^\circ)'$ is chosen appropriately. Denoting by $(\cdot,\cdot)$ the standard L^2 inner product we introduce the L^2 projections $Q_0 : \mathcal{S}_j^\circ \to V_0$, $Q_\lambda : \mathcal{S}_j^\circ \to V_\lambda$ and the representation operators $A_0 : V_0 \to V_0$, $A_\lambda : V_\lambda \to V_\lambda$, $\lambda \in \Lambda_H$ defined by

$$(Q_\nu w, v) = (w, v) \ , \ v \in V_\nu$$

for each $w \in \mathcal{S}_j^\circ$ and

$$(A_\nu w, v) = a(w, v) \ , \quad v \in V_\nu \ ,$$

for each $w \in V_\nu$, $\nu = 0, \lambda$. Then the operator P may be rewritten as

$$P = H_j A_j$$

where H_j stands for the preconditioner

$$H_j = A_0^{-1} Q_0 + \sum_{\lambda \in \Lambda_H} A_\lambda^{-1} Q_\lambda \ ,$$

and A_j is the representation operator of $a(\cdot,\cdot)$ on $\mathcal{S}_j^\circ \times \mathcal{S}_j^\circ$. Evaluation of $A_\lambda^{-1} Q_\lambda$ leads to

$$H_j^{(\mu)} = (A_0^{(\mu)})^{-1} Q_0^{(\mu)} + \sum_{\lambda \in \Lambda_H^{(\mu)}} \frac{(\cdot, \lambda)}{a(\lambda, \lambda)} \lambda \ , \quad \mu = 1, 2 \ . \tag{3.9}$$

In view of Remark 3.1 we will refer to $H_j^{(1)}$ and its variants as the "non-symmetric" preconditioners and to $H_j^{(2)}$ as the "symmetric" preconditioner, respectively.

Let us briefly discuss some modifications of the preconditioners $H_j^{(\mu)}$, $\mu = 1, 2$. The evaluation of $(A_0^{(\mu)})^{-1}Q_0^{(\mu)}$ requires the solution of a linear system for the stiffness matrix given by $a(\cdot, \cdot)$ restricted to $V_0^{(\mu)} \times V_0^{(\mu)}$, $\mu = 1, 2$. Due to the definition (3.5) the entries of $A_0^{(1)}$ and $a(\lambda, \lambda)$, $\lambda \in \Lambda_H^{(1)}$ may change with each step of the outer iteration. To avoid the corresponding evaluations of the quadratic form $a(\cdot, \cdot)$ the preconditioner $H_j^{(1)}$ may be replaced by

$$\tilde{H}_j^{(1)} = T_j \tilde{A}_0^{-1} \tilde{Q}_0 + \sum_{\lambda \in \hat{\Lambda}_H} \frac{(\cdot, T_j \lambda)}{a(\lambda, \lambda)} T_j \lambda \,, \tag{3.10}$$

where $\tilde{A}_0$ is the representation of $a(\cdot, \cdot)$ restricted to $\mathcal{S}_0^\circ \times \mathcal{S}_0^\circ$ and $\tilde{Q}_0$ denotes the L^2 projection to $\mathcal{S}_0^\circ$, respectively. Note that a related modification of $H_j^{(2)}$ is not necessary as only the selection and not the shape of the involved hierarchical basis functions is depending on the actual active set $\mathcal{P}_j^\bullet$. Still the linear system on the coarsest level is supposed to change with each outer iteration step, each time causing a Cholesky decomposition of the new coefficient matrix. To reduce the computational effort, we may replace the matrix by its diagonal or even by the identity matrix (see [36] for a further discussion). In the case of rapidly varying coefficients, frequently occurring in practical problems, the jumps should be incorporated in the preconditioners. We refer to Yserentant [37] for details.

Note that existing implementations of the standard hierarchical basis preconditioner are easily changed to (3.10) by simply neglecting the contributions from active points [20]. For a similar application of truncated hierarchical basis functions to obstacle problems we refer to [34].

The final part of this chapter will provide condition number estimates both for the nonsymmetric and for the symmetric case. The subsequent analysis will be guided by the following lemma on abstract additive Schwarz methods.

Lemma 3.1 *i) Assume that for all $v \in \mathcal{S}_j^\circ$ there is a splitting $v = v_0 + \sum_{\lambda \in \Lambda_H} v_\lambda$ such that*

$$c\{a(v_0, v_0) + \sum_{\lambda \in \Lambda_H} a(v_\lambda, v_\lambda)\} \leq a(v, v). \tag{3.11}$$

holds for some fixed positive constant c. Then we have the estimate

$$c\,a(v, v) \leq a(Pv, v)\ , \quad v \in \mathcal{S}_j^\circ .$$

ii) Assume that for all splittings $v = v_0 + \sum_{\lambda \in \Lambda_H} v_\lambda$ of $v \in \mathcal{S}_j^\circ$ the estimate

$$a(v, v) \leq C\{a(v_0, v_0) + \sum_{\lambda \in \Lambda_H} a(v_\lambda, v_\lambda)\} \tag{3.12}$$

holds for some fixed positive constant C. Then we have the estimate

$$a(Pv,v) \le Ca(v,v)\ , \quad v \in \mathcal{S}_j^\circ.$$

.

Proof. The assertion i) is the well–known lemma of P.L. Lions [27]. To prove the second assertion we apply (3.12) to the splitting $Pv = P_0 v + \sum_{\lambda \in \Lambda_H} P_\lambda v$ for some fixed $v \in \mathcal{S}_j^\circ$ to obtain

$$a(Pv,Pv) \le C\{a(P_0 v, P_0 v) + \sum_{\lambda \in \Lambda_H} a(P_\lambda v, P_\lambda v)\} = Ca(Pv,v)$$

which completes the proof. ■

Remark 3.2 The assumptions (3.11) and (3.12) can be regarded as an asymptotic orthogonality of the subspaces V_0, V_λ. Note that (3.12) is frequently established by a strengthened Cauchy–Schwarz inequality with respect to $a(\cdot,\cdot)$ or any other spectrally equivalent quadratical form.

In addition to the usual (semi) norms $\|\cdot\|_0$ and$|\cdot|_1$ of $L^2(\Omega)$ and $H^1(\Omega)$ we will make use of the semi–inner product

$$(v,w)_{1,\Omega_0} = \sum_{i=1}^{2} \int_{\Omega_0} \partial_i v \partial_i w \, dx\ , \quad v,w \in H^1(\Omega_0)$$

for measurable $\Omega_0 \subset \Omega$ with the induced semi–norm $|v|_{1,\Omega_0} = (v,v)_{1,\Omega_0}^{1/2}$. We introduce the interpolation operators $I_k : \mathcal{S}_j \to \mathcal{S}_k$ by

$$I_k v = \sum_{p \in \mathcal{P}_k} v(p)\lambda_p^{(k)}\ , \quad 0 \le k \le j.$$

and call $p,q \in \mathcal{P}_k$ k–neighbors if there is an edge $e = (p,q) \in \mathcal{E}_k$. Finally, constants depending only on the ellipticity (1.3) and the shape regularity of $\mathcal{T}_0$ will be denoted by c or C. Other parameters will be indicated explicitly.

We take up the analysis of the preconditioners with the following technical lemma

Lemma 3.2 *Let $p \in \mathcal{P}_k^\circ$ have a k–neighbor $q \in \mathcal{P}_k^\bullet$. Then the estimate*

$$|v(p)\lambda_p^{(k)}|_1 \le c|I_k v|_{1,\mathrm{supp}\lambda_p^{(k)}}$$

holds for all $v \in \mathcal{S}_j^\circ$.

Proof. Let $t_{p,q} \in \mathcal{T}_k$ be a triangle with vertices p and q. Then from the assumptions on p, q we have

$$|I_k v|_{1,t_{p,q}} = 0 \Leftrightarrow I_k v|_{t_{p,q}} = 0 \ , \quad v \in \mathcal{S}_j^{\circ}.$$

Now the assertion follows by exploiting the equivalence of norms on the reference triangle. ■

The following assumption on the splitting $\mathcal{P}_j = \mathcal{P}_j^{\bullet} \cup \mathcal{P}_j^{\circ}$ will be crucial for the analysis of the nonsymmetric preconditioners $H_j^{(1)}$ and $\tilde{H}_j^{(1)}$:

(R) There is a nonnegative constant k_0 independent of k such that

$$T_j v = T_{k+k_0} v \ , \quad v \in \mathcal{S}_k^{\circ} \ , \quad j \geq k + k_0. \tag{3.13}$$

Remark 3.3 The condition (R) states that subsequent truncation of level k functions uniformly becomes stationary after k_0 steps. We can expect from heuristic arguments that (R) is satisfied with k_0 independent of j by all splittings arising from the outer iteration, if the free boundary is a lower dimensional manifold which is approximated properly by the finite element discretization and the underlying active set strategy.

The condition (R) will be typically applied as in the proof of the following lemma.

Lemma 3.3 *Assume that* (R) *is satisfied. Then there exist constants* $c(k_0)$, $C(k_0)$ *with the property*

$$c(k_0) a(\lambda, \lambda) \leq a(T_j \lambda, T_j \lambda) \leq C(k_0) a(\lambda, \lambda) \ , \quad \lambda \in \hat{\Lambda}_H. \tag{3.14}$$

Proof. Let $1 \leq k \leq j-1$. Then it follows by the usual affine transformation technique that

$$c|\lambda|_{1,t} \leq |T_{k+1}\lambda|_{1,t} \leq C|\lambda|_{1,t} \tag{3.15}$$

holds for all $\lambda \in \hat{\Lambda}_k$ and $t \in \mathcal{T}_k$. Now apply (R) to obtain the factorization

$$T_j = T_j T_{j-1} \dots T_{k+1} = T_{k+\bar{k}} \dots T_{k+1}$$

with $\bar{k} = \min(k_0, j-k)$. Together with (3.15) we have

$$c^{\bar{k}} |\lambda|_{1,t} \leq |T_j \lambda|_{1,t} \leq C^{\bar{k}} |\lambda|_{1,t} \tag{3.16}$$

for all $\lambda \in \hat{\Lambda}_k$ and $t \in \mathcal{T}_k$. Now the assertion follows from the ellipticity of $a(\cdot,\cdot)$. ■

We are ready to establish lower and upper bounds for the nonsymmetric preconditioners $H_j = H_j^{(1)}, \tilde{H}_j^{(1)}$.

Theorem 3.1 *Assume that the regularity condition* (R) *holds. Then there exist constants* K_0*,* K_1 *depending only on* α_0*,* α_1 *in* (1.3)*, the shape regularity of* $\mathcal{T}_0$ *and the constant* k_0 *in* (R) *such that the estimate*

$$K_0(j+1)^{-2}a(v,v) \le a(H_jA_jv,v) \le K_1 a(v,v)\ , \quad H_j = H_j^{(1)}, \tilde{H}_j^{(1)}$$

holds for all $v \in \mathcal{S}_j^\circ$*.*

Proof. Let us first consider the case $H_j = H_j^{(1)}$. To verify the assumption of Lemma 3.1 i) we consider the splitting

$$v = v_0 + \sum_{\lambda\in\Lambda_H^{(1)}} v_\lambda\ , \quad v_0 \in V_0^{(1)}, v_\lambda \in V_\lambda^{(1)} \tag{3.17}$$

of some fixed $v \in \mathcal{S}_j^\circ$. As (3.8) provides a direct splitting of $\mathcal{S}_j^\circ$, this representation is unique. We can find $\hat{v}_0 \in \hat{V}_0$, $v_\lambda \in \hat{V}_\lambda$ with the property

$$v_0 = T_j\hat{v}_0\ , \quad v_\lambda = T_j\hat{v}_\lambda\ , \quad \lambda \in \hat{\Lambda}_H \tag{3.18}$$

and define $\hat{v} \in \mathcal{S}_j$ by

$$\hat{v} = \hat{v}_0 + \sum_{\lambda\in\hat{\Lambda}_H} \hat{v}_\lambda. \tag{3.19}$$

Note that in general $\hat{v} \notin \mathcal{S}_j^\circ$. By the arguments applied in the proof of Lemma 3.3, it follows from (R) that

$$|\hat{v}|_1 \le c(k_0)|T_j\hat{v}|_1 = c(k_0)|v|_1. \tag{3.20}$$

Coming back to the assumption of Lemma 3.1 i), we have from Lemma 3.3 and the equivalence of norms on $\mathcal{S}_j$ that

$$a(v_0,v_0) + \sum_{\lambda\in\Lambda_H^{(1)}} a(v_\lambda,v_\lambda) \le c(k_0)\{a(\hat{v}_0,\hat{v}_0) + \sum_{\lambda\in\hat{\Lambda}_H} a(\hat{v}_\lambda,\hat{v}_\lambda)\}.$$

Hence, in view of (3.20) the lower bound follows from

$$a(\hat{v}_0,\hat{v}_0) + \sum_{\lambda\in\hat{\Lambda}_H} a(\hat{v}_\lambda,\hat{v}_\lambda) \le C(j+1)^2 a(\hat{v},\hat{v}). \tag{3.21}$$

Assume for the moment that $\hat{V}_0 \ne \emptyset$. Then we only have to collect the well–known results of Yserentant [37] to show

$$\begin{aligned}\sum_{\lambda\in\hat{\Lambda}_H} a(\hat{v}_\lambda,\hat{v}_\lambda) &\le \alpha_1 \sum_{\lambda\in\hat{\Lambda}_H} |\hat{v}_\lambda|_1^2 \le c\sum_{k=1}^{j} 4^k \sum_{\lambda\in\hat{\Lambda}_k} v(\lambda)\|\lambda\|_0^2 \\ &\le 2c\sum_{k=1}^{j} 4^k\|(I_k - I_{k-1})\hat{v}\|_0^2 \le C(j+1)^2|\hat{v}|_1^2\end{aligned}$$

where $v(\lambda)$ denotes the value of v at the supporting point of λ. In particular, we have employed an inverse inequality (Lemma 3.3 of [37]), the spectral equivalence of the incorporated quadrature rule with the L^2–norm and the approximation of unity by the interpolation operators I_k (Theorem 3.2 of [37]). The remaining estimate

$$a(\hat{v}_0, \hat{v}_0) = a(I_0\hat{v}, I_0\hat{v}) \leq C(j+1)|\hat{v}|_1^2$$

easily follows from the stability of the interpolation (Theorem 3.1 of [37]).

We still have to consider the case

$$\hat{\Lambda}_i \neq \emptyset \ , \ \ \hat{\Lambda}_{i-1} = \emptyset \tag{3.22}$$

for some $i > 0$. Changing the initial level from 0 to i the assertion (3.21) is immediately obtained from

$$\sum_{\lambda \in \hat{\Lambda}_i} |\hat{v}_\lambda|_1^2 \leq C|I_i\hat{v}|_1^2 \tag{3.23}$$

and the stability of the interpolation cited above. As a consequence of (3.22) every point $p \in \mathcal{P}_i^\circ$ has an i–neighbor $q \in \mathcal{P}_i^\bullet$ so that (3.23) follows from Lemma 3.2. This completes the proof of the lower bound of $a(H_jA_jv, v)$.

To prove an upper bound by Lemma 3.1 ii), it is sufficient to show that

$$a(v,v) \leq K_1\{a(v_0, v_0) + \sum_{\lambda \in \Lambda_H^{(1)}} a(v_\lambda, v_\lambda)\} \tag{3.24}$$

holds for the splitting (3.17) of some fixed $v \in \mathcal{S}_j^\circ$. Recall that the splitting is unique. Using the arguments of the proof of Lemma 2.4 in [36] and the proof of Lemma 3.3 we can show that

$$c(k_0) \sum_{p \in \mathcal{P}_k \cap t} |w(p)|^2 \leq |T_l w|_{1,t}^2 \leq C(k_0) \sum_{p \in \mathcal{P}_k \cap t} |w(p)|^2 \ , \ \ t \in \mathcal{T}_k, \tag{3.25}$$

for all $w \in \operatorname{span}\{\hat{\Lambda}_k\}$. Based on this norm equivalence, we can extend the proof of the strengthened Cauchy–Schwarz inequality in [36] to truncated functions, giving

$$(w_l, w_k)_1 \leq c(k_0) \left(\frac{1}{\sqrt{2}}\right)^{|l-k|-k_0} |w_l|_1 |w_k|_1 \tag{3.26}$$

for all $w_l \in \operatorname{span}\{\Lambda_l^{(1)}\}$, $w_k \in \operatorname{span}\{\Lambda_k^{(1)}\}$ and $|l-k| \geq k_0$. From (3.26) the estimate

$$|v|_1^2 \leq C(k_0)\{|v_0|_1^2 + \sum_{\lambda \in \Lambda_H^{(1)}} |v_\lambda|_1^2\} \tag{3.27}$$

can be derived by well–known arguments. Finally, (3.24) is an immediate consequence of (3.27) and the ellipticity of $a(\cdot,\cdot)$.

By Lemma 3.3 and the equivalence of norms on $\mathcal{S}_0$ it is obvious that the preconditioner $\tilde{H}_j^{(1)}$ is just a spectrally equivalent modification of $H_j^{(1)}$. This completes the proof of the theorem. ∎

For the symmetric preconditioner $H_j^{(2)}$ we can state a related result without any regularity assumptions imposed on the active set.

Theorem 3.2 *There exist constants K_0, K_1 depending only on α_0, α_1 in* (1.3) *and the shape regularity of $\mathcal{T}_0$ such that the estimate*

$$K_0(j+1)^{-2}a(v,v) \le a(H_j^{(2)}A_jv,v) \le K_1 a(v,v)$$

holds for all $v \in \mathcal{S}_j^\circ$.

Proof. Let $v \in \mathcal{S}_j^\circ$. Based on the unique splitting

$$v = v_0 + \sum_{\lambda\in\Lambda^{(2)}} v_\lambda\ ,\quad v_0 \in V_0^{(2)},\ v_\lambda \in V_\lambda^{(2)},$$

we can follow the arguments in the proof of Theorem 3.1 with the important difference that the corresponding results on hierarchical bases can be applied directly. We only have to take care of the case

$$\Lambda_i^{(2)} \neq \emptyset\ ,\quad \Lambda_{i-1}^{(2)} = \emptyset \tag{3.28}$$

for some $i > 0$. But as in the case (3.22) we find at least one i–neighbor $q \in \mathcal{P}_i^\bullet$ for each point $p \in \mathcal{P}_i^{\circ,\mathrm{reg}}$ so that Lemma 3.2 can be applied. ∎

Remark 3.4 Recall that the construction of the preconditioners is independent of the construction of the disjoint splitting $\mathcal{P}_j = \mathcal{P}_j^\circ \cup \mathcal{P}_j^\bullet$. In particular, if we are solving an unconstrained elliptic problem, we can define the active set $\mathcal{P}_j^\bullet$ as the set of all nodes on which the iterative error is considered small enough. A corresponding strategy has been proposed in [33]. In this case we can not expect k_0 in condition (R) to be uniformly bounded (c.f. Remark 3.3) so that only the symmetric preconditioner should be used.

Remark 3.5 In the proofs of Theorems 3.1 and 3.2 we have extended well–known results on hierarchical bases from the unconstrained to the constrained case by suitable properties of the truncation operators T_j or the restriction of the active set $\mathcal{P}_j^\bullet$. The same technique can be applied to other multilevel additive Schwarz methods as for example the BPX preconditioner to obtain related results in three space dimensions [7].

Theorems 3.1 and 3.2 show that under reasonable assumptions, all preconditioners under consideration are spectrally equivalent. Still in the nonsymmetric case the actual constants depend heavily on the constant k_0 , while the behaviour of the symmetric preconditioner $H_j^{(2)}$ is expected to be more robust with respect to the choice of $\mathcal{P}_j^\bullet$. This theoretical reasoning will be supported by the numerical results presented in Chapter 5.

4. Semi–Local and Local Error Estimates

Let $u \in H_0^1(\Omega)$ denote the exact solution of (1.2) and $u_j \in \mathcal{S}_j$ the exact solution of the approximate problem (2.1) with respect to $\mathcal{T} = \mathcal{T}_j$. Expecting that only an approximation $\tilde{u}_j \in \mathcal{S}_j$ of u_j is known in actual computations, we are interested in a–posteriori error estimates $\tilde{\varepsilon}$ for the total error ε,

$$\varepsilon = \|u - \tilde{u}_j\| := a(u - \tilde{u}_j, u - \tilde{u}_j)^{1/2}$$

which are efficient and reliable in the sense that

$$\gamma_0 \tilde{\varepsilon} \leq \|u - \tilde{u}_j\| \leq \gamma_1 \tilde{\varepsilon} \tag{4.1}$$

holds with positive coefficients γ_0, γ_1 depending only moderately on the refinement level j. The local contributions to $\tilde{\varepsilon}$ will be used as local error indicators in the adaptive refinement process. This concept of adaptivity is well established for linear elliptic equations and has been used by a variety of authors. See [3, 13, 21, 26, 35] for further references. Extending the approach of Deuflhard, Leinen and Yserentant [13, 26] to obstacle problems, we will proceed in two main steps:

- Step 1: Replace the exact solution u in (4.1) by the piecewise quadratic approximation $U_j \in H_0^1(\Omega)$.
- Step 2: Localize the computation of U_j to obtain $\tilde{U}_j$ with $\tilde{\varepsilon} := |\tilde{U}_j - \tilde{u}_j|$ satisfying (4.1).

The first step is settled by the following lemma which is a consequence of the triangle inequality.

Lemma 4.1 *Assume that the piecewise quadratic approximation U_j is of higher accuracy in the sense that*

$$\|u - U_j\| \leq q\|u - u_j\| \ , \quad 0 \leq q < 1 \ , \quad j = 0, 1, \ldots \tag{4.2}$$

and $\tilde{u}_j \in \mathcal{S}_j$ satisfies

$$\|u - u_j\| \leq \sigma \|u - \tilde{u}_j\| \ , \quad j = 0, 1, \ldots \tag{4.3}$$

with $q\sigma < 1$ and q, σ not depending on j. If $\tilde{\varepsilon}$ satisfies

$$\tilde{\gamma}_0 \tilde{\varepsilon} \leq \|\tilde{u}_j - U_j\| \leq \tilde{\gamma}_1 \tilde{\varepsilon} \tag{4.4}$$

then (4.1) *holds with $\gamma_0 = \tilde{\gamma}_0/(1 + q\sigma)$ and $\gamma_1 = \tilde{\gamma}_1/(1 - q\sigma)$.*

Remark 4.1 Note that (4.3) holds with $\sigma = 1$ if no obstacle is present. In general, (4.3) follows from

$$\|u_j - \tilde{u}_j\| \leq (1 - 1/\sigma)\|u - u_j\|.$$

which may be regarded as an accuracy assumption on $\tilde{u}_j$. Recall that for sufficiently smooth data the piecewise quadratic approximation is of higher order than piecewise linear elements (c.f. [12]). In this case (4.2) is trivial, if the initial triangulation $\mathcal{T}_0$ is chosen fine enough.

In the sequel we assume that the assumptions of Lemma 4.1 are satisfied to concentrate on the derivation of $\tilde{\varepsilon}$ with the property (4.4).

Let $\mathcal{Q}_j \subset H_0^1(\Omega)$ denote the subspace of piecewise quadratic functions on $\mathcal{T}_j$ vanishing at the boundary and

$$K_j^{\mathcal{Q}} = \left\{ v \in \mathcal{Q}_j | \ v(p) \leq \varphi^L(p), p \in \mathcal{P}_j, \ v(e) \leq \varphi^Q(e), e \in \mathcal{E}_j \right\}$$

the corresponding approximation of the constraints K. Here we used $v(e) := v(\text{midpoint of } e), \ e \in \mathcal{E}_j$, for functions $v : \Omega \to \mathbb{R}$ and suitable restrictions φ^L, φ^Q of the obstacle φ to $\mathcal{P}_j$ and $\mathcal{E}_j$, respectively. Now U_j can be computed from

$$\text{Find } U_j \in K_j^{\mathcal{Q}} \text{ such that } \quad a(U_j, U_j - v) \leq \ell(U_j - v) \ , \quad v \in K_j^{\mathcal{Q}}. \tag{4.5}$$

For notational convenience the index j will be suppressed in the following notations. In view of Lemma 4.1 we are interested in the defect $d = U_j - \tilde{u}_j \in \mathcal{Q}_j$ which is the unique solution of

$$\text{Find } d \in D \text{ such that } \quad a(d, d - v) \leq r(d - v) \ , \quad v \in D. \tag{4.6}$$

The constraints are given by

$$D = D(\tilde{u}_j) := \{ v \in \mathcal{Q}_j | \ v + \tilde{u}_j \in K_j^{\mathcal{Q}} \}$$

and the right–hand side is the residual $r := \ell - a(\tilde{u}_j, \cdot)$.

As d is not available at reasonable computational cost the remainder of this chapter will be devoted to the localization of the defect problem (4.6). A possible way is indicated in the next lemma, showing that (4.1) is preserved by spectrally equivalent modifications of $a(\cdot, \cdot)$.

Lemma 4.2 *Let $\tilde{d}$ be the solution of*

$$\text{Find } \tilde{d} \in D \text{ such that } \quad \tilde{a}(\tilde{d}, \tilde{d} - v) \leq r(\tilde{d} - v) \ , \quad v \in D \tag{4.7}$$

with a symmetric form $\tilde{a}(\cdot, \cdot)$ satisfying

$$c_0 \tilde{a}(v, v) \leq a(v, v) \leq c_1 \tilde{a}(v, v) \ , \quad v \in \mathcal{Q}_j. \tag{4.8}$$

with positive constants c_0, c_1. Then

$$C_0 \tilde{a}(\tilde{d}, \tilde{d}) \leq a(d, d) \leq C_1 \tilde{a}(\tilde{d}, \tilde{d}) \tag{4.9}$$

holds with $C_0 = \left(c_0^{-1} + 2c_1(1 + c_0^{-1})\right)^{-1}, \ C_1 = c_1 + 2c_0^{-1}(1 + c_1)$.

Proof. By symmetry arguments it is sufficient to establish the right inequality in (4.9). Together with (4.8) we obtain from (4.6) that

$$a(d,d) \le c_1 \tilde{a}(\tilde{d},\tilde{d}) + 2r(d-\tilde{d}).$$

Now the assertion follows from

$$r(d-\tilde{d}) \le c_0^{-1}(1+c_1)\tilde{a}(\tilde{d},\tilde{d}). \tag{4.10}$$

To show (4.10) observe that the choice $v = d$ in (4.7) leads to

$$r(d-\tilde{d}) \le \tilde{a}(\tilde{d}, d-\tilde{d}). \tag{4.11}$$

Hence, in view of Cauchy's inequality it remains to prove

$$|d-\tilde{d}|_{\tilde{a}} \le c_0^{-1}(1+c_1)|\tilde{d}|_{\tilde{a}} \tag{4.12}$$

with $|\cdot|_{\tilde{a}}$ denoting the energy–norm induced by $\tilde{a}(\cdot,\cdot)$. It is obvious that $\tilde{d}$ is the solution of the original problem (4.6) with r replaced by a modified right–hand side $\tilde{r}$ defined by

$$\tilde{r} := r + a(\tilde{d},\cdot) - \tilde{a}(\tilde{d},\cdot).$$

As the solution of variational inequalities depends Lipschitz–continuously on the right–hand side with Lipschitz constant c_0^{-1} (c.f.[23]), we obtain (4.12) from

$$|d-\tilde{d}|_{\tilde{a}} \le c_0^{-1} \sup_{|v|_{\tilde{a}}=1} |a(\tilde{d},v) - \tilde{a}(\tilde{d},v)| \le c_0^{-1}(1+c_1)|\tilde{d}|_{\tilde{a}}$$

This completes the proof. ■

Note that Lemma 4.2 is valid for arbitrary convex constraints and arbitrary space dimensions.

To construct suitable quadratic forms $\tilde{a}(\cdot,\cdot)$ we introduce the two–level splitting

$$\mathcal{Q}_j = \mathcal{S}^L \oplus \mathcal{S}^Q \tag{4.13}$$

consisting of the linear part $\mathcal{S}^L = \mathcal{S}_j$ and the remaining quadratic part $\mathcal{S}^Q$. Note that the quadratic bubbles $\mu_e \in \mathcal{Q}_j,\ e \in \mathcal{E}_j$ defined by

$$\mu_e(p) = 0\ ,\ \ p \in \mathcal{P}_j\ ,\ \ \mu_e(\text{midpoint of } g) = \delta_{e,g}\ ,\ \ g \in \mathcal{E}_j$$

form a basis of $\mathcal{S}^Q$. Following (4.13) we split $v \in \mathcal{Q}_j$ according to

$$\begin{aligned} &v = v^L + v^Q \\ &v^L \in \mathcal{S}^L\ ,\ \ v^Q = \textstyle\sum_{e\in\mathcal{E}_j} v_e\mu_e\ \in \mathcal{S}^Q \end{aligned} \tag{4.14}$$

Then we obtain the quadratic form $b(\cdot,\cdot)$,

$$b(v,w) = a(v^L,w^L) + a^Q(v^Q,w^Q)\ ,\quad a^Q(v^Q,w^Q) := \sum_{e\in\mathcal{E}_j} v_e w_e a(\mu_e,\mu_e) \quad (4.15)$$

by neglecting the coupling of $\mathcal{S}^L$, $\mathcal{S}^Q$ and $\mu_e,\mu_g, e\neq g$, respectively. Using additionally the preconditioner $\hat{a}(\cdot,\cdot)$ resulting from the standard hierarchical basis decomposition of $\mathcal{S}^L = \mathcal{S}_j$, we end up with

$$\hat{b}(v,w) = \hat{a}(v^L,w^L) + a^Q(v^Q,w^Q) \quad (4.16)$$

Summarising these results we obtain the first important result of this chapter.

Theorem 4.1 *Assume that condition* (Q) *is satisfied. Let $\hat{d}$ be the solution of the semi–local problem*

$$\text{Find } \hat{d}\in D \ \ \textit{such that} \ \ \hat{b}(\hat{d},\hat{d}-v) \le r(\hat{d}-v)\ ,\ \ v\in D. \quad (4.17)$$

Then (4.1) *holds for*

$$\tilde{\varepsilon} = |\hat{d}|_{\hat{b}}$$

and $\gamma_0 = \hat{\gamma}_0/(j+1)$, $\gamma_1 = \hat{\gamma}_1$. Here $\hat{\gamma}_0,\hat{\gamma}_1$ are depending only on the shape regularity and the local ellipticity of $a(\cdot,\cdot)$.

Proof. Theorem 4.1 is an immediate consequence of the Lemmas 4.1 and 4.2 together with the Lemma on p. 14 in [13]. ∎

Remark 4.2 The error estimate (4.17) is called semi–local, because the frequencies of $\hat{d}$ are decoupled with respect to the quadratic form but coupled by the set of constraints D. Of course (4.17) reduces to the error estimate proposed in [13], if the obstacle is not active.

Remark 4.3 The simplified defect problem (4.17) may be solved approximately using the active set strategy described above. As the preconditioners proposed in the preceding chapter are just truncated versions of $\hat{a}(\cdot,\cdot)$, we can expect the corresponding linear subproblems to be solved very efficiently. Note that by Theorem 3.1 or 3.2 any preconditioned cg–iterate of the linear subproblem satisfies (4.1) as soon as the active set is determined correctly.

To derive a less robust but local error estimate we consider the simplified defect problem

$$\text{Find } \tilde{d}\in D \text{ such that} \quad b(\tilde{d},\tilde{d}-v) \le r(\tilde{d}-v)\ ,\quad v\in D. \quad (4.18)$$

Recall that

$$c_0 b(v,v) \le a(v,v) \le c_1 b(v,v)\ ,\quad v\in\mathcal{Q}_j \quad (4.19)$$

with positive constants c_0, c_1 independent of j (c.f. [13]). According to (4.2) the solution $\tilde{d}$ of (4.18) provides an error estimate with the property (4.1). Now (4.18) is decoupled by one block Gauss–Seidel iteration step applied to the initial iterate zero, i.e., we compute an estimate $\delta = \delta^L + \delta^Q$ from

$$\text{Find } \delta^L \in D^L \text{ such that } a(\delta^L, \delta^L - v) \le r^L(\delta^L - v)\ , \quad v \in D^L \tag{4.20}$$

and

$$\begin{aligned}&\text{Find } \delta^Q \in D^Q(\delta^L) \text{ such that}\\ &\qquad a^Q(\delta^Q, \delta^Q - v) \le r^Q(\delta^Q - v)\ , \quad v \in D^Q(\delta^L)\end{aligned} \tag{4.21}$$

where $r^L,\ r^Q$ denote the restriction of r to $\mathcal{S}^L, \mathcal{S}^Q$ and $D^L,\ D^Q(\delta^L)$ are defined by

$$D^L = \mathcal{S}^L \cap D\ , \quad D^Q(w^L) = \{v^Q \in \mathcal{S}^Q |\ v^Q + w^L \in D\}\ , \quad w^L \in \mathcal{S}^L$$

Note that the linear defect problem is recovered by (4.20) with the consequence $\delta^L = 0$. Moreover, each component of δ^Q can be computed separately so that

$$\tilde{\varepsilon} = |\delta^Q|_{a^Q} \tag{4.22}$$

provides a local error estimate.
Let us introduce the interpolation operator $\pi : \mathcal{S}^L \to \mathcal{S}^Q$ defined by

$$\pi(v^L)(\text{midpoint of } e) = (v^L(p_1) + v^L(p_2))/2\ , \quad e = (p_1, p_2) \in \mathcal{E}_j\ , \quad v^L \in \mathcal{S}^L$$

We now show that (4.22) provides a lower bound for the total error.

Theorem 4.2 *Assume that*

$$\text{(L)} \qquad |\pi(\tilde{d}^L)|_{a^Q} \le \beta \|\tilde{d}^L\|$$

holds with a positive constant β independent of j. Then

$$\gamma_0 |\delta^Q|_{a^Q} \le \|\tilde{u}_j - u\|$$

holds with a positive constant γ_0 depending only on β, the shape regularity of $\mathcal{T}_0$ and the local ellipticity of $a(\cdot,\cdot)$.

Proof. Obviously $\tilde{d}^Q$ is the solution of

$$\begin{aligned}&\text{Find } \tilde{d}^Q \in D^Q(\tilde{d}^L) \text{ such that}\\ &\qquad a^Q(\tilde{d}^Q, \tilde{d}^Q - v) \le r(\tilde{d}^Q - v)\ , \quad v \in D^Q(\tilde{d}^L)\end{aligned} \tag{4.23}$$

with $\tilde{d} = \tilde{d}^L + \tilde{d}^Q$. Representing (4.23) as a complementary problem it is easily verified that (4.21) and (4.23) are symmetric with respect to the obstacle and the right–hand side. More precisely (4.21) and (4.23) can be replaced by

$$\begin{aligned} &\text{Find } \delta^Q \in R \text{ such that} \\ &\quad a^Q(\delta^Q, \delta^Q - v) \leq a^Q(\varphi^Q - \pi(\tilde{u}_j), \delta^Q - v) \;, \quad v \in R \end{aligned} \tag{4.24}$$

and

$$\begin{aligned} &\text{Find } \tilde{d}^Q \in R \text{ such that} \\ &\quad a^Q(\tilde{d}^Q, \tilde{d}^Q - v) \leq a^Q(\varphi^Q - \pi(\tilde{u}_j + \tilde{d}^L), \tilde{d}^Q - v) \;, \quad v \in R \end{aligned} \tag{4.25}$$

with constraints

$$R = \{v \in \mathcal{S}^Q | \; v_e \leq r_e^Q / a(\mu_e, \mu_e), \; e \in \mathcal{E}_j\}$$

Again we use that (4.24), (4.25) are Lipschitz with respect to the right–hand side to obtain

$$|\delta^Q - \tilde{d}^Q|_{a^Q} \leq |\pi(\tilde{d}^L)|_{a^Q} \tag{4.26}$$

Now the assertion follows from the triangle inequality and the assumption (L). ∎

Remark 4.4 Obviously (L) can be replaced by

$$|\tilde{d}^L|_{\ell^2} \leq \beta' \|\tilde{d}^L\| \tag{4.27}$$

which may be regarded as a regularity condition imposed on the free boundary. Indeed, regarding $\tilde{d}^L$ as a perturbation of the decoupled solution $\delta^L = 0$ by the coupling with $\tilde{d}^Q$ at the free boundary, condition (L) requires that these perturbations remain local with increasing j. Note that if condition (L) is not satisfied, (4.21) still provides a lower bound for the error though it may deteriorate with successive refinement.

Remark 4.5 The error estimate (4.21) has been originally proposed in [24, 25] for the adaptive solution of a special obstacle problem arising in semiconductor device simulation. In this special problem we can expect from the physical data that the error is dominated uniformly in j by contributions generated away from the free boundary, suffering only minor effects from the localization (4.20), (4.21). In particular, the nonactive region can be always resolved with sufficient accuracy on the initial triangulation $\mathcal{T}_0$. Under these assumptions we can easily prove that (4.21) is reliable in the sense of (4.1), in particular that (4.21) also provides a uniform upper bound of the exact error ε.

However, simple examples show that (4.21) may deliver $\tilde{\varepsilon} = |\delta^Q|_{a^Q} = 0$ though we have $d \neq 0$, showing that an upper bound cannot be obtained from (4.21) without further assumptions such as mentioned above. Together with Theorem 4.2 this indicates that (4.21) is likely to underestimate the error, a behaviour which will be confirmed by numerical experiments reported in the next chapter.

5. Numerical Results

In this chapter we compose an adaptive Multilevel Method from the modules described above. This method is then applied to a challenging model problem confirming the properties expected from the theoretical considerations.

On each refinement level j we apply the active–set strategy given in Chapter 2 until the active set is left invariant. The iteration is started with the interpolated approximation from the previous level with the value at each node having at least one active neighbor projected to the obstacle. On the first level the obstacle function is used as initial iterate. Each step of the outer iteration requires the solution of the linear subproblem (2.8) which is performed iteratively by cg–iterations preconditioned by the reduced hierarchical basis preconditioners introduced above. This inner iteration is stopped as soon as the estimated linear iteration error κ satisfies

$$\kappa \leq \kappa_0 \ . \tag{5.1}$$

where estimate κ is computed as described in [13]. Recall that the threshold κ_0 has to be chosen small enough to ensure the convergence of the outer iteration (c.f. Remark 2.1). In the following example $\kappa_0 = 10^{-3}$ is used.

The same algorithm with κ_0 replaced by $\kappa_0' = 10^{-2}$ is applied to the solution of the semi–local defect problem providing the error estimate

$$\varepsilon^s = |\hat{d}|_{\hat{b}} \ ,$$

with $\hat{d}$ computed approximately from (4.17). A local error estimate

$$\varepsilon^l = |\delta|_b$$

is obtained by approximating (4.20) and evaluating (4.21). The iterative solution of the semi–local defect problem is started with the local estimate $(0, \delta^Q)$. Given some approximation $\theta = \hat{d}, \delta$ of the defect d, an edge $e \in \mathcal{E}_j$ is refined if its contribution η_e,

$$\eta_e = (\theta_e^Q)^2 \ a(\mu_e, \mu_e),$$

exceeds a certain threshold $\bar{\eta}$. To determine $\bar{\eta}$ we extrapolate η_e as proposed in [2] (see [24] for details). A new triangulation is constructed by red refinements and green closures referring to [3, 26, 31, 32] for details.

Now we apply the algorithm to a well-known problem describing the elasto–plastic torsion of a cylindrical bar with quadratical cross–section $\Omega = (0,1) \times (0,1)$ which is twisted at its upper end around the longitudinal axis in such a way that the lateral surface remains stress free. Modelling the plastic region according to the von Mieses yield criterion and normalizing physical constants, it has been shown in [11] that for positive twist angle C per unit length the stress potential u is the solution of the variational inequality (1.2) with $a(\cdot,\cdot)$, $\ell(\cdot)$ given by

$$a(v,w) = \int_\Omega (\frac{\partial v}{\partial x}\frac{\partial w}{\partial x} + \frac{\partial v}{\partial y}\frac{\partial w}{\partial y}) d(x,y) \ , \quad \ell(v) = 2C \int_\Omega v \ d(x,y)$$

and constraints K,

$$K = \{v \in H_0^1(\Omega) | \ v(x,y) \leq \mathrm{dist}((x,y),\partial\Omega), \ \text{ f.a.a. } (x,y) \in \Omega\}.$$

The active points characterize the plastic region while the material is considered elastic in nonactive points. We refer to [15, 17] for the numerical treatment and to [30] for a theoretical analysis of the problem.

Note that the problem has singular perturbation character with respect to the elastic region which is located along the diagonals and is shrinking for increasing C.

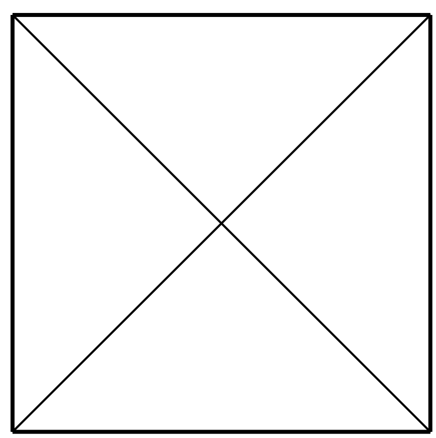

Figure 5.1 Initial Triangulation $\mathcal{T}_0$

Starting with the initial triangulation $\mathcal{T}_0$ depicted in Figure 5.1 and choosing $C = 15$, all nodal points remain active up to the 3rd (uniform) refinement level, rendering a quite challenging problem for an adaptive multilevel method.

In Table 1.1 we report the number of iterations required by the solution process. The data are presented in the form "number of outer iterations / average number of inner iterations" both needed for the solution and the semi–local error estimate, respectively. In both cases the symmetric version

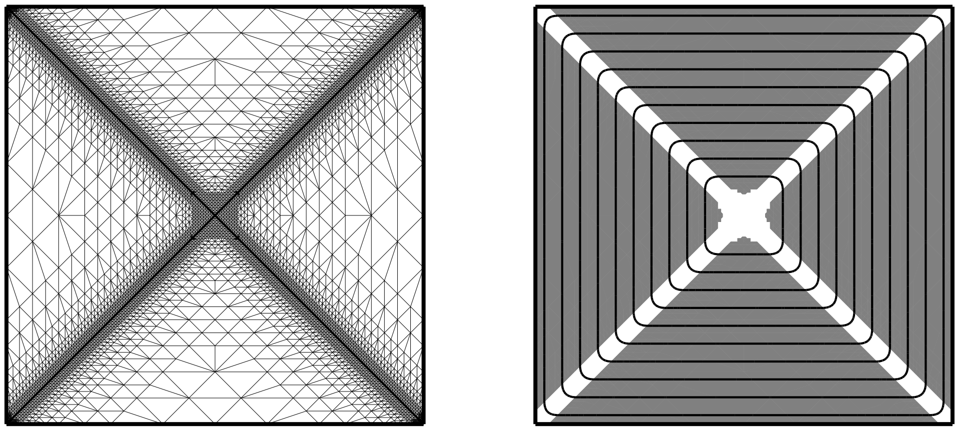

Figure 5.2 Final Triangulation $\mathcal{T}_{10}$ and Solution U_{10}

of the hierarchical basis preconditioner is used. The difficulty of detecting the elastic region leads to the difference between depth and refinement level arising from level 5 to 7. Note that $\mathcal{T}_7$ finally allows for a satisfying resolution of the elastic zone. Up to this level the computational work is dominated by the error estimation providing the local error indicators for the adaptive refinement process.

Level	Depth	Nodes	Iterations	
			Solution	Error Estimate
0	0	5	1/0.0	2/0.5
1	1	13	1/0.0	2/1.0
2	2	29	1/0.0	3/1.0
3	3	57	2/0.5	3/2.3
4	4	153	2/2.5	3/3.6
5	5	381	2/5.0	4/2.0
6	5	541	3/3.0	3/2.0
7	5	749	3/3.3	1/0.0
8	6	1605	3/4.3	2/0.0
9	7	5793	4/5.5	2/0.0
10	8	6265	3/6.0	2/0.0

Table 5.1 Iteration History

On the subsequent levels the semi–local error estimate automatically reduces to the local error estimate. Indeed, the outer iterations do not change the initial guess and may be skipped.

The final triangulation $\mathcal{T}_{10}$ is depicted in Figure 5.2 together with the level curves and the elastic region of the corresponding solution.

The behaviour of both error estimates is illustrated in more detail in Figure 5.3. Again it is obvious that the situation changes at level 7 (749 nodes), showing a significant decrease of the "exact" error and both estimates. To compute the "exact" error we performed a uniform refinement of $\mathcal{T}_{10}$ and computed the difference to the corresponding solution. Note that only the semi–local estimate provides satisfactory results on lower levels. In fact, due to the very coarse initial grid the local error estimate fails in this example providing $\varepsilon_j^l = 0$ for $j = 0, 1, 2$. Recall that the performance of both error estimates could be expected from the theoretical considerations in the preceding chapter. In particular, the local estimate (4.21) should not be used until the underlying triangulation is fine enough to detect all parts of the inactive region but works very effectively from this moment on.

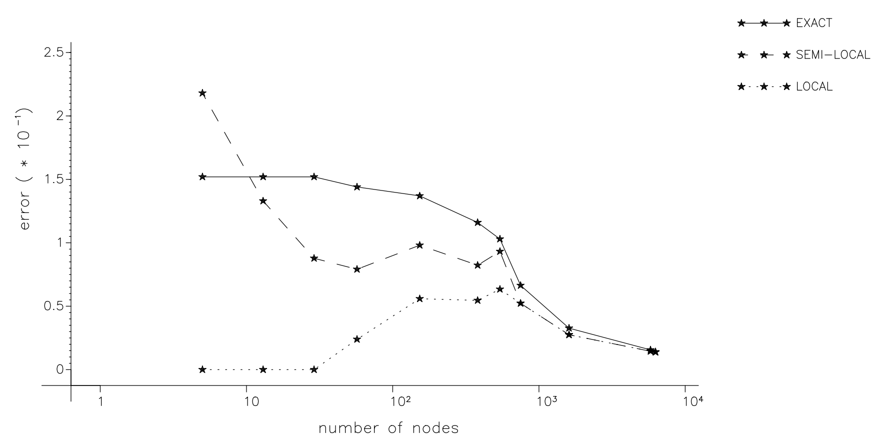

Figure 5.3 Comparison of the Error Estimates

The final Figure 5.4 gives a comparison of both versions of the hierarchical basis preconditioners. To amplify the different behaviour we choose ε_0 very small, i.e., $\varepsilon_0 = 10^{-8}$ and the initial iterate is fixed to the upper obstacle for all inner iterations. For each outer iteration we report the number of (preconditioned) cg iterations for the linear subsystem involving the maximal number of unknowns.

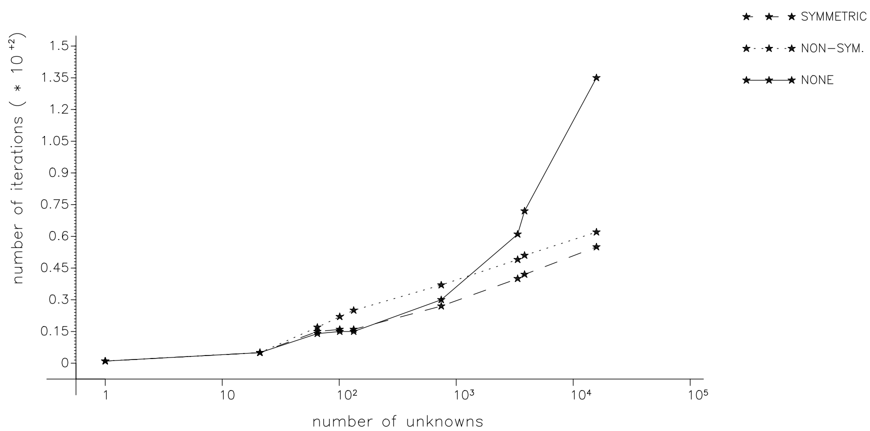

Figure 5.4 Comparison of the Preconditioners

As expected, multilevel preconditioning does not improve the convergence of the cg iteration as long as the actual problem allows no suitable representation on the coarser triangulations. Obviously, the nonsymmetric version even deteriorates the convergence until the contribution of non–truncated hierarchical basis functions becomes dominant on level 9. On the other hand the symmetric version immediately takes advantage of the good resolution on level 7 (133 unknowns) and does not deteriorate the convergence on lower levels. Note that in both cases the number of iterations becomes a linear function of the refinement level j, if j is large enough. This is exactly the behaviour predicted by the theoretical results derived in Chapter 3.

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