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STEADY, SHALLOW ICE SHEETS AS OBSTACLE PROBLEMS: WELL-POSEDNESS AND FINITE ELEMENT APPROXIMATION*

GUILLAUME JOUVET† AND ED BUELER‡

Abstract. We formulate steady, shallow ice sheet flow as an obstacle problem, the unknown being the ice upper surface and the obstacle being the underlying bedrock topography. This generates a free-boundary defining the ice sheet extent. The obstacle problem is written as a variational inequality subject to the positive-ice-thickness constraint. The corresponding PDE is a highly nonlinear elliptic equation which generalizes the p-Laplacian equation. Our formulation also permits variable ice softness, basal sliding, and elevation-dependent surface mass balance. Existence and uniqueness are shown in restricted cases which we may reformulate as a convex minimization problem. In the general case we show existence by applying a fixed point argument. Using continuity results from that argument, we construct a numerical solution by solving a sequence of obstacle p-Laplacian-like problems by finite element approximation. As a real application, we compute the steady-state shape of the Greenland ice sheet in a steady present-day climate.

Key words. ice sheet model, shallow ice approximation, obstacle problem, variational inequality, p-Laplace, finite elements

AMS subject classifications. 35J20, 35J60, 35J70, 65J15, 65K15, 65N12, 65Z05

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1. Introduction.

Obstacle problems [27] are free-boundary problems for partial differential equations (PDEs), wherein an inequality constrains the solution to be on one side of an identified function, the obstacle. Such problems may be formulated as variational inequalities [21] and sometimes as constrained minimization of a functional. This paper locates the standard continuum model for shallow ice sheets [13, 16] in the class of obstacle problems which may be formulated as variational inequalities but which generally do not correspond to a minimization. Time-dependent, two-dimensional, isothermal ice sheets on flat beds already have a weak formulation [6], but here we allow arbitrary bed topography, elevation-dependent accumulation, basal sliding, and variable ice softness within a three-dimensional ice mass. These features are essential for effective modeling.

The model here takes, as time-independent inputs, the bedrock topography under the ice sheet and the climate, which determines the accumulation/ablation (surface mass balance) function [13]. If ice flow is in balance with these inputs, then the geometry of the ice sheet is steady. We characterize this steady geometry.

The simplest standard description of large ice sheets is the "shallow ice approximation" (SIA) [13, 17], a free surface lubrication approximation in which viscous shear stresses balance gravitational body forces. In the steady isothermal case it is usually stated as PDE (2.6) below. It applies to polar ice sheets because they mostly flow by shear deformation in vertical planes [16]. Fast sliding portions of such ice sheets, typically smaller regions near margins and possibly in contact with the ocean,

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†Department of Mathematics and Computer Science, Free University of Berlin, 14195 Berlin, Germany (guillaume.jouvet@fu-berlin.de). This author's work was supported by the Swiss National Science Foundation.
‡Department of Mathematics and Statistics, and Geophysical Institute, University of Alaska, Fairbanks, AK 99775 (ffelb@uaf.edu). This author's work was supported by NASA grant NNX09AJ38G.

are modeled by a membrane-like flow regime not addressed here [29].

The ice surface elevation satisfies the SIA on the region with positive ice thickness, a region which is generally unknown as a function of climatic conditions. We therefore state our model on a larger domain as an obstacle problem, where the ice sheet surface elevation must remain on or above the bedrock elevation. This constraint describes the grounded margin in a systematic way, replacing ad hoc descriptions of ice sheet margin boundary conditions in the numerical modeling literature [19, 25], and explaining why both Dirichlet and Neumann boundary conditions apply at ablation margins in a well-posed model [5, 28].

The obstacle problem is developed in section 2 into variational inequality form. In section 3 we first choose suitable function spaces. Then we restrict the problem to the case with no basal sliding, flat bedrock, and where both the mass balance and the ice softness are elevation-independent. Because the variational inequality is equivalent to minimization in this restricted case, we can show well-posedness. However, these restrictions are too severe when modeling real ice sheets, but we are able to show existence of solutions in the general case by a fixed point strategy. This approach leaves open most questions of uniqueness. In subsection 3.5 we identify a qualitative property of our variational inequality which distinguishes it from other obstacle problems: in glaciological terms, where it snows perennially, even locally, there will be an ice sheet or glacier. In section 4, we approximate by finite elements, first in cases where the variational inequality is equivalent to a convex minimization problem. An iterative technique addresses the general case. In the restricted case we prove convergence and an a priori error estimate. In section 5 we first measure the convergence rate for an exact solution which solves a nonflat bedrock case, and then we illustrate the numerical method by computing the steady-state shape of the Greenland ice sheet based on observed bedrock elevation and surface mass balance.

2. Model. Let $\Omega \subset \mathbb{R}^2$ be a bounded open region with Lipschitz boundary. In this paper ∇ denotes the gradient and $\nabla\cdot$ the divergence in (horizontal) coordinates $\mathbf{x} = (x, y)$ on Ω. The bedrock elevation is a function $z = b(\mathbf{x})$ in Ω. The unknown function is the elevation $z = h(\mathbf{x})$ of the upper ice surface, and thus $h \geq b$ on Ω; see Figure 2.1. The ice thickness $H = h - b$ is nonnegative in Ω.

In addition to the bed elevation, the inputs to the model include a source function a called the surface mass balance, which is the yearly-averaged accumulation (ablation) rate of deposition (removal) of ice by snowfall (by melting with runoff). We

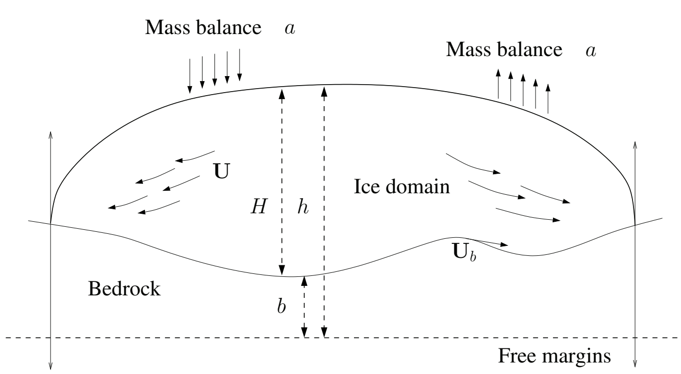

FIG. 2.1. *Vertical section of an ice sheet with notation.*

assume that this function depends on horizontal location and surface elevation, so $a = a(\mathbf{x}, z)$ for $\mathbf{x} \in \Omega$ and $z \in \mathbb{R}$. Also, the base of the ice may slide at a horizontal velocity $\mathbf{U}_b = \mathbf{U}_b(\mathbf{x})$, or the base of the ice may be frozen so that $\mathbf{U}_b = 0$.

2.1. The shallow ice approximation (SIA). Ice sheets and glaciers are incompressible, non-Newtonian, gravity-driven slow flows [13, 16]. The nonlinear viscosity (shear-thinning) property of ice may be described by Glen's power law (equation (4.16) in [16]) with coefficient $A(\mathbf{x}, z)$ and exponent $p > 2$,

$$D_{ij} = A(\mathbf{x}, z)\tau^{p-2}\tau_{ij}, \tag{2.1}$$

where D_{ij} is the strain rate tensor and τ_{ij} is the deviatoric strain rate tensor. Glaciologists write the power using $n \geq 1$, where $p = n + 1$. Laboratory experiments [15] suggest that $1.8 \leq n \leq 4$.

The ice softness $A(\mathbf{x}, z)$ generally depends on temperature. Allowing functional form $A = A(\mathbf{x}, z)$ here is motivated by the prospect of coupling with a conservation-of-energy model. Such a model would determine a temperature or enthalpy field [16] and thus softness A through standard parameterizations [17]. While we do not consider conservation of energy here, fixed point iterations with such a model could determine the thermo-mechanically coupled steady state of the ice sheet.

Momentum conservation yields a nonlinear Stokes problem incorporating (2.1). Its lubrication approximation, the SIA considered here, is obtained by expanding the problem in powers of $\epsilon = d/L$, where d is a typical thickness and L is a typical horizontal extent, and then dropping terms of order ϵ^2 and higher [13, Chapter 18]. The horizontal velocity $\mathbf{U}$ is then computable from the surface elevation h and its gradient [16, equation (5.84)],

$$\mathbf{U}(\mathbf{x}, z) = -2(\rho g)^{p-1}\left[\int_b^z A(s)(h-s)^{p-1}ds\right]|\nabla h|^{p-2}\nabla h + \mathbf{U}_b. \tag{2.2}$$

Here ρ is the density of ice and g is the acceleration of gravity.

Though we allow a given basal sliding velocity $\mathbf{U}_b(\mathbf{x})$ in (2.2), we do not consider any relation between the glaciological driving stress $-\rho g(h-b)\nabla h$ [16] and the sliding velocity; there is no SIA-type sliding law [20]. Instead, similar to the ice softness field above, inclusion of a sliding velocity is motivated by the prospect of coupling with a more physical sliding model, such as one which uses the shallow shelf approximation as its sliding law [4]. Such coupling is of greatest interest in ice stream zones where SIA-type sliding laws cannot account for fast basal sliding. Of course, the simple nonsliding case $\mathbf{U}_b \equiv 0$ is representative of the majority of ice sheets by area, and our theory loses nothing important when restricted to that case.

An important part of this Stokes problem, in determining the shape of ice sheets, is the free surface equation ("kinematic boundary condition"; equation (5.21) in [16]). This equation relates the movement of the ice surface to the ice velocity and the mass balance data $a(\mathbf{x}, z)$. Equivalently for this incompressible flow, in steady state a continuity equation applies to the flow (the "Saint-Venant equation" [20]),

$$\nabla \cdot \mathbf{q} = a, \tag{2.3}$$

where

$$\mathbf{q} := \int_b^h \mathbf{U}(z)dz \tag{2.4}$$

defines the total horizontal ice flux. Inclusion of (2.2) and (2.4) into (2.3) gives an equation for the surface elevation h:

$$-\nabla \cdot \left(2(\rho g)^{p-1} \left[\int_b^h A(s)(h-s)^p ds \right] |\nabla h|^{p-2} \nabla h - (h-b)\mathbf{U}_b \right) = a. \tag{2.5}$$

If $A(\mathbf{x}, z) = A_0$ is constant (e.g., the isothermal case), then (2.5) reduces to a PDE,

$$-\nabla \cdot \big(\Lambda (h-b)^{p+1} |\nabla h|^{p-2} \nabla h - (h-b)\mathbf{U}_b \big) = a, \tag{2.6}$$

where $\Lambda > 0$ is constant. One may also rewrite (2.5) in terms of ice thickness $H = h - b$:

$$-\nabla \cdot \left(2(\rho g)^{p-1} \left[\int_0^H A(s+b)(H-s)^p ds \right] |\nabla (H+b)|^{p-2} (\nabla (H+b)) - H\mathbf{U}_b \right) = a. \tag{2.7}$$

Equations (2.5), (2.6), and (2.7) hold only on the domain where there is ice, that is, on an unknown subregion of Ω. Determining that subregion is part of any complete formulation of the ice sheet problem.

2.2. Reformulation as an obstacle problem. The obvious fact that the surface elevation of the ice sheet equals or exceeds that of the bed underneath it is now given the interpretation that the bed is an "obstacle" [27] to the surface position of the ice. The solutions h, H to (2.5), (2.7), respectively, are in fact constrained by the equivalent inequalities

$$h \geq b \qquad \iff \qquad H \geq 0. \tag{2.8}$$

In regions with no ice we set $h = b$ and $H = 0$, so h and H are defined on all of Ω.

Constraint (2.8) generally implies the existence of a free boundary [18]. Let

$$\Omega_+ = \{h > b\} = \{H > 0\}$$

be the subregion of Ω where ice is present, the (open) support of the ice sheet. A nonempty free boundary $\Gamma = \Omega \cap \partial \Omega_+$ is said to be a *free (grounded) ice sheet margin*. Locating the free margin Γ is an important part of any ice sheet problem in which ice sheet extent is a function of climate. Only a weak formulation of the problem, in which constraint (2.8) is incorporated from the beginning correctly describes the dependence of Γ and Ω_+ on the problem data.

For a steady ice sheet with $\Omega_+ \neq \emptyset$, the source function a must be positive (accumulation) in some part of Ω_+. If there is a nonempty free margin and a is continuous, however, a must be negative (ablation) outside Ω_+, i.e., on $\Omega_- = \Omega \setminus (\Omega_+ \cup \Gamma)$. Ice flows outward from areas with accumulation into areas with ablation, and the ice sheet thins to zero thickness at the margin.

The weak formulation of problem (2.7) on the whole domain Ω requires us to suppose that the following boundary conditions hold on Γ:

$$H = 0, \qquad \mathbf{q} \cdot \mathbf{n} = 0, \tag{2.9}$$

where $\mathbf{n}$ is the outward unit normal vector along Γ. The value of $\mathbf{q}$ on Γ should be understood as its limit from the interior of Ω_+. As expected in an obstacle problem, extra conditions at the free boundary are needed to to determine its location. In fact, on Ω_- we extend the flux by zero, $\mathbf{q} = \mathbf{0}$. Let v be a test function satisfying $v \geq b$ everywhere. Now apply the Gauss theorem:

$$-\int_\Omega \mathbf{q} \cdot \nabla(v-h) = \int_{\Omega_-} (\nabla \cdot \mathbf{q})(v-h) + \int_{\Omega_+} (\nabla \cdot \mathbf{q})(v-h) - \int_\Gamma [(v-h)\mathbf{q} \cdot \mathbf{n}]_-^+,$$

where $[\,]_-^+$ denotes the difference across Γ. Note that $\mathbf{q} = \mathbf{0}$ and $a \leq 0$ on Ω_-, so that $\nabla \cdot \mathbf{q} \geq a$ on Ω_-. Also using (2.3) on Ω_+ and $\mathbf{q} \cdot \mathbf{n} = 0$ on Γ, we obtain

$$-\int_\Omega \mathbf{q} \cdot \nabla(v-h) \geq \int_\Omega a(v-h). \tag{2.10}$$

The boundary Γ is no longer explicit in this reformulation. Expanding $\mathbf{q}$ as in (2.5), we have this variational inequality generalization of the SIA:

$$\int_\Omega \left(2(\rho g)^{p-1} \left[\int_b^h A(s)(h-s)^p ds \right] |\nabla h|^{p-2} \nabla h - (h-b)\mathbf{U}_b \right) \cdot \nabla(v-h) \geq \int_\Omega a(v-h). \tag{2.11}$$

Note that boundary conditions (2.9) apply at grounded free margins of steady ice sheets (i.e., at ablation-zone margins [5]) but not at the grounding lines of marine ice sheets where ice shelves may be attached [16]. At such grounding lines thickness H may be relatively small but the flux $\mathbf{q}$ is significant, and so mathematical formulations are quite different (see, for example, [30]).

Weak form (2.11) is a degenerate extension of the p-Laplacian obstacle problem because thickness $H = h - b$ goes to zero at the free boundary (margin) Γ. This is the reason why the solution of (2.11) (or (2.5)) is well known to exhibit infinite gradients at the margin Γ [5, 16]. A further reformulation, (2.19), that recovers nondegenerate p-Laplacian form is proposed in the next subsection, but, while this formulation involves only a flat obstacle, it acquires a "tilt" which destroys monotonicity of the variational form. In subsection 3.5 we identify a qualitative property which is common to model (2.11) and its apparently different, though actually equivalent, version (2.19).

2.3. Ellipticity and transformed ice thickness. For clarity, consider the simplest SIA, (2.6). If we were to remove the power of thickness ("$(h-b)^{p+1}$") from the coefficient, then the problem would be uniformly elliptic as a p-Laplace problem. The corresponding obstacle problem would not generate a singular gradient at the free margin [7].

A transformation of thickness, following [6, 26], restores such uniform ellipticity to (2.11), though it significantly modifies the p-Laplace form when the bedrock is not flat. We apply the following change of variables:

$$H = u^{(p-1)/(2p)}. \tag{2.12}$$

Equation (2.7) becomes

$$-\nabla \cdot \big(\mu(\mathbf{x}, u)|\nabla u - \Phi(\mathbf{x}, u)|^{p-2}(\nabla u - \Phi(\mathbf{x}, u)) - \Psi(\mathbf{x}, u)\big) = \alpha(\mathbf{x}, u), \tag{2.13}$$

where

$$\mu(\mathbf{x},u) := 2\left(\frac{\rho g(p-1)}{2p}\right)^{p-1}\int_0^1 A(\mathbf{x}, b+su^{(p-1)/(2p)})(1-s)^p ds, \tag{2.14}$$

$$\Phi(\mathbf{x},u) := -\left(\frac{2p}{p-1}\right) u^{(p+1)/(2p)}\nabla b(\mathbf{x}), \tag{2.15}$$

$$\Psi(\mathbf{x},u) := u^{(p-1)/(2p)}\,\mathbf{U}_b(\mathbf{x}), \tag{2.16}$$

$$\alpha(\mathbf{x},u) := a(\mathbf{x}, b+u^{(p-1)/(2p)}). \tag{2.17}$$

Constraint (2.8) becomes

$$u \geq 0 \quad \text{on } \Omega; \tag{2.18}$$

that is, the ice thickness is nonnegative. Assuming ice softness $A(\mathbf{x},z)$ satisfies reasonable bounds (below), (2.13) has the form of an elliptic p-Laplace equation with a "tilt" Φ, that is, a tilt determined by the bedrock gradient ∇b.

We now apply transformation (2.12) to rewrite (2.11) as a variational inequality for u. Of course, (2.13) is the interior condition where $u > 0$. The solution $u \geq 0$ satisfies

$$\int_\Omega \big(\mu(\mathbf{x},u)|\nabla u - \Phi(\mathbf{x},u)|^{p-2}(\nabla u - \Phi(\mathbf{x},u)) - \Psi(\mathbf{x},u)\big)\cdot\nabla(v-u) \geq \int_\Omega \alpha(\mathbf{x},u)(v-u) \tag{2.19}$$

for all $v \geq 0$ on Ω.

We identify six nonlinearities in problem (2.19), making the mathematical theory nontrivial:

(i) A p-Laplace-type nonlinearity induced by Glen's flow law. Though it is not a case considered in this paper, this nonlinearity is removed when $p = 2$ so that $|\nabla u - \Phi(\mathbf{x},u)|^{p-2} = 1$.

(ii) A solution-dependent diffusion coefficient $\mu(\mathbf{x},u)$ generated by the ice softness properties. This nonlinearity vanishes if $A(\mathbf{x},z) = A(\mathbf{x})$ is elevation-independent, in which case $\mu(\mathbf{x},u) = \mu_0$ is constant.

(iii) A nonlinearity in $\Phi(\mathbf{x},u)$ due to bedrock gradient ∇b. If the bed is flat, then $\Phi(\mathbf{x},u) = 0$.

(iv) A nonlinearity in $\Psi(\mathbf{x},u)$ due to basal sliding. If $\mathbf{U}_b = 0$, then $\Psi(\mathbf{x},u) = \mathbf{0}$.

(v) A nonlinearity driven by elevation-dependent mass balance $\alpha(\mathbf{x},u)$. This nonlinearity vanishes if $a = a(\mathbf{x})$ is elevation-independent.

(vi) Obstacle problems are inherently nonlinear. Even if the above nonlinearities are removed, the solution space is not affine.

In the next section, we address the function spaces, and the mathematical framework, in which to study the well-posedness of (2.19).

3. Existence, uniqueness, and regularity. In subsection 3.1 we choose function spaces and state assumptions on data that make (2.19) precise. Subsection 3.2 treats well-posedness of (2.19) by using convex analysis, but only in a restricted case with fewer nonlinearities. Existence in the general case follows by using a fixed point argument (subsection 3.3). Then in later parts we address regularity, a novel qualitative property of the solutions, and boundedness. From now on we suppress explicit dependence on $\mathbf{x}$, but we always show dependence on the solution u when present.

3.1. Mathematical setting. Let $\mathcal{X} = W_0^{1,p}(\Omega)$, and define the convex admissible subset

$$\mathcal{K} := \{u \in \mathcal{X}, u \geq 0\}. \tag{3.1}$$

For $g \in L^p(\Omega)$ and $f \in W^{1,p}(\Omega)$ we have norms

$$\|g\|_{L^p} := \left(\int_\Omega |g|^p\right)^{1/p}, \qquad \|f\|_{W^{1,p}} := \left(\int_\Omega |f|^p\right)^{1/p} + \left(\int_\Omega |\nabla f|^p\right)^{1/p}.$$

Since $\Omega \subset \mathbb{R}^2$ is bounded, the Poincaré inequality [11] applies: there is $C_1 > 0$ so that

$$\int_\Omega |v|^p \leq C_1 \int_\Omega |\nabla v|^p \qquad \forall v \in \mathcal{X}. \tag{3.2}$$

It follows that norms $\|\cdot\|_{W^{1,p}}$ and $\|\nabla(\cdot)\|_{L^p}$ are equivalent on $\mathcal{X}$. Recall that $W^{1,p}(\Omega) \hookrightarrow \mathcal{C}(\overline{\Omega})$ is continuous and compact when $p > 2$ [8, p. 114]. Thus for each $p > 2$ there is a constant $C_2 > 0$ so that

$$\|v\|_{L^\infty} \leq C_2 \|v\|_{W^{1,p}} \qquad \forall v \in \mathcal{X}. \tag{3.3}$$

We will use the following technical inequalities [3, Lemma 2.1].

LEMMA 3.1. *For all $p > 1$, there exist $M_1, M_2 > 0$ such that*

$$\left||\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right| \leq M_1(|\xi| + |\eta|)^{p-2}|\xi - \eta|, \tag{3.4}$$

$$\left(|\xi|^{p-2}\xi - |\eta|^{p-2}\eta\right) \cdot (\xi - \eta) \geq M_2(|\xi| + |\eta|)^{p-2}|\xi - \eta|^2 \tag{3.5}$$

for all $(\xi, \eta) \in \mathbb{R}^2 \times \mathbb{R}^2$.

Let $q = p/(p-1) \in (1, 2)$ be the exponent conjugate to p. In order to define the notion of weak solution for the variational inequality corresponding to (2.13), we make the following hypotheses on the data:

(H1) $A(\mathbf{x}, z)$ is a positive measurable function, and there exist $A_1, A_2 > 0$ so that for all $(\mathbf{x}, z) \in \Omega \times \mathbb{R}$,

$$0 < A_1 \leq A(\mathbf{x}, z) \leq A_2. \tag{3.6}$$

(H2) $b \in W^{1,p}(\Omega)$.

(H3) $\mathbf{U}_b \in [L^q(\Omega)]^2$.

(H4) $a : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is measurable in the first argument and continuous in the second argument, and there exists $M > 0$ such that

$$|a(\mathbf{x}, z)| \leq M \qquad \forall (\mathbf{x}, z) \in \Omega \times \mathbb{R}. \tag{3.7}$$

Regarding (H1), while ice temperature is observed to vary continuously within glaciers, there is no mathematical or physical need to assume that the ice softness A is continuous, though certainly A is observed to be bounded. Assumptions (H2) and (H3) imply that bedrock elevation and sliding velocity have the minimal regularity needed to pose the problem in $W^{1,p}$. In practice $\|b\|_{W^{1,p}}$ may be large because subglacial topography can be mountainous. Hypothesis (H4) requires choosing an artificial bound M on the maximum amount of accumulation (ablation) independent of geometric factors like the surface elevation solution. Such a bound is not provided by existing information about climate, but observation-based simulations with prescribed surface mass balance, as foreseen here, always allow such a bound.

It follows from (H1) and (2.14) that there exists μ_i, $i = 1, 2$, such that

$$0 < \mu_1 \leq \mu(\mathbf{x}, u) \leq \mu_2 \quad \forall \mathbf{x} \in \Omega, \ \forall u \in \mathcal{K}. \tag{3.8}$$

Moreover, one can check that μ is continuous with respect to argument u. Owing to (H2) and (H3), we have

$$\|\Phi(v)\|_{L^p} \leq \left(\frac{2p}{p-1}\right) \|v\|_{L^\infty}^{(p+1)/(2p)} \|b\|_{W^{1,p}}, \qquad \|\Psi(v)\|_{L^q} \leq \|v\|_{L^\infty}^{(p-1)/(2p)} \|\mathbf{U}_b\|_{L^q} \tag{3.9}$$

for all $v \in \mathcal{K}$, where Φ and Ψ are defined by (2.15) and (2.16). It follows that

$$\Phi(\cdot, v) \in [L^p(\Omega)]^2, \quad \Psi(\cdot, v) \in [L^q(\Omega)]^2 \tag{3.10}$$

for all $v \in \mathcal{K}$. Note also that (H4) implies $\alpha(\cdot, v) \in L^\infty(\Omega)$ for all $v \in \mathcal{K}$. Furthermore, functions μ, Φ, and Ψ are each measurable in their first arguments and continuous in their second arguments.

By the above hypotheses on data we can define constrained ($u \geq 0$) weak solutions to problem (2.13), as follows.

Definition 3.2. *A function $u \in \mathcal{K}$ solves the (transformed) steady shallow ice sheet problem if u satisfies* (2.19) *for all $v \in \mathcal{K}$. The ice thickness solution is given by $H = u^{(p-1)/(2p)}$.*

By property (3.8) we say that variational inequality (2.19) is a uniformly elliptic nonlinear p-Laplacian obstacle problem with additional modification $\nabla u \mapsto \nabla u - \Phi(\mathbf{x}, u)$, which we call a "tilt" of the p-Laplacian form.

As noted, problem (2.19) possesses several explicit nonlinearities. Well-posedness is not obvious. In the restricted cases where we may rewrite (2.19) as equivalent to a minimization, we can show existence and uniqueness using convex analysis tools. However, only a constant tilt of the p-Laplace form can be addressed that way. Also for most of the other nonlinearities, (2.13) cannot be rewritten as the critical point equations of a minimization. (If that were the case, differentiation of (2.13) with respect to a direction w, i.e., a proposed second variation of the functional, would lead to a symmetric bilinear form for variables v and w. Formal computations show that this is not so.) This motivates the fixed point approach in section 3.3.

3.2. Minimization and monotone operator. Consider $u \in \mathcal{K}$ satisfying

$$\int_\Omega \left(k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})\right) \cdot \nabla(v - u) \geq \int_\Omega f(v - u) \quad \forall v \in \mathcal{K}, \tag{3.11}$$

where $k \in L^\infty(\Omega)$, $k(\mathbf{x}) \geq \bar{k} > 0$, $\mathbf{Z} \in [L^p(\Omega)]^2$, and $f \in L^\infty(\Omega)$. These simplifications of (2.19) remove several nonlinearities. Nonetheless variational inequality (3.11) with tilt $\nabla u \mapsto \nabla u - \mathbf{Z}$, and with the p-Laplace operator, is not to our knowledge addressed in the literature. Therefore we sketch how the classical existence and uniqueness proof for the p-Laplace problem [8] applies to (3.11).

Inequality (3.11) has an equivalent minimization problem. Define the functional

$$J(v) := \frac{1}{p} \int_\Omega k|\nabla v - \mathbf{Z}|^p - fv \quad \forall v \in \mathcal{K}. \tag{3.12}$$

One may show that $J(v)$ is Gâteaux differentiable,

$$\langle J'(v), w\rangle = \int_\Omega k|\nabla v - \mathbf{Z}|^{p-2}(\nabla v - \mathbf{Z}) \cdot \nabla w - fw \quad \forall w \in \mathcal{K}. \tag{3.13}$$

Variational inequality (3.11) can be stated "$\langle J'(u), v-u\rangle \geq 0$ for all $v \in \mathcal{K}$," which says that the directional derivative of J at u in the direction $v-u$ is nonnegative, including when u is on the boundary of $\mathcal{K}$. One can show that $u \in \mathcal{K}$ solves (3.11) if and only if it solves the minimization problem

$$J(u) = \min_{v\in\mathcal{K}} J(v). \tag{3.14}$$

We may therefore prove well-posedness of (3.11) by showing it for problem (3.14). On the one hand, J is continuous, is strictly convex, and satisfies the coercivity property that there exist $C_1, C_2 > 0$ so that

$$J(v) \geq C_1 \int_\Omega |\nabla v - \mathbf{Z}|^p - C_2 \qquad \forall v \in \mathcal{X}. \tag{3.15}$$

On the other hand, weak lower semicontinuity of J follows from the lower boundedness of J and the convexity of function $s \mapsto p^{-1}|s-\mathbf{Z}|^p - fv$ [11, section 8.2.2, Theorem 1]. We have all the ingredients to prove the following theorem [8, 11], and also a corollary which applies to (2.19) in restricted cases.

THEOREM 3.3. *There exists a unique solution $u \in \mathcal{K}$ to problem* (3.14).

COROLLARY 3.4. *Assume that ice softness is elevation-independent ($A = A(\mathbf{x})$), the bed is flat (b constant), there is no basal sliding ($\mathbf{U}_b = \mathbf{0}$), and the mass balance is elevation-independent ($a = a(\mathbf{x})$). There exists a unique solution $u \in \mathcal{K}$ to* (2.19)*, and thus a unique ice sheet thickness $H = u^{(p-1)/(2p)}$.*

The major application of Theorem 3.3 in this paper is not the above corollary, however, but instead in the fixed point argument in the next subsection, where $\mathbf{Z} \neq 0$. Before that argument, let us mention an alternative way to prove Theorem 3.3 by using the theory of monotone operators. If we define the operator $\mathcal{M} = J' : W^{1,p} \to (W^{1,p})'$ by (3.13), then we can show, using [3, Lemma 2.1] and Hölder's inequality, that there exist $C_1, C_2 > 0$ such that for all $u, v, w \in \mathcal{K}$

$$|(\mathcal{M}u - \mathcal{M}v)(w)| \leq C_1(\|u\|_{W^{1,p}} + \|v\|_{W^{1,p}} + \|\mathbf{Z}\|_{L^{1,p}})^{p-2}\|u-v\|_{W^{1,p}}\,\|w\|_{W^{1,p}}, \tag{3.16}$$

$$(\mathcal{M}u - \mathcal{M}v)(u-v) \geq C_2\|u-v\|^p_{W^{1,p}}. \tag{3.17}$$

Property (3.16) states the Lipschitz continuity, for bounded arguments, of $\mathcal{M}$ in its first argument, and the continuity of $\mathcal{M}$ in its second argument. Property (3.17) shows that $\mathcal{M}$ is a strictly monotone operator. Owing to Corollary III.1.8 in [21], this implies existence and uniqueness of $u \in \mathcal{K}$ solving (3.11).

A monotone approach is slightly more general than our minimization approach. Specifically, if we make the glaciologically unrealistic assumption that a depends on ice thickness in a *nonincreasing* manner, then monotonicity property (3.17) is still satisfied with the corresponding nonincreasing function $f(u)$, and existence and uniqueness follow.

3.3. Fixed points: Existence. In order to apply Schaefer's fixed point theorem [11, section 9.2.2, Theorem 4] to prove the existence of a solution to (2.19), we first establish a preliminary lemma, a continuity result. Define the set

$$B = \{\mu \in L^\infty(\Omega)\,,\, \mu_1 \leq \mu(\mathbf{x}) \leq \mu_2\}.$$

Define the map

$$\mathcal{T} : B \times [L^p(\Omega)]^2 \times [L^q(\Omega)]^2 \times L^q(\Omega) \longrightarrow \mathcal{K}, \tag{3.18}$$

which takes the tuple $(\mu, \Phi, \Psi, \alpha)$ to the unique $u \in \mathcal{K}$ solving the variational inequality

$$\int_\Omega \left(\mu|\nabla u - \Phi|^{p-2}(\nabla u - \Phi) - \Psi\right) \cdot \nabla(v-u) \geq \int_\Omega \alpha(v-u) \qquad \forall v \in \mathcal{K}. \tag{3.19}$$

Theorem 3.3 ensures that $\mathcal{T}$ is well defined. The next lemma gives an estimate of the difference between two solutions of (3.19) deriving from different data, from which continuity of $\mathcal{T}$ follows.

LEMMA 3.5. *Let* $\tilde{\mu}, \overline{\mu} \in B$, $\tilde{\Phi}, \overline{\Phi} \in L^p(\Omega)$, $\tilde{\Psi}, \overline{\Psi} \in [L^q(\Omega)]^2$, $\tilde{\alpha}, \overline{\alpha} \in L^q(\Omega)$, *and* $\tilde{u}, \overline{u} \in \mathcal{K}$ *such that* $\mathcal{T}(\tilde{\mu}, \tilde{\Phi}, \tilde{\Psi}, \tilde{\alpha}) = \tilde{u}$ *and* $\mathcal{T}(\overline{\mu}, \overline{\Phi}, \overline{\Psi}, \overline{\alpha}) = \overline{u}$. *Then there exists a constant* $D_1 > 0$ *that depends on* μ_1, μ_2, *but is independent of* $\tilde{\mu}$, $\tilde{\Phi}$, $\tilde{\Psi}$, a, $\overline{\mu}$, $\overline{\Phi}$, $\overline{\Psi}$, *and* $\overline{\alpha}$, *such that*

$$\begin{aligned} D_1\|\tilde{u} - \overline{u}\|_{W^{1,p}}^{p-1} \leq \|\nabla\overline{u} - \overline{\Phi}\|_{L^p}^{p-1}\|\overline{\mu} - \tilde{\mu}\|_{L^\infty} + \|\tilde{\alpha} - \overline{\alpha}\|_{L^q} + \|\tilde{\Psi} - \overline{\Psi}\|_{L^q} \\ + \|\nabla\overline{u} - \overline{\Phi}\|_{L^p}^{p-2}\|\tilde{\Phi} - \overline{\Phi}\|_{L^p} + \|\tilde{\Phi} - \overline{\Phi}\|_{L^p}^{p-1}. \end{aligned} \tag{3.20}$$

Moreover, if $\tilde{\Phi} = \overline{\Phi}$ *and* $\tilde{\mu} = \overline{\mu}$, *then there exists a constant* $D_2 > 0$ *independent of* $\tilde{\Psi}$, $\tilde{\alpha}$, $\overline{\Psi}$, *and* $\overline{\alpha}$ *such that*

$$D_2\|\tilde{u} - \overline{u}\|_{W^{1,p}}^p \leq \int_\Omega (\tilde{\alpha} - \overline{\alpha})(\tilde{u} - \overline{u}) + \int_\Omega (\tilde{\Psi} - \overline{\Psi}) \cdot \nabla(\tilde{u} - \overline{u}). \tag{3.21}$$

Proof. Applying (3.19) for $\mathcal{T}(\tilde{\mu}, \tilde{\Phi}, \tilde{\Psi}, \tilde{\alpha}) = \tilde{u}$ with $v = \overline{u} \in \mathcal{K}$ and (3.19) for $\mathcal{T}(\overline{\mu}, \overline{\Phi}, \overline{\Psi}, \overline{\alpha}) = \overline{u}$ with $v = \tilde{u} \in \mathcal{K}$, we obtain, respectively,

$$\int_\Omega \left(\tilde{\mu}|\nabla\tilde{u} - \tilde{\Phi}|^{p-2}(\nabla\tilde{u} - \tilde{\Phi}) - \tilde{\Psi}\right) \cdot \nabla(\overline{u} - \tilde{u}) \geq \int_\Omega \tilde{\alpha}(\overline{u} - \tilde{u}), \tag{3.22}$$

$$\int_\Omega \left(\overline{\mu}|\nabla\overline{u} - \overline{\Phi}|^{p-2}(\nabla\overline{u} - \overline{\Phi}) - \overline{\Psi}\right) \cdot \nabla(\tilde{u} - \overline{u}) \geq \int_\Omega \overline{\alpha}(\tilde{u} - \overline{u}). \tag{3.23}$$

For the sake of convenience, we call $\tilde{X} = \nabla\tilde{u} - \tilde{\Phi}$ and $\overline{X} = \nabla\overline{u} - \overline{\Phi}$ and define

$$|||Y||| = \int_\Omega (|\overline{X}| + |Y|)^{p-2}|Y|^2 \tag{3.24}$$

for any $Y \in [L^p(\Omega)]^2$. Using inequality (3.5), we obtain

$$\begin{aligned} |||\tilde{X} - \overline{X}||| &\leq 2^{p-2}\int_\Omega (|\overline{X}| + |\tilde{X}|)^{p-2}|\overline{X} - \tilde{X}|^2 \\ &\leq \frac{2^{p-2}}{M_2}\int_\Omega \left(|\tilde{X}|^{p-2}\tilde{X} - |\overline{X}|^{p-2}\overline{X}\right) \cdot (\tilde{X} - \overline{X}) = E_1 + E_2, \end{aligned} \tag{3.25}$$

where

$$\begin{aligned} E_1 &:= \frac{2^{p-2}}{M_2}\int_\Omega \left(|\tilde{X}|^{p-2}\tilde{X} - |\overline{X}|^{p-2}\overline{X}\right) \cdot \nabla(\tilde{u} - \overline{u}), \\ E_2 &:= -\frac{2^{p-2}}{M_2}\int_\Omega \left(|\tilde{X}|^{p-2}\tilde{X} - |\overline{X}|^{p-2}\overline{X}\right) \cdot (\tilde{\Phi} - \overline{\Phi}). \end{aligned}$$

Since $\tilde{\mu} \in [\mu_1, \mu_2]$ (see (3.8)), we have

$$E_1 \leq \frac{2^{p-2}}{M_2\mu_1}\int_\Omega \left(\tilde{\mu}|\tilde{X}|^{p-2}\tilde{X} - \tilde{\mu}|\overline{X}|^{p-2}\overline{X}\right) \cdot \nabla(\tilde{u} - \overline{u}). \tag{3.26}$$

From (3.22) and (3.23), we obtain

$$\int_\Omega \left(\tilde\mu|\tilde X|^{p-2}\tilde X - \overline{\mu}|\overline{X}|^{p-2}\overline{X}\right)\cdot\nabla(\tilde u - \overline{u}) \le \int_\Omega (\tilde\alpha - \overline{\alpha})(\tilde u - \overline{u}) + \int_\Omega (\tilde\Psi - \overline{\Psi})\cdot\nabla(\tilde u - \overline{u}). \tag{3.27}$$

From (3.26) and (3.27), we have

$$\begin{aligned} E_1 \le\ & \frac{2^{p-2}}{M_2\mu_1}\left(\int_\Omega \left(\overline{\mu}|\overline{X}|^{p-2}\overline{X} - \tilde\mu|\overline{X}|^{p-2}\overline{X}\right)\cdot\nabla(\tilde u - \overline{u})\right. \\ & \left. + \int_\Omega (\tilde\alpha - \overline{\alpha})(\tilde u - \overline{u}) + \int_\Omega (\tilde\Psi - \overline{\Psi})\cdot\nabla(\tilde u - \overline{u})\right). \end{aligned} \tag{3.28}$$

From Hölder's inequality, it follows that

$$E_1 \le \frac{2^{p-2}}{M_2\mu_1}\left(\|\overline{X}\|_{L^p}^{p-1}\|\overline{\mu} - \tilde\mu\|_{L^\infty} + \|\tilde\alpha - \overline{\alpha}\|_{L^q} + \|\tilde\Psi - \overline{\Psi}\|_{L^q}\right)\|\tilde u - \overline{u}\|_{W^{1,p}}. \tag{3.29}$$

Using inequality (3.4), we have

$$\begin{aligned} E_2 &\le \frac{2^{p-2}M_1}{M_2}\int_\Omega (|\overline{X}| + |\tilde X|)^{p-2}|\tilde X - \overline{X}||\tilde\Phi - \overline{\Phi}| \\ &\le \frac{2^{2(p-2)}M_1}{M_2}\int_\Omega (|\overline{X}| + |\tilde X - \overline{X}|)^{p-2}|\tilde X - \overline{X}||\tilde\Phi - \overline{\Phi}|. \end{aligned}$$

By using the inequality (see Lemma 2.2 in [22])

$$(a+r)^{p-2}rs \le \epsilon(a+r)^{p-2}r^2 + \epsilon^{-1}(a+s)^{p-2}s^2 \qquad \forall a,r,s\ge 0,\ \forall \epsilon\in(0,1],$$

with $a = |\overline{X}|$, $r = |\tilde X - \overline{X}|$, $s = |\tilde\Phi - \overline{\Phi}|$, we obtain, for all $\epsilon\in[0,1]$,

$$E_2 \le \frac{2^{2(p-2)}M_1}{M_2}\left(\epsilon|||\tilde X - \overline{X}||| + \epsilon^{-1}\int_\Omega (|\overline{X}| + |\tilde\Phi - \overline{\Phi}|)^{p-2}|\tilde\Phi - \overline{\Phi}|^2\right).$$

Setting $\epsilon = \frac{M_2}{M_1\, 2^{2p-3}}$, and using Hölder's inequality, there exists $C_1 > 0$ such that

$$E_2 \le \frac{1}{2}|||\tilde X - \overline{X}||| + C_1\left(\|\overline{X}\|_{L^p}^{p-2}\|\tilde\Phi - \overline{\Phi}\|_{L^p}^2 + \|\tilde\Phi - \overline{\Phi}\|_{L^p}^p\right). \tag{3.30}$$

From (3.25), (3.29), and (3.30), we have

$$\begin{aligned} \frac{1}{2}|||\tilde X - \overline{X}||| \le & \frac{2^{p-2}}{M_2\mu_1}\left(\|\overline{X}\|_{L^p}^{p-1}\|\overline{\mu} - \tilde\mu\|_{L^\infty} + \|\tilde\alpha - \overline{\alpha}\|_{L^q} + \|\tilde\Psi - \overline{\Psi}\|_{L^q}\right)\|\tilde u - \overline{u}\|_{W^{1,p}} \\ & + C_1\left(\|\overline{X}\|_{L^p}^{p-2}\|\tilde\Phi - \overline{\Phi}\|_{L^p}^2 + \|\tilde\Phi - \overline{\Phi}\|_{L^p}^p\right). \end{aligned}$$

Clearly, there exists $C_2 > 0$ such that

$$\|\tilde u - \overline{u}\|_{W^{1,p}}^p \le C_2(\|\tilde X - \overline{X}\|_{L^p}^p + \|\tilde\Phi - \overline{\Phi}\|_{L^p}^p) \le C_2(|||\tilde X - \overline{X}||| + \|\tilde\Phi - \overline{\Phi}\|_{L^p}^p),$$

and therefore, we have

$$\begin{aligned} C_3\|\tilde u - \overline{u}\|_{W^{1,p}}^p \le & \left(\|\overline{X}\|_{L^p}^{p-1}\|\overline{\mu} - \tilde\mu\|_{L^\infty} + \|\tilde\alpha - \overline{\alpha}\|_{L^q} + \|\tilde\Psi - \overline{\Psi}\|_{L^q}\right)\|\tilde u - \overline{u}\|_{W^{1,p}} \\ & + \|\overline{X}\|_{L^p}^{p-2}\|\tilde\Phi - \overline{\Phi}\|_{L^p}^2 + \|\tilde\Phi - \overline{\Phi}\|_{L^p}^p \end{aligned} \tag{3.31}$$

for a constant $C_3 > 0$. It is then easy to check that (3.31) implies (3.20) in each of the cases $\|\tilde{u}-\overline{u}\|_{W^{1,p}} \leq \|\tilde{\Phi}-\overline{\Phi}\|_{L^p}$ and $\|\tilde{\Phi}-\overline{\Phi}\|_{L^p} < \|\tilde{u}-\overline{u}\|_{W^{1,p}}$.

If $\tilde{\Phi} = \overline{\Phi}$ and $\tilde{\mu} = \overline{\mu}$, then $E_2 = 0$, and it follows from (3.25) and (3.28) that there exist $C_4, C_5 > 0$ such that

$$\|\tilde{u}-\overline{u}\|^p_{W^{1,p}} \leq C_4 |||\tilde{X}-\overline{X}||| \leq C_5 \left(\int_\Omega (\tilde{\alpha}-\overline{\alpha})(\tilde{u}-\overline{u}) + \int_\Omega (\tilde{\Psi}-\overline{\Psi}) \cdot \nabla(\tilde{u}-\overline{u}) \right),$$

proving (3.21). $\square$

Now we apply the fixed point argument.

THEOREM 3.6. *There exists at least one solution to* (2.19). *Moreover, there exists $C > 0$ depending only on $p > 2$, μ_1, μ_2, M, $\|\mathbf{U}_b\|_{L^q}$, and $\|b\|_{W^{1,p}}$ such that any solution u of* (2.19) *satisfies $\|u\|_{W^{1,p}} < C$.*

Proof. Define the map $\mathcal{A} : C(\overline{\Omega}) \to C(\overline{\Omega})$, which takes the function w to the unique u solving

(3.32)
$$\int_\Omega \big(\mu(w)|\nabla u - \Phi(w)|^{p-2}(\nabla u - \Phi(w)) - \Psi(w)\big) \cdot \nabla(v-u) \geq \int_\Omega \alpha(w)(v-u) \quad \forall v \in \mathcal{K},$$

the problem addressed by Theorem 3.3. The map can be decomposed as $\mathcal{A} = i \circ \mathcal{T} \circ \mathcal{R}$, where i is compact. That is,

$$\mathcal{A}:\ w \in C(\overline{\Omega}) \xrightarrow{\mathcal{R}} \begin{pmatrix} \mu(w) \in B \\ \Phi(w) \in [L^p(\Omega)]^2 \\ \Psi(w) \in [L^q(\Omega)]^2 \\ \alpha(w) \in L^q(\Omega) \end{pmatrix} \xrightarrow{\mathcal{T}} u \in \mathcal{X} \overset{i}{\hookrightarrow} u \in C(\overline{\Omega}).$$

The first and the last components of $\mathcal{R}$ are continuous, since both μ and α are pointwise continuous and bounded (by μ_2 and M, respectively). From definitions (2.15) and (2.16) and assumptions (H2) and (H3) there exists $C > 0$ such that

$$\|\Phi(v) - \Phi(w)\|_{L^p} \leq C \left(\frac{2p}{p-1}\right) \|v-w\|^{(p+1)/(2p)}_{L^\infty} \|b\|_{W^{1,p}}, \tag{3.33}$$

$$\|\Psi(v) - \Psi(w)\|_{L^q} \leq C \|v-w\|^{(p-1)/(2p)}_{L^\infty} \|\mathbf{U}_b\|_{L^q} \tag{3.34}$$

for all $v, w \in \mathcal{K}$. The continuity of the second and the third factors of $\mathcal{R}$ results from (3.33) and (3.34). Continuity of $\mathcal{T}$ is stated in Lemma 3.5. Since $p > 2$, the last embedding is compact [8, p. 114]. As a consequence, $\mathcal{A}$ is continuous and compact.

We now show that

$$S = \{w \in C(\overline{\Omega})\,,\, w = \lambda \mathcal{A}(w) \text{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. By using Lemma 3.5 with $\tilde{\mu} = \mu(w)$, $\tilde{\Phi} = \Phi(w)$, $\tilde{\Psi} = \Psi(w)$, $\tilde{\alpha} = \alpha(w)$, $\tilde{u} = \mathcal{A}(w)$, $\overline{\mu} = 1$, $\overline{\Phi} = 0$, $\overline{\Psi} = 0$, $\overline{\alpha} = 0$, $\overline{u} = 0$, (3.7), and (3.9), we obtain

$$\begin{aligned} D_1 \|\mathcal{A}(w)\|^{p-1}_{W^{1,p}} &\leq \|\alpha(w)\|_{L^q} + \|\Psi(w)\|_{L^q} + \|\Phi(w)\|^{p-1}_{L^p} \\ &\leq M|\Omega|^{\frac{1}{q}} + \|\mathbf{U}_b\|_{L^q} \|w\|^{\frac{p-1}{2p}}_{L^\infty} + \left(\frac{2p}{p-1}\right) \|b\|^{p-1}_{W^{1,p}} \|w\|^{\frac{(p-1)(p+1)}{2p}}_{L^\infty}, \end{aligned} \tag{3.35}$$

for any $w \in C(\overline{\Omega})$. Using (3.3), for $w \in S$ we have

$$\|w\|_{L^\infty} \leq \|\mathcal{A}(w)\|_{L^\infty} \leq C_2 \|\mathcal{A}(w)\|_{W^{1,p}}. \tag{3.36}$$

Since $0 < \frac{p-1}{2p} < \frac{(p-1)(p+1)}{2p} < p-1$, it follows from (3.35) and (3.36) that there exists $C > 0$ so that $\|w\|_{L^\infty} < C$. Thus S is bounded and, by Schaefer's theorem [11, section 9.2.2, Theorem 4], $u \in C(\bar{\Omega})$ exists so that $u = \mathcal{A}(u)$ and u solves (2.19).

It remains to show the a priori bound for any solution u of (2.19). By using (3.35) with $w = u \in C(\overline{\Omega})$, $\mathcal{A}(u) = u$, and (3.3), we obtain

$$D_1 \|u\|_{W^{1,p}}^{p-1} \le M|\Omega|^{\frac{1}{q}} + C_2 \|\mathbf{U}_b\|_{L^q} \|u\|_{W^{1,p}}^{\frac{p-1}{2p}} + \left(\frac{2p}{p-1}\right) C_2 \|b\|_{W^{1,p}}^{p-1} \|u\|_{W^{1,p}}^{\frac{(p-1)(p+1)}{2p}}.$$

Again using $0 < \frac{p-1}{2p} < \frac{(p-1)(p+1)}{2p} < p-1$, we conclude that $C > 0$ exists. ∎

Uniqueness in Theorem 3.6 remains open in any cases significantly beyond those covered by Corollary 3.4. However, Lemma 3.5 provides a second proof of uniqueness in the cases which can be addressed by monotone operator methods. Assumptions can be slightly relaxed. Indeed, assume $\mu = \mu(\mathbf{x})$, $\Phi = 0$, $\Psi = 0$, and that $\alpha = \alpha(\mathbf{x}, u)$ is a nonincreasing function of u (i.e., $a = a(\mathbf{x}, h)$ is a nonincreasing function of h). If both $u_1 \in \mathcal{K}$ and $u_2 \in \mathcal{K}$ solve (2.19), then setting $\tilde{u} = u_1$, $\overline{u} = u_2$, $\tilde{\alpha} = \alpha(u_1)$, and $\overline{\alpha} = \alpha(u_2)$ in Lemma 3.5, we obtain from (3.21)

$$D_2 \|u_1 - u_2\|_{W^{1,p}}^p \le \int_\Omega (\alpha(u_1) - \alpha(u_2))(u_1 - u_2) \le 0, \tag{3.37}$$

and then $u_1 = u_2$. Also using Lemma 3.5, uniqueness could be shown in the sliding case if the sliding basal velocity field $\mathbf{U}_b$ satisfies

$$\int_\Omega (\Psi(u) - \Psi(w)) \cdot (\nabla(u - w)) \le 0 \qquad \forall v, w \in \mathcal{K}, \tag{3.38}$$

recalling that $\Psi(u) = u^{(p-1)/(2p)} \mathbf{U}_b$. Unfortunately, nonincreasing functions $u \mapsto \alpha(\mathbf{x}, u)$ and sliding fields $\mathbf{U}_b$ satisfying condition (3.38) represent cases not commonly seen in glaciological applications. In contrast, the case of an increasing function α was addressed, for a flat bed only, in [20]. It appears that monotonicity arguments cannot be used to significantly extend the well-posedness result in Corollary 3.4.

3.4. Interior condition and regularity. Suppose that $u \in \mathcal{K}$ solves (2.19). Define an open set $O = \{\mathbf{x} \in \Omega,\, u(\mathbf{x}) > 0\}$, and suppose $\varphi \in C_c^\infty(O)$. Since u and φ are both continuous, there exists $\delta > 0$ such that $v = u + \epsilon\varphi \in \mathcal{K}$ for all $\epsilon \in [-\delta, \delta]$. Because (2.19) applies for both positive and negative ϵ, a PDE weak form holds:

$$\int_O \mu(u) \left(|\nabla u - \Phi(u)|^{p-2} (\nabla u - \Phi(u)) - \Psi(u)\right) \cdot \nabla\varphi = \int_O \alpha(u)\varphi \tag{3.39}$$

for any $\varphi \in C_c^\infty(O)$. If the solution u is also sufficiently differentiable, and if μ, Φ, Ψ, and α are sufficiently differentiable, then it follows by integration-by-parts that the strong form

$$-\nabla \cdot \left(\mu(u) |\nabla u - \Phi(u)|^{p-2} (\nabla u - \Phi(u)) - \Psi(u)\right) = \alpha(u) \tag{3.40}$$

applies at points in O. In glaciological terms, therefore, the ice thickness $H = u^{(p-1)/(2p)}$ solves the SIA, PDE (2.7), where it is positive.

This brings us to the regularity of solutions to (2.19). For the classical obstacle problem, namely $p = 2$, $\mu = 1$, $\Phi = 0$, and $\Psi = 0$ in (2.19), the solution is in $W^{2,\infty}(\Omega)$ [21]. This result does not apply to all p-Laplacian problems when $p > 2$, however.

Indeed, for the unobstructed p-Laplacian problem in dimension one, Glowinski and Marroco [14] show that $u \notin W^{2,p}(\Omega)$ if $p \geq (3+\sqrt{5})/2 \approx 2.6$, though $u \in W^{2,p}(\Omega)$ for smaller p.

Other authors [7, 24] show that if u solves the obstacle problem (2.19) with $\mu = 1$, $\Phi = 0$, and $\Psi = 0$, then there is $0 < \epsilon < 1$ so that $u \in C^{1,\epsilon}(\Omega)$. That is, u at least has a Hölder-continuous first derivative in the better-understood restricted cases.

3.5. On a property special to the ice sheet problem. An obvious property of a steady ice sheet is that if the steady surface mass balance a is positive at some point ($a(\mathbf{x}) > 0$ for $\mathbf{x} \in \Omega$), and if a is continuous, then $h(\mathbf{x}) > b(\mathbf{x})$ at that point. That is, if, in a steady climate, it snows at some location more than it melts, then there will be an ice sheet there. We may write $\{a > 0\} \subset \Omega_+$, where $\Omega_+ = \{h > b\}$ is defined in section 2. Though this is intuitive for an ice sheet in a steady climate, this property is outstanding among obstacle problems, as we now explain.

Consider a uniformly elliptic p-Laplacian variational inequality with $1 < p < \infty$, namely, an inequality of the form

$$\int_\Omega k|\nabla u|^{p-2}\nabla u \cdot \nabla(v-u) \geq \int_\Omega f(v-u) \tag{3.41}$$

for $k \in L^\infty(\Omega)$, $f \in L^q(\Omega)$, and $q = p/(p-1)$, where $k \geq \bar{k}$ a.e. for some $\bar{k} > 0$. The associated obstacle problem [7] seeks $u \in \mathcal{A}$ so that (3.41) holds for all $v \in \mathcal{A}$, where $\mathcal{A} = \{v \geq \psi\} \subset W_0^{1,p}(\Omega)$ is the (convex) admissible set and $\psi \in W^{1,p}(\Omega)$ with $\psi|_{\partial\Omega} \leq 0$ is the obstacle.

These obstacle problems generally allow upward force to be applied to the membrane (i.e., $f > 0$ in an open subset of Ω) even at locations where the membrane is in contact with the obstacle (i.e., $u = \psi$ where $f > 0$). If the obstacle is *flat*, $\psi \equiv 0$, however, and under the apparently technical assumption that $u \in W^{2,\infty}(\Omega)$, if u solves (3.41), then a "blistering" property holds: $f \in C(\Omega)$ and $f(\mathbf{x}) > 0$ implies $u(\mathbf{x}) > 0$. We demonstrate this as follows. Suppose $f(\mathbf{x}) > 0$ and $u(\mathbf{x}) = 0$. We first show that $u \equiv 0$ in a ball around $\mathbf{x}$. Because f is continuous, there is $\delta_0 > 0$ for which $f \geq \delta_0$ on a ball $B_\epsilon(\mathbf{x})$ with $\epsilon > 0$. Consider test functions $v = u + \varphi$ in (3.41), with $\varphi \in W_0^{1,p}(B_\epsilon(\mathbf{x}))$ nonnegative and extended by zero so that $\varphi \in W_0^{1,p}(\Omega)$ (e.g., section 5.4 of [11]). By the technical assumption we can integrate (3.41) by parts, which consists now of integrals on $B_\epsilon(\mathbf{x})$ only. We deduce

$$-\nabla \cdot (k|\nabla u|^{p-2}\nabla u) \geq f > 0 \qquad \text{a.e. in } B_\epsilon(\mathbf{x}).$$

The strong maximum principle for the p-Laplace operator [31, Theorem 5] now says that u cannot reach its minimum in $B_\epsilon(\mathbf{x})$ unless u is constant in this ball. Since $u(\mathbf{x}) = 0$ and $u \geq \psi \equiv 0$, we have $u \equiv 0$ on $B_\epsilon(\mathbf{x})$. Now let $U \in W_0^{1,p}(\Omega)$ such that $U = 0$ on $\Omega \setminus B_\epsilon(\mathbf{x})$ and $U > 0$ on $B_\epsilon(\mathbf{x})$. By setting $v = u + U \in \mathcal{A}$ in (3.41), we obtain

$$0 = \int_\Omega k|\nabla u|^{p-2}\nabla u \cdot \nabla U \geq \int_\Omega fU > 0,$$

a contradiction. The function $U \geq 0$ which arises above could be called a "blister."

Generic, nonflat obstacles may have a shape so that the solution u is against the obstacle even in areas where $f > 0$, however. Specifically, for each uniformly elliptic p-Laplacian variational inequality (3.41) there is an obstacle ψ so that the blistering property fails for that obstacle problem. To prove this, consider constant upward force

$f = \delta_0 > 0$ on all of Ω. Solve (3.41) (in the unconstrained case), yielding solution $\tilde{u} \in W_0^{1,p}(\Omega)$. Note $\tilde{u}(\mathbf{x}) > 0$ for every $\mathbf{x} \in \Omega$ by the strong maximum principle. Now let $\psi = \tilde{u}$. Then $u = \psi$ is the solution of (3.41) with obstacle ψ and $f = \delta_0 > 0$. The blistering property does not apply. In conclusion, the qualitative property of "blistering" is not generic across this large class of variational inequality obstacle problems, though it applies when the obstacle is flat.

The blistering property is essential to any realistic mathematical model of steady-state ice sheets because ice sheets exist even in highly localized areas of perennial snowfall. In form (2.19) this property is now not surprising because the obstacle is flat. In form (2.11) this property still holds because (2.11) is equivalent to (2.19), but it is much less obvious. Inequality (2.11) is not uniformly elliptic because of the extra nonlinearity in the coefficient (as pointed out at the start of subsection 2.3). Making (2.11) uniformly elliptic by a regularization $(h-b)^{p+1} \to (h-b+\epsilon)^{p+1}$ would destroy the blistering property.

3.6. Fixed points: Bounded iteration. The proof of Theorem 3.6 suggests that we should be able to construct a solution of (2.19) as the limit of a sequence of solutions of problem (3.11). More precisely, given $u_0 \in \mathcal{K}$, let $\{u_i\} \subset \mathcal{K}$ be the sequence constructed iteratively by finding $u_{i+1} \in \mathcal{K}$ such that

$$\int_\Omega \big(\mu(u_i)|\nabla u_{i+1} - \Phi(u_i)|^{p-2}(\nabla u_{i+1} - \Phi(u_i)) - \Psi(u_i)\big) \cdot \nabla(v - u_{i+1}) \geq \int_\Omega \alpha(u_i)(v - u_{i+1}) \tag{3.42}$$

for all $v \in \mathcal{K}$ and $i \in \mathbb{N}$. Indeed, each problem (3.42) is of type (3.11) with $k = \mu(u_i)$, $Z = \Phi(u_i)$, and $f = \alpha(u_i) - \nabla \cdot \Psi(u_i)$. The convergence of $\{u_i\}$ to a solution of (2.19) for a sufficiently close first iterate would follow from Banach's theorem if the map $\mathcal{A}$ defined in the proof of Theorem 3.6 were a contraction mapping. Although we cannot prove that property, at least the sequence is bounded.

THEOREM 3.7. *Given $u_0 \in \mathcal{K}$, let $\{u_i\} \subset \mathcal{K}$ be defined by* (3.42). *Then there exists $C > 0$ depending only on p, μ_1, μ_2, M, $\|\mathbf{U}_b\|_{L^q}$, $\|b\|_{W^{1,p}}$, and $\|u_0\|_{W^{1,p}}$ such that*

$$\|u_i\|_{W^{1,p}} < C \qquad \forall i \in \mathbb{N}. \tag{3.43}$$

Proof. In the notation of Lemma 3.5, note that $\overline{u} = \mathcal{T}(\overline{\mu}, \overline{\Phi}, \overline{\Psi}, \overline{\alpha}) = 0$ if $\overline{\mu} = 1$, $\overline{\Phi} = 0$, $\overline{\Psi} = 0$, and $\overline{\alpha} = 0$. Let $\tilde{u} = \mathcal{T}(\tilde{\mu}, \tilde{\Phi}, \tilde{\Psi}, \tilde{\alpha})$, where $\tilde{\mu} = \mu(u_i)$, $\tilde{\Phi} = \Phi(u_i)$, $\tilde{\Psi} = \Psi(u_i)$, and $\tilde{\alpha} = \alpha(u_i)$, so that $\tilde{u} = u_{i+1}$. From estimates (3.3), (3.7), (3.9), and Lemma 3.5 we obtain

$$\begin{aligned} D_1\|u_{i+1}\|_{W^{1,p}}^{p-1} &\leq \|\alpha(u_i)\|_{L^q} + \|\Psi(u_i)\|_{L^q} + \|\Phi(u_i)\|_{L^p}^{p-1} \\ &\leq M|\Omega|^{\frac{1}{q}} + c_1\|\mathbf{U}_b\|_{L^q}\|u_i\|_{W^{1,p}}^{\frac{p-1}{2p}} + c_2\|b\|_{W^{1,p}}^{p-1}\|u_i\|_{W^{1,p}}^{\frac{(p-1)(p+1)}{2p}} \end{aligned}$$

for some $c_1, c_2 > 0$. Taking the $\rho = 1/(p-1) \in (0,1]$ power, and noting $(a+b)^\rho \leq a^\rho + b^\rho$ for $a, b \geq 0$, we obtain

$$\|u_{i+1}\|_{W^{1,p}} \leq c_3 + c_4\|u_i\|_{W^{1,p}}^{\frac{1}{2p}} + c_5\|u_i\|_{W^{1,p}}^{\frac{p+1}{2p}}$$

for c_i depending on the identified quantities. Therefore we may choose $\bar{C} > 0$ so that

$$\|u_{i+1}\|_{W^{1,p}} \leq \frac{\bar{C}}{3}\left(1 + \|u_i\|_{W^{1,p}}^{\frac{1}{2p}} + \|u_i\|_{W^{1,p}}^{\frac{p+1}{2p}}\right).$$

Let $m = (p+1)/(2p)$. If $\|u_i\|_{W^{1,p}} \leq 1$, then $\|u_{i+1}\|_{W^{1,p}} \leq \bar{C}$, while if $\|u_i\|_{W^{1,p}} > 1$, then $\|u_{i+1}\|_{W^{1,p}} < \bar{C}\|u_i\|^m_{W^{1,p}}$, because $m > 1/(2p)$. We have

$$\|u_{i+1}\|_{W^{1,p}} \leq \bar{C} \max(1, \|u_i\|^m_{W^{1,p}}).$$

Now we assume, without loss of generality, that $\bar{C} > 1$. By induction,

$$\|u_{i+1}\|_{W^{1,p}} \leq \bar{C}^{1+m+\cdots+m^i} \max(1, \|u_0\|^{m^{i+1}}_{W^{1,p}}).$$

Noting $m < 1$ so $1 + m + \cdots + m^i \leq 1/(1-m)$ and $\|u_0\|^{m^{i+1}}_{W^{1,p}} \leq \max(1, \|u_0\|_{W^{1,p}})$, the result follows. □

4. Finite element approximation. The nonlinear problem (2.19) cannot be directly approximated by finite element techniques. Indeed, if we replace the functional spaces in the variational formation by conforming approximation spaces, the nonlinearities that are not representable by minimization cannot be solved using standard techniques. For this reason, we approximate the nonlinear problem (2.19) by combining the iteration scheme defined by (3.42) and a finite element method [8] to solve each individual problem of form (3.11). The current section is devoted to the numerical approximation of the subproblems of type (3.11).

Suppose Ω is polygonal and that we have a family of triangulations $\mathcal{T}_{\mathtt{h}}$ which exactly cover Ω. Let $\mathtt{h}_\tau$ be the largest edge length of a triangle τ, and let ρ_τ be the diameter of an inscribed circle in τ. Each triangulation $\mathcal{T}_{\mathtt{h}}$ is assumed to be regular in the sense that the ratio $\mathtt{h}_\tau/\rho_\tau$ is uniformly bounded above, so that triangles do not become arbitrarily flat. Let $\mathtt{h} = \max_{\tau\in\mathcal{T}_{\mathtt{h}}} \mathtt{h}_\tau$, and suppose the family $\mathcal{T}_{\mathtt{h}}$ includes the $\mathtt{h} \to 0$ limit of uniform refinement. Let $\mathcal{X}_{\mathtt{h}} \in \mathcal{X}$ be space of the continuous functions which are linear on each triangle of $\mathcal{T}_{\mathtt{h}}$ ("P_1 finite elements" [8]) and have boundary values zero. Let $\pi_{\mathtt{h}} : C(\overline{\Omega}) \to \mathcal{X}_{\mathtt{h}}$ denote the interpolation operator such that for all $v \in C(\overline{\Omega})$, $v = \pi_{\mathtt{h}}(v)$ at each node of $\mathcal{T}_{\mathtt{h}}$. We recall the following standard approximation result [8, Theorem 3.1.6]: There exists $C > 0$ such that for $m = 0, 1$, all $\bar{p} \in [p, +\infty]$, and all $v \in W^{2,\bar{p}}$ we have

$$\|v - \pi_{\mathtt{h}}(v)\|_{W^{m,p}} \leq Ch^{2-m}|\Omega|^{(1/p)-(1/\bar{p})}\|v\|_{W^{2,\bar{p}}}. \tag{4.1}$$

(Actually, [8, Theorem 3.1.6] gives the local version of (4.1) on each element, but since the triangulations $\mathcal{T}_{\mathtt{h}}$ exactly cover Ω we obtain the global approximation result by summing the local estimates over all elements.)

Let $\mathcal{K}_{\mathtt{h}} = \mathcal{X}_{\mathtt{h}} \cap \mathcal{K}$ be the restriction of $\mathcal{X}_{\mathtt{h}}$ to positive functions. Since $\mathcal{K}_{\mathtt{h}}$ is a closed convex set of $\mathcal{K}$, the existence and uniqueness result proved in the continuous case is still valid with $\mathcal{K}$ replaced by $\mathcal{K}_{\mathtt{h}}$. Let $u_{\mathtt{h}} \in \mathcal{K}_{\mathtt{h}}$ be the unique solution of

$$\int_\Omega k|\nabla u_{\mathtt{h}} - \mathbf{Z}|^{p-2}(\nabla u_{\mathtt{h}} - \mathbf{Z}) \cdot \nabla(v_{\mathtt{h}} - u_{\mathtt{h}}) \geq \int_\Omega f(v_{\mathtt{h}} - u_{\mathtt{h}}) \qquad \forall v_{\mathtt{h}} \in \mathcal{K}_{\mathtt{h}}, \tag{4.2}$$

for $k \in L^\infty(\Omega)$, $\mathbf{Z} \in [L^p(\Omega)]^2$, and $f \in L^q(\Omega)$. Equivalently,

$$J(u_{\mathtt{h}}) \leq J(v_{\mathtt{h}}) \qquad \forall v_{\mathtt{h}} \in \mathcal{K}_{\mathtt{h}}, \tag{4.3}$$

where J is defined by (3.12). Applying Lemma 3.5 with $\tilde{\mu} = k$, $\tilde{\Phi} = \mathbf{Z}$, $\tilde{\Psi} = 0$, $\tilde{\alpha} = f$, $\tilde{u} = u_{\mathtt{h}}$, $\overline{\mu} = 1$, $\overline{\Phi} = 0$, $\overline{\Psi} = 0$, $\overline{\alpha} = 0$, $\overline{u} = 0$, and $\mathcal{K}_{\mathtt{h}}$ in place of $\mathcal{K}$, there exists $C > 0$ that depends continuously on data $\|f\|_{L^q}$, $\|\mathbf{Z}\|_{L^p}$ such that

$$\|u_{\mathtt{h}}\|_{W^{1,p}} \leq C. \tag{4.4}$$

Thus we have a bound independent of $\mathtt{h}$ on norms of solutions to (4.2).

The convergence of the finite element solution to the continuous solution can be established by adapting the proof of Theorem 5.3.2 in [8].

THEOREM 4.1. *For the solutions $u \in \mathcal{K}$ and $u_{\mathtt{h}} \in \mathcal{K}_{\mathtt{h}}$ to* (3.11) *and* (4.2)*, respectively, we have $u_{\mathtt{h}} \to u$ in $W^{1,p}(\Omega)$ as $\mathtt{h} \to 0$. That is, the finite element approximation converges in $W^{1,p}(\Omega)$ and thus in $L^\infty(\Omega)$.*

The result is abstract in the sense that we do not have a bound on the error $\|u - u_{\mathtt{h}}\|$ for a given $\mathtt{h} > 0$. We now prove an a priori error estimate by following the quasi norm technique used by Barrett and Liu [3, 23]. Error estimates for linear variational inequalities were first established in [12]; see also [8]. A priori error estimates for a p-Laplace problem were obtained in [3]. The p-Laplacian obstacle problem was considered in [23].

The next lemma is a generalization of Cea's lemma [8]. For its proof, define the solution-dependent functional

$$v \mapsto |||v|||_u := \int_\Omega k(|\nabla u - \mathbf{Z}| + |\nabla v|)^{p-2} |\nabla v|^2. \tag{4.5}$$

We can show [3] that $|||\cdot|||_u$ is a quasi norm; i.e., it satisfies all properties of the norm except homogeneity. Moreover, using Hölder's inequality, we can show that there exist $D_1, D_2 > 0$ such that, for all $v \in W^{1,p}(\Omega)$, we have

$$D_1 \|v\|_{W^{1,p}}^p \le |||v|||_u \le D_2 \Big[\|\nabla u - \mathbf{Z}\|_{L^p} + \|\nabla v\|_{L^p} \Big]^{p-2} \|v\|_{W^{1,p}}^2. \tag{4.6}$$

LEMMA 4.2. *Let u be the solution of* (3.11) *and $u_{\mathtt{h}}$ be the solution of* (4.2)*. Moreover, assume $\nabla \cdot (k|\nabla u - \mathbf{Z}|^{p-1}(\nabla u - \mathbf{Z})) + f \in L^q$. Then there exists $D > 0$ such that, for all $v_{\mathtt{h}} \in \mathcal{K}_{\mathtt{h}}$,*

$$\begin{aligned} D\|u - u_{\mathtt{h}}\|_{W^{1,p}}^p \le {} & [\|\nabla u - \mathbf{Z}\|_{L^p} + \|\nabla(u - v_{\mathtt{h}})\|_{L^p}]^{p-2} \|u - v_{\mathtt{h}}\|_{W^{1,p}}^2 \\ & + \|\nabla \cdot (k|\nabla u - \mathbf{Z}|^{p-1}(\nabla u - \mathbf{Z})) + f\|_{L^q} \|u - v_{\mathtt{h}}\|_{L^p}. \end{aligned} \tag{4.7}$$

Proof. By definition (4.5), we have

$$|||u - u_{\mathtt{h}}|||_u = \int_\Omega k(|\nabla u - \mathbf{Z}| + |\nabla(u - u_{\mathtt{h}})|)^{p-2} |\nabla(u - u_{\mathtt{h}})|^2 \tag{4.8}$$

$$\le 2^{p-2} \int_\Omega k(|\nabla u - \mathbf{Z}| + |\nabla u_{\mathtt{h}} - \mathbf{Z}|)^{p-2} |(\nabla u - \mathbf{Z}) - (\nabla u_{\mathtt{h}} - \mathbf{Z})|^2. \tag{4.9}$$

Owing to inequality (3.5), we have

$$\begin{aligned} & |||u - u_{\mathtt{h}}|||_u \\ & \le \frac{2^{p-2}}{M_2} \int_\Omega k(|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z}) - |\nabla u_{\mathtt{h}} - \mathbf{Z}|^{p-2}(\nabla u_{\mathtt{h}} - \mathbf{Z})) \cdot \nabla(u - u_{\mathtt{h}}) \\ & = \frac{2^{p-2}}{M_2}(E_1 + E_2), \end{aligned}$$

where (adding and subtracting $v_{\mathtt{h}}$)

$$\begin{aligned} E_1 &:= \int_\Omega k(|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z}) - |\nabla u_{\mathtt{h}} - \mathbf{Z}|^{p-2}(\nabla u_{\mathtt{h}} - \mathbf{Z})) \cdot \nabla(u - v_{\mathtt{h}}), \\ E_2 &:= \int_\Omega k(|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z}) - |\nabla u_{\mathtt{h}} - \mathbf{Z}|^{p-2}(\nabla u_{\mathtt{h}} - \mathbf{Z})) \cdot \nabla(v_{\mathtt{h}} - u_{\mathtt{h}}). \end{aligned}$$

For the sake of simplicity, E_1 and E_2 are handled separately. Owing to inequality (3.4), we have

$$\begin{aligned} E_1 &\le M_1 \int_\Omega k(|\nabla u - \mathbf{Z}| + |\nabla u_{\mathtt{h}} - \mathbf{Z}|)^{p-2} |\nabla(u - u_{\mathtt{h}})||\nabla(u - v_{\mathtt{h}})| \\ &\le 2^{p-2} M_1 \int_\Omega k(|\nabla u - \mathbf{Z}| + |\nabla u - \nabla u_{\mathtt{h}}|)^{p-2} |\nabla(u - u_{\mathtt{h}})||\nabla(u - v_{\mathtt{h}})|. \end{aligned}$$

By using the inequality (see Lemma 2.2 in [22])

$$(a+r)^{p-2}rs \le \epsilon(a+r)^{p-2}r^2 + \epsilon^{-1}(a+s)^{p-2}s^2 \qquad \forall a,r,s \ge 0,\ \forall \epsilon \in (0,1],$$

with $a = |\nabla u - \mathbf{Z}|$, $r = |\nabla u - \nabla u_{\mathtt{h}}|$, $s = |\nabla u - \nabla v_{\mathtt{h}}|$, we obtain, for all $\epsilon \in [0,1]$,

$$E_1 \le 2^{p-2}M_1(\epsilon|||u-u_{\mathtt{h}}|||_u + \epsilon^{-1}|||u-v_{\mathtt{h}}|||_u). \tag{4.10}$$

On the other hand, E_2 may be rewritten as

$$\begin{aligned}E_2 = &\int_\Omega k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})\cdot\nabla(v_{\mathtt{h}} - u) + \int_\Omega k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})\cdot\nabla(u - u_{\mathtt{h}})\\ &- \int_\Omega k|\nabla u_{\mathtt{h}} - \mathbf{Z}|^{p-2}(\nabla u_{\mathtt{h}} - \mathbf{Z})\cdot\nabla(v_{\mathtt{h}} - u_{\mathtt{h}}).\end{aligned}$$

Using (3.11) with $v = u_{\mathtt{h}}$ and (4.2), we obtain

$$E_2 \le \int_\Omega k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})\cdot\nabla(v_{\mathtt{h}} - u) - \int_\Omega f(v_{\mathtt{h}} - u).$$

After integrating by parts, and applying Hölder's inequality, we obtain

$$E_2 \le \|\nabla\cdot(k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})) + f\|_{L^q}\|u - v_{\mathtt{h}}\|_{L^p}. \tag{4.11}$$

Fixing ϵ small enough, we obtain from (4.10) and (4.11)

$$|||u-u_{\mathtt{h}}|||_u \le C\{|||u-v_{\mathtt{h}}|||_u + \|\nabla\cdot(k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})) + f\|_{L^q}\|u-v_{\mathtt{h}}\|_{L^p}\}, \tag{4.12}$$

where $C > 0$ is a constant independent of u, $u_{\mathtt{h}}$, and $v_{\mathtt{h}}$. Inequality (4.7) follows from (4.6) and (4.12). ∎

The lemma above, when combined with the standard approximation result (4.1), the a priori bound (4.4), and the continuity of $\pi_{\mathtt{h}}$, gives the following convergence theorem.

THEOREM 4.3. *Let u be the solution of* (3.11) *and $u_{\mathtt{h}}$ be the solution of* (4.2). *Moreover, assume $u \in W^{2,p}$ and $\nabla\cdot(k|\nabla u - \mathbf{Z}|^{p-2}(\nabla u - \mathbf{Z})) + f \in L^q$. There exists $D > 0$ that depends on u but not on $\mathtt{h}$ such that*

$$\|u - u_{\mathtt{h}}\|_{W^{1,p}} \le D\,\mathtt{h}^{2/p}. \tag{4.13}$$

Convergence gets worse with increasing p. Also, Theorem 4.3 requires $W^{2,p}$ regularity that we cannot ensure; see section 3.4. However, if we have lower regularity, we can still give a weaker error estimate following [3, section 4]. Specifically, if $u \in W^{2,s}$ for $s \in [1,2]$, then the same conclusion (4.13) holds but with power "$\mathtt{h}^{s/p}$" in the bound.

5. Numerical applications. We first investigate the convergence of the finite element approximation, relative to the prediction of Theorem 4.3, using an exact solution to the $k = 1$ and $\mathbf{Z} = \mathbf{0}$ case of problem (3.11). Specifically, define the radial function $\tilde{u}(r)$ on the square $\Omega = [-1,1]^2$, where $r = |\mathbf{x}|$, $R = 0.75$, $s = r/R$, by the formula

$$\tilde{u}(r) = 1 - \frac{p-1}{p-2}\left(s^{p/(p-1)} - (1-s)^{p/(p-1)} + 1 - \left(\frac{p}{p-1}\right)s\right) \tag{5.1}$$

if $r < R$, and as zero if $r \geq R$ [16, subsection 5.6.3]. This defines $u(\mathbf{x}) = \tilde{u}(r)$, which is not in $W^{2,2}(\Omega)$ but is in $W^{2,\lambda}(\Omega)$ for any $\lambda < \frac{p-1}{p-2}$. The right-hand side f of (3.11) is chosen to make the PDE (interior condition) true on $r < R$, and it is extended by the value $f(R)$; thus f is continuous on Ω.

We build a coarse regular triangulation of $\Omega = [-1, 1]^2$ with a cell size of $h = 0.25$, and then refine uniformly five times by $\mathtt{h}' = \mathtt{h}/2$. For each mesh, a numerical solution $u_{\mathtt{h}}$ of (4.2) is computed using a projected nonlinear Gauß–Seidel method [9]. The estimate of Theorem 4.3, namely $\mathcal{O}(\mathtt{h}^{2/p})$ convergence, is tested by computing $\|u - u_{\mathtt{h}}\|_{W^{1,p}}$ for these meshes. Figure 5.1 displays errors in cases $p = 3$, $p = 4$, and $p = 6$. The observed convergence is stronger than anticipated from Theorem 4.3, which is therefore not optimal, but the rate of convergence appears to decrease with increasing p, as expected.

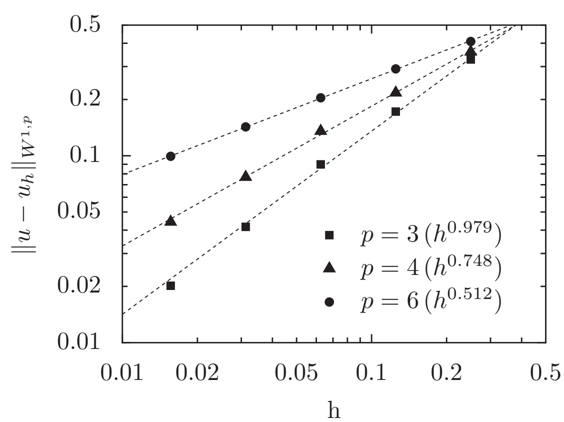

FIG. 5.1. *Convergence of the finite element method for an exact solution.*

5.1. Radial case with nonflat bedrock. Now we solve the ice sheet problem (2.19) in the case of smooth but nonflat bedrock. This illustrates the convergence of fixed point iteration (3.42). The projected nonlinear Gauß–Seidel method [9] is again used for the generalized p-Laplace obstacle problem (3.11) at each step. For simplicity in this and the next subsection, we assume that no basal sliding occurs, that the mass balance is elevation-independent, and that the ice softness A is constant. We use the following values from [19]: $p = 4$, $\rho = 910$ kg m^{-3}, $g = 9.81$ m s^{-2}, and $A = 10^{-16}$ Pa^{-3} a^{-1}.

To define a radial ice sheet, let $u(r) = H_0^{2p/(p-1)}\tilde{u}(r)$ and $b(r) = -b_0 \cos(z_0 \pi r/R)$, where $\tilde{u}(r)$ is defined by (5.1) and where $H_0 = 3$ km, $R = 750$ km, $b_0 = 500$ m, and $z_0 = 1.2$. These functions are defined on $\Omega = [-L, L]^2$, with $L = 1000$ km. The right-hand side of (2.19), namely the function α (i.e., the mass balance a), is found using symbolic computation so that the interior condition is satisfied. Thus u exactly solves (2.19).

Let $u_0 = 0$ be the initial iterate, and run scheme (3.42). The left column of Figure 5.2 shows iterates u_1, u_2, u_3 resulting from scheme (3.42), along with the exact solution u. Figure 5.3 displays the L^2 relative error in u and in the surface elevation $h = b + H = b + u^{(p-1)/(2p)}$. The convergence rate in surface elevation is worse than in variable u, due to the singular surface gradient near the ice sheet margin. Closer inspection shows large L^∞ error just inside the circle of radius R; compare [5] in the

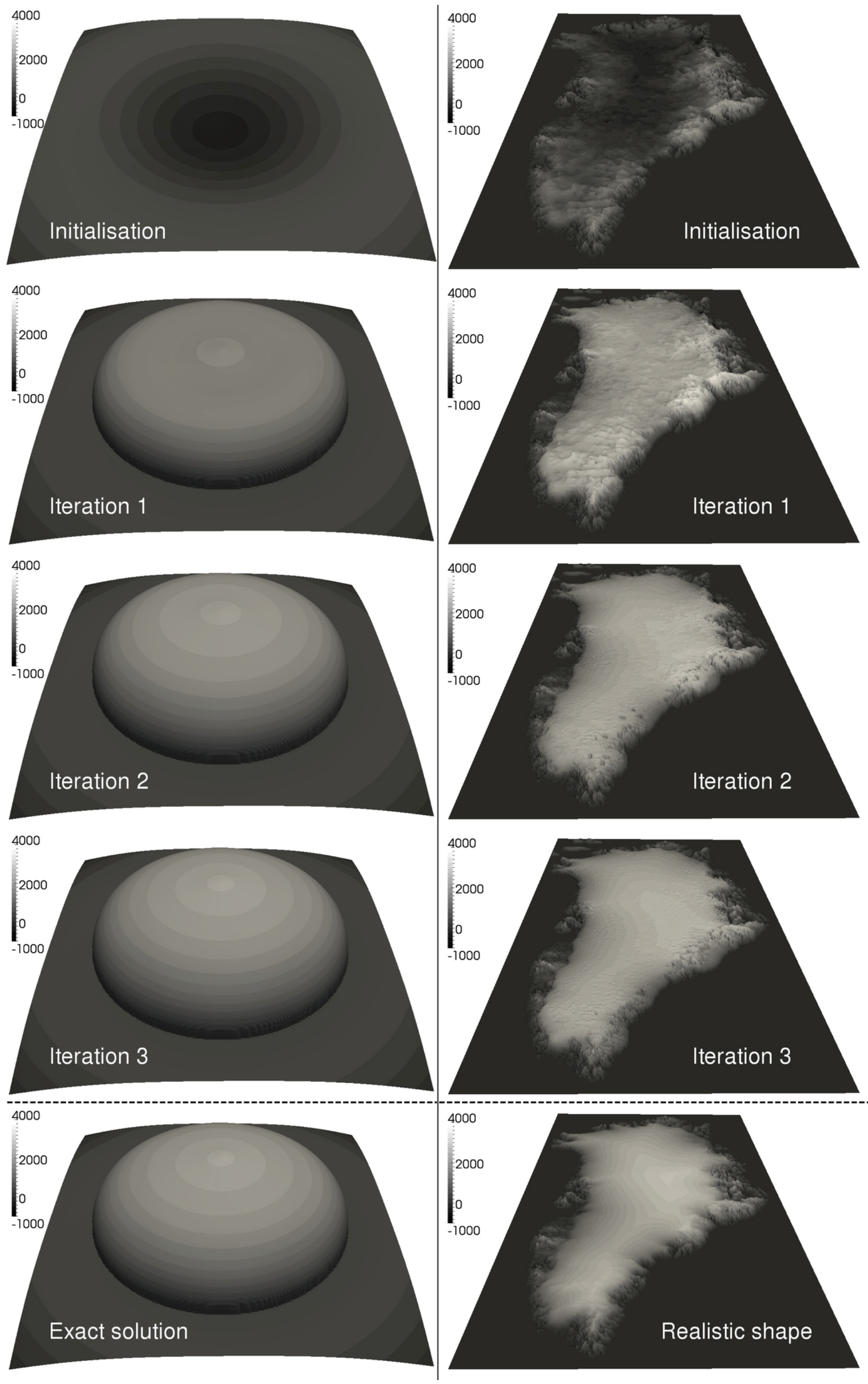

FIG. 5.2. *The four first iterates of the fixed point algorithm. Left: case of smooth, radial bedrock; exact solution at bottom. Right: Greenland case with bedrock and mass balance data; observed surface elevation at bottom.*

flat bedrock case. The steady solution is observed to be insensitive to the choice of initial iterate (not shown), which suggests the uniqueness of the solution of (2.19) under these conditions with a smooth bedrock; compare [20] for flat bedrock.

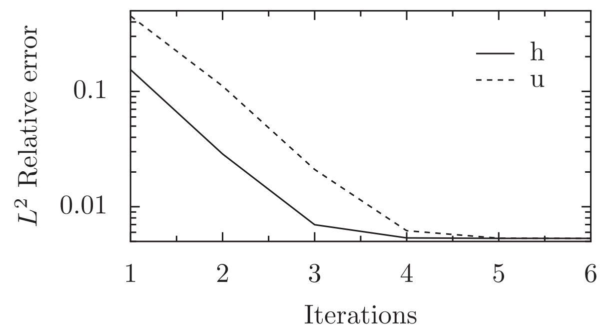

FIG. 5.3. *L^2 relative numerical error for transformed thickness ($u = H^{(2p)/(p-1)}$) and surface elevation ($h = b + H$), in the case of a radial ice sheet.*

5.2. Greenland. The Greenland ice sheet is almost 2400 km long in a north-south direction, and its greatest width is 1100 km. Gridded data for the bedrock topography b [2] and the mass balance a [10] are available at 5 km resolution. These were used in the computation, which was otherwise the same as in subsection 5.1. Figure 5.2 (right column) shows the first iterates from the fixed point algorithm (3.42). The steady state shape is similar to the observed surface elevation [2].

6. Conclusion. The numerical modeling of shallow ice sheets and glaciers has, until now, proceeded via computation of time-dependent geometry changes, even when the simulation goal is steady geometry [19, 25]. Excepting references [6, 20], which apply only to flat bedrock, all such simulations have used ad hoc treatment of the grounded margin shape as a boundary condition. The current paper describes, in contrast, a mathematical formulation of the steady state of shallow ice sheets on nontrivial bedrock topography. This weak formulation says that particular boundary conditions (zero thickness and zero normal flux) apply at margins, because these margins are the free boundaries in a problem which is globally an obstacle problem. We also include variable ice softness, basal sliding, and elevation-dependent surface mass balance within the same formulation. We make progress toward the well-posedness of this formulation by showing that solutions exist. Fundamental issues of uniqueness, regularity, and stability remain.

We conceive of bedrock topography as an obstacle to the surface elevation of the ice sheet. Equivalently, ice sheet thickness must be nonnegative. Bedrock topography is also a barrier to mathematical progress, however, because the p-Laplacian form is "tilted" by the change $\nabla u \mapsto \nabla u - \mathbf{Z}$, with $\mathbf{Z}$ proportional to the bedrock gradient, in the weak form of the problem. This apparently defeats certain mathematical tools, including the monotone operator theory. It has blocked mathematical progress in other models also [1]. This barrier is partially overcome here, as we find sufficient continuity to apply a fixed point argument.

Our steady results can be applied to the time-dependent case by taking implicit time-steps, and, with small modifications, the methods of this paper show existence for each step in a time semidiscretized problem. As another extension of this work, one could couple the ice sheet flow model used here to a model for the temperature of ice through fixed point iterations involving the ice softness. Similarly one could couple it to a membrane stress balance which determines the sliding velocity [4]. We believe that a similar fixed point approach could be applied, provided that the existence and the uniqueness of each single model is established, and together with a continuity result with respect to data similar to Lemma 3.5.

We also demonstrate convergence of, and an a priori error bound for, a finite element approximation of each subproblem in a fixed point iteration to solve the complete model. The realistic Greenland ice sheet calculation in section 5 is, to our knowledge, the first computation of ice sheet geometry on nontrivial bedrock, in balance with climate, which avoids computing a time-dependent sequence of physical states. A few iterations in our scheme avoids thousands of model years of stability-limited explicit time-steps as is done in existing ice sheet models.

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