Heterogeneous Domain Decomposition of Surface and Porous Media Flow

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Abstract. We present a heterogeneous domain decomposition approach to the Richards equation coupled with surface water flow. Assuming piecewise constant soil parameters in the constitutive equations for saturation and relative permeability, we present a novel domain decomposition approch to the Richards equation involving on fast and robust subdomain solver based on optimization techniques. The coupling of ground and surface water is resolved by a Dirichlet–Neumann-type iteration.

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A SOLVER-FRIENDLY DISCRETIZATION OF RICHARDS EQUATION IN HOMOGENEOUS SOIL

The Richards equation [1, 2, 3]

$$n\,\theta(p)_t + \mathrm{div}\mathbf{v}(p) = 0\,, \qquad \mathbf{v}(p) = -Kkr(\theta(p))\nabla(p-z) \tag{1}$$

is a well-accepted mathematical model of the saturated–unsaturated groundwater flow in homogeneous soil. Here, p is the unknown capillary pressure on $\Omega \times (0,T)$ for a time $T > 0$ and a domain $\Omega \subset \mathbb{R}^3$ inhibited by the porous medium, n is the porosity and K stands for the hydraulic conductivity. The coordinate in the direction of gravity is denoted by z. The saturation θ, the relative permeability kr and p are related by state equations suggested, e.g., by van Genuchten [4] or Brooks and Corey [5]. To fix the ideas, we consider on the Brooks–Corey functions given by

$$\theta(p) = \begin{cases} \theta_m + (\theta_M - \theta_m)\left(\frac{p}{p_b}\right)^{-\lambda} & \text{for } p \le p_b \\ \theta_M & \text{for } p \ge p_b \end{cases}, \qquad kr(\theta) = \left(\frac{\theta - \theta_m}{\theta_M - \theta_m}\right)^{3+\frac{2}{\lambda}}, \quad \theta \in [\theta_m, \theta_M]\,. \tag{2}$$

where the minimal and maximal saturation θ_m, $\theta_M \in [0,1]$, λ, and the bubbling pressure p_b are soil parameters. Note that 1 degenerates to an elliptic problem for $p \ge p_b$, because $\theta(p) = \theta_M$ is constant in this case. This excludes explicit time stepping. It is a long-standing problem in unsaturated porous media flow simulations that "most discretization approaches for Richards' equation lead to nonlinear systems that are large and difficult to solve" [6] and that "poor iterative solver performance ... [is] often reported" [7]. Apart from the degeneracy resulting from $kr(\theta) \to 0$ this is due to the fact that the parameter functions degenerate to step functions for extreme soil parameters. On this background we suggest a discretization of (1) that allows to use arguments from convex optimization instead of linearization in the iterative finite element solution of the spatial problems.

The starting point is the reformulation

$$M(u)_t - \mathrm{div}\Big(\nabla u - kr(M(u))e_z\Big) = 0 \tag{3}$$

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of (1) in terms of a generalized pressure u obtained by Kirchhoff transformation

$$\kappa : p \mapsto u := \int_0^p kr(\theta(q))\,dq$$

and a generalized saturation $M(u) = \theta(\kappa^{-1}(u))$. Here we set $n = K = 1$, for simplicity. This transformation leads to a separation of ill-conditioning (now located in the inverse Kirchhoff transformation $p = \kappa^{-1}(u)$) and the numerical solution process of the remaining semilinear problem (3). Weak formulation and subsequent time-discretization by a lumped implicit Euler scheme of the second order terms and explicit upwinding of the first order terms together with a finite element discretization of the resulting spatial problems leads to discrete problems of the form

$$u_h \in \mathscr{S}_h : \quad \int_\Omega I_{\mathscr{S}_h}(M(u_h)v)\,dx + \int_\Omega \tau \nabla u_h \nabla v\,dx = \ell(v) \quad \forall v \in \mathscr{S}_h \tag{4}$$

to be solved in each time step. Here, $\tau > 0$ is the time step size, $\mathscr{S}_h$ denotes the space of piecewise linear finite elements with respect to a triangulation $\mathscr{T}_h$ with mesh size h, nodal interpolation $I_{\mathscr{S}_h} : C(\overline{\Omega}) \to \mathscr{S}_h$, and the functional ℓ involves the approximation from the previous time step and boundary data. We emphasize that (4) can be reformulated as a strictly convex minimization problem of the form

$$u_h \in \mathscr{S}_h : \ \ \mathscr{J}(u_h) + \phi_h(u_h) \leq \mathscr{J}(v) + \phi_h(v) \quad \forall v \in \mathscr{S}_h$$

with quadratic energy $\mathscr{J}(v) = \frac{1}{2}(\tau \nabla v, \nabla v) - \ell(v)$ and the convex, lower semicontinuous, proper functional $\phi_h(v) = \int_\Omega I_{\mathscr{S}_h}(\Phi(v))\,dx$ generated by a nonlinear convex function Φ satisfying $\Phi' = M$. We emphasize that this formulation even extends to step functions $\theta(p)$ and $kr(\theta)$ and therefore is robust with respect to the soil parameters. Moreover, multigrid solvers are available that are robust with respect to the smoothness of Φ and thus with respect to the soil parameters and provide similar efficiency as in the linear self-adjoint case [8, 9].

As we are certainly interested in approximations of the physical pressure p we conclude the approximation process by discrete inverse Kirchhoff transformation

$$p_h := I_{\mathscr{S}_h} \kappa^{-1}(u_h)\ . \tag{5}$$

We also introduce the approximation $\theta_h(p_h) := I_{\mathscr{S}_h} M(u_h)$ of the saturation θ.

Our solver-friendly discretization thus consists of the following three steps: 1) Kirchhoff transformation into generalized pressure u. 2) Finite element discretization with algebraic (multigrid) solution provides u_h. 3) Discrete inverse Kirchhoff transformation of u_h into physical pressure p_h. We now explain how this discretization can be obtained directly in terms of p_h. To this end, we use the mean-value theorem on a reference triangle to find x_T, y_T on the boundary of each triangle T such that the discrete chain rule

$$\nabla u_h = kr_T(p_h) \nabla p_h\ , \qquad kr_T(p_h) = \begin{pmatrix} kr(\theta(p_h(x_T))) & 0 \\ 0 & kr(\theta(p_h(y_T))) \end{pmatrix}$$

holds true. Hence, p_h can be equivalently obtained from the standard finite element discretization

$$p_h \in \mathscr{S}_h : \quad \int_\Omega I_{\mathscr{S}_h}(\theta_h(p_h)v)\,dx + \tau \int_\Omega kr_h(p_h) \nabla p_h \nabla v dx = \ell(v) \quad \forall v \in \mathscr{S}_h$$

with numerical integration $kr_h(p_h)| = kr_T(p_h)$ for all $T \in \mathscr{T}_h$.

We close this section with some convergence results, where we will make use of the non-degeneracy condition

$$kr(\cdot) \geq c > 0\ . \tag{6}$$

It can be achieved by suitable regularization of the corresponding Brooks–Corey function.

Theorem 1. Assume that the boundary data are sufficiently smooth and that the family of triangulations $\mathscr{T}_h$ with $h \to 0$ is shape regular. Then $u_h \to u$ in $H^1(\Omega)$ and $\theta_h \to \theta$ in $L^2(\Omega)$ Moreover, if the non-degeneracy condition (6) holds, then $M(u_h) \to M(u)$ in $H^1(\Omega)$ and $p_h \to p$ in $L^2(\Omega)$.

For a proof, we refer to Berninger et al. [9]. Numerical experiments also carried out in this paper even suggest optimal order of convergence. Theoretical justification is the subject of future research.

A MULTIDOMAIN DISCRETIZATION OF RICHARDS EQUATION IN HETEROGENEOUS SOIL

In the case of space–dependent soil parameters Richards equation takes the form

$$n\,\theta(x,p)_t + \mathrm{div}\mathbf{v}(p) = 0\,, \qquad \mathbf{v}(p) = -Kkr(x,\theta(x,p))(\nabla p - z)\;. \tag{7}$$

Assume that the soil parameters $\theta_{m,i}$, $\theta_{M,i} \in [0,1]$, λ_i, and the bubbling pressure $p_{b,i}$ are constant on subdomains Ω_i of Ω (7) can be rewritten as

$$n_i\,\theta_i(p_i)_t - \mathrm{div}\Big(K_i\,kr_i(\theta_i(p_i))\nabla(p_i - z)\Big) = 0 \quad \text{on } \Omega_i \times (0,T) \tag{8}$$

with $p_i = p|\Omega_i$ and interface conditions imposing the continuity of p and of the flux $K_i kr_i(\theta_i(p_i))$ across interior boundaries. After Kirchhoff transformation in each of the subdomains, we obtain the following multidomain version of (3)

$$n_i\,M_i(u_i)_t - \mathrm{div}\big(K_i(\nabla u_i - kr_i(M_i(u_i))e_z)\big) = 0 \quad \text{on } \Omega_i \times (0,T) \tag{9}$$

with nonlinear interface conditions

$$\kappa_i^{-1}u_i \;=\; \kappa_j^{-1}u_j \tag{10}$$

$$K_i(\nabla u_i - kr_i(M_i(u_i))e_z)\cdot\mathbf{n}_{ij} \;=\; K_j(\nabla u_j - kr_j(M_j(u_j))e_z)\cdot\mathbf{n}_{ij}. \tag{11}$$

on the subdomain boundaries $\Gamma_{ij} = \Omega_i \cap \Omega_j$. Here,

$$\kappa_i : p_i \mapsto u_i := \int_0^{p_i} kr_i(\theta_i(q))\,dq$$

is the Kirchhoff transformation and $u_i = \kappa_i(p_i)$ denotes the generalized pressure in each subdomain. Discretization in time and space along the line of the previous section leads to discrete interface problems for approximations $u_{i,h}$ of the generalized pressure. Note that these problems can be solved by nonlinear versions of well-known substructuring techniques with the fast and robust multigrid methods mentioned in the preceding section as subdomain solvers. We refer to Berninger [10] for further information. Discrete inverse Kirchhoff transformation provides the desired approximations of the physical pressure.

COUPLING WITH SURFACE WATER

Let us first assume that the surface water is non-moving with horizontal water table, uniquely determined by the height $h = h(x,t)$ of water over the surface γ over the soil. The hydrostatic pressure $p_\gamma = h\rho g$ provides a Dirichlet boundary condition for the Richards equation. For given geometry γ, the height h determines the mass $m(t)$ of surface water in the reservoir and vice versa. Denoting the outward normal to γ by $\mathbf{n}$, mass conservation

$$\frac{d}{dt}m(t) = \rho\int_\gamma \mathbf{v}(x,t)\cdot\mathbf{n}\,d\sigma(x)\,. \tag{12}$$

relates $m(t)$ to the flux $\mathbf{v}(x,t)$. Near the water table of the lake one can observe seepage faces where water can flow out (and the water pressure vanishes), whereas further away, one usually has noflow conditions (with a nonpositive water pressure). This complementarity condition is often called Signorini-type or outflow condition [11, 12]. It reads

$$p \;\le\; 0\,, \quad \mathbf{v}\cdot\mathbf{n} \;\ge\; 0\,, \quad p\cdot(\mathbf{v}\cdot\mathbf{n}) = 0 \qquad \text{on } \gamma(t)\;. \tag{13}$$

A priori, it is unknown where we have outflow and where noflow occurs. Apart from the Dirichlet boundary conditions given by the hydrostatic pressure of surface water and the Signorini-type boundary conditions one usually has Neumann boundary conditions $\mathbf{v}\cdot\mathbf{n} = f_N(t)$ for some function $f_N(t)$ on the rest of $\partial\Omega$.

After explicit time discretization of (12), the mass m^{k+1} can be computed from the flux $\mathbf{v}^k$. Then the new flux $\mathbf{v}^{k+1}$ is obtained from the Richards eqution with Signorini boundary condition. All considerations of the preceding sections apply to this case provided that γ intersects the boundary of nor more than one subdomain Ω_i. Implicit time discretization gives rise to a heterogeneous iteration of Dirichlet–Neumann-type.

Moving surface water can be described by the shallow water equations. As hydrostatic pressure is part of the modelling assumptions, we can use the same interface condition with a similar heterogeneous domain decomposition strategie.

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