The 2–Lagrange Multiplier Method Applied to Nonlinear Transmission Problems for the Richards Equation in Heterogeneous Soil with Cross Points

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THE 2-LAGRANGE MULTIPLIER METHOD APPLIED TO NONLINEAR TRANSMISSION PROBLEMS FOR THE RICHARDS EQUATION IN HETEROGENEOUS SOIL WITH CROSS POINTS

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Abstract. We formulate the 2-Lagrange multiplier method for the Richards equation in heterogeneous soil. This allows a rigorous formulation of a discrete version of the Richards equation on subdomain decompositions involving cross points. Using Kirchhoff transformation, the individual subdomain problems can be transformed to convex minimization problems and solved efficiently using a monotone multigrid method. We discuss and compare weak formulations of the time-discrete and fully discretized multi-domain problem. It is shown that in the case of two subdomains, when solving the resulting discrete system with a Richardson iteration, the new method is equivalent to the Robin method for the Richards equation proposed in [6]. We give numerical results for a problem with realistic soil parameters.

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1. Introduction. The Richards equation for unsaturated porous media flow poses considerable numerical difficulties [1, 12, 17, 20, 34, 40]. In particular, the straightforward treatment by Newton's method suffers from ill-conditioned Newton matrices. This ill-conditioning is inherent in the problem, caused by large derivatives of some of the hydrological parameter functions.

A different approach was proposed in [8]. There it was shown that, upon rescaling the time-discrete problem pointwise by Kirchhoff transformation, the transformed problem is equivalent to a convex minimization problem. This minimization problem can be solved robustly and efficiently by the monotone multigrid method [22]. An inverse Kirchhoff transformation then yields the solution of the original problem.

Unfortunately, in general the Kirchhoff transformation only leads to a convex minimization problem if the saturation and permeability functions do not depend on space. This is a fairly restrictive homogeneity assumption. In [7], Berninger et al. proposed a solution for the case where the functions are space-independent only on each subdomain of a partition of the full domain. Then Kirchhoff transformation could be applied separately on each subdomain. The individual subproblems were tied together by nonlinear transmission conditions, and the resulting coupled system could be solved by a variety of domain decomposition solvers, see also [4, 6].

In [4, 6], this domain decomposition approach was formulated for the case of layered soil. It was not clear in the discrete setting how that approach should be formulated in the presence of cross points, i.e., points where more than two subdomains meet. Layered soil types can be justified only in a small subset of geohydrological applications. It is the purpose of this paper to extend the domain decomposition approach to situations which include cross points. To this end, we generalize the 2-Lagrange multiplier method in [24] from the linear setting to our nonlinear problem.

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The regular Schwarz iteration combines local subdomain solutions by using Dirichlet boundary conditions on the interfaces between the subdomains; see, e.g., [29, 38]. Instead of using Dirichlet boundary conditions on the interface between the subdomains, one can use Robin data (a linear combination of the Dirichlet and Neumann data). Under certain circumstances, an optimal value for the parameter in the linear combination can be estimated analytically. The resulting method is called optimized Schwarz method (OSM) (cf. [14, 18] and references therein).

The 2-Lagrange multiplier (2LM) method [16, 32] is a method which is "dual" to the optimized Schwarz method in the following sense. It also uses Robin problems for the subdomain problems, but the iterates are Robin traces $\lambda_k^{(j)} \in H^{-\frac{1}{2}}(\partial\Omega_k)$ instead of functions $u_k^{(j)} \in H^1(\Omega_k)$. One can go from one to the other by solving local Robin problems. By itself, the 2LM method is just a reformulation of the original problem in terms of the Robin traces $\lambda_1, \ldots, \lambda_N$. The reformulated problem can be solved with a number of Krylov-type methods. If instead the Richardson iteration is used, then the resulting algorithm is equivalent to the optimized Schwarz method [19]. The analysis of the 2LM method for general domains and subdomains can be found in [13, 24, 25].

The 2LM formulation allows to treat cross points in a natural way. The issue of cross points has long been solved in FETI algorithms [15], and they can be treated fairly easily for additive Schwarz methods. However, for technical reasons methods with Robin boundary conditions have had more difficulty dealing with cross points and it is only recently that a systematic approach has been devised [24]. Our method is based on this approach and hence we are able to handle cross points for the Richards equation in the same way.

The basis of this article is the following fundamental observation: The 2LM formulation relies on the unique solvability of local Robin subdomain problems, but it does not rely on the linearity of these problems. This insight allows us to introduce a 2LM method for the Richards equation. It is the first time that the 2LM method is used to solve a nonlinear problem. More concretely, if the parameter functions of the Richards equation are space-independent on each subdomain of a nonoverlapping decomposition of the domain, then we can formulate the 2LM method with respect to this decomposition. We can then again turn each subdomain problem into a convex minimization problem using the Kirchhoff transformation. This allows us to efficiently solve the Richards equation on partitions with cross points, and still use the fast multigrid method of Berninger et al. [8] for each subdomain problem. Note that this approach extends easily to other equations with a similar structure, such as Stefan problems [22].

For the Richards and Stefan problems the 2LM formulation allows efficient solution algorithms by splitting the domain into subdomains with space-independent parameter functions. However, the method can also be used to parallelize these equations on distributed machines. As shown in Section 5.3, the different subdomain problems can be distributed easily on different processors. From this point of view, the method becomes attractive even for nonlinear problems without physically given interfaces, as, e.g., the Allen–Cahn equation for phase transitions. In such cases, the domain would be decomposed only to distribute the load across multiple processors.

The linear 2LM problem is usually solved with a Krylov method such as GMRES [33]. This does not work in the nonlinear situation. Instead of these Krylov space algorithms, we use the Minimum Polynomial Extrapolation [36] convergence acceleration technique which allows to treat nonlinear problems.

It is well known from the linear setting that a domain decomposition method will not scale well to a large number of subdomains unless there is an extra coarse grid correction step. The 2LM method as presented here is no exception and, therefore, we only give numerical results for a small number of subdomains. An efficient coarse grid correction for the linear 2LM method has been proposed in [25]. Ongoing work on this topic is to extend that approach to our nonlinear setting.

We proceed as follows. In Chapter 2, we introduce a multi-domain problem for the Richards equation after application of our time discretization. We prove equivalence results for different weak multi-domain formulations revealing that, in contrast to fully discretized settings, cross points do not play a significant role in spatially continuous problems. In Chapter 3, we then make the assumption that the parameter functions are space-independent on each subdomain. After Kirchhoff transformation and space discretization we arrive at a proper algebraic form of the domain decomposition problem with cross points. Chapter 4 reformulates this problem in the 2-Lagrange multiplier way. The new variables are now multi-valued traces on the skeleton of the partition, and can be interpreted as Robin boundary values for the individual subdomain problems. Chapter 5 discusses how we solve the 2LM system. We briefly describe the Minimal Polynomial Extrapolation (MPE) method to speed up the convergence of a Richardson iteration, and show how the 2LM operator can be evaluated efficiently. Chapter 6 shows that if the resulting system for the multi-valued trace is solved with a Richardson iteration, and if the partition contains only two subdomains, then the resulting algorithm is equivalent to the nonlinear Robin method of [6]. This generalizes a corresponding result for the linear case [19]. We close with a numerical example in Chapter 7.

2. Multi-domain formulation for the Richards equation. In this section we introduce the Richards equation, and a particular time discretization. We then set up a nonoverlapping partition of the domain and reformulate the Richards equation as a set of subdomain problems and suitable coupling conditions. We prove equivalence of this formulation to the weak form of the global problem.

2.1. The model problem. Let Ω be a Lipschitz domain in $\mathbb{R}^d$, $d = 1, 2, 3$. We consider the Richards equation [3, 11, 31]

$$n(x)\,\frac{\partial}{\partial t}\,\theta(x,p) + \operatorname{div}\mathbf{v}(x,p) = 0\,, \quad \mathbf{v}(x,p) = -K_h(x)\,\mathrm{kr}(x,\theta(x,p))\nabla(p-z)$$

for a scalar pressure field p on Ω. For simplicity we assume homogeneous Dirichlet boundary conditions. The bounded functions $n(x) \in (0,1]$ and $K_h(x) > 0$ are the porosity and the hydraulic conductivity, respectively, and z is the vertical component of $x \in \Omega$, directed downwards.

At each $x \in \Omega$, the saturation θ is a function of p, and the relative permeability kr is a function of θ, both given by certain equations of state depending on the soil type. In principle, it is enough to require θ to be Lipschitz continuous and increasing with respect to p, and kr to be a positive L^∞-function with respect to θ. For certain assertions we also need kr to be bounded away from zero. In our numerical example

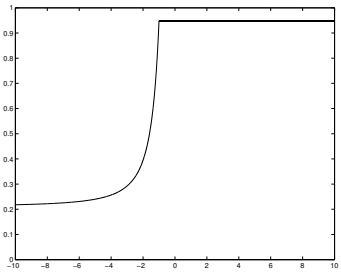

FIG. 2.1. $p \mapsto \theta(x,p)$

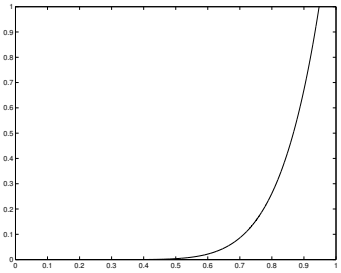

FIG. 2.2. $\theta \mapsto \mathrm{kr}(x,\theta)$

we choose the parameter functions according to Brooks and Corey [10, 39] given by

$$\theta(x,p) = \begin{cases} \theta_m + (\theta_M - \theta_m)\left(\frac{p}{p_b}\right)^{-\lambda} & \text{for } p \le p_b\,, \\ \theta_M & \text{for } p \ge p_b\,, \end{cases} \tag{2.1}$$

$$\mathrm{kr}(x,\theta) = \left(\frac{\theta - \theta_m}{\theta_M - \theta_m}\right)^{3+\frac{2}{\lambda}} \qquad \text{for } \theta \in [\theta_m, \theta_M]\,,$$

which typically look like the functions displayed in Figures 2.1 and 2.2. The soil parameters $\theta_m, \theta_M \in [0,1]$ are the minimal and maximal saturation, respectively, p_b is the bubbling pressure, and λ the pore size distribution factor. In the general case, all these parameters can depend on the position x.

2.2. Time discretization. Let $0 = t_0 < t_1 < \cdots < t_L = T$ be a partition of the time interval $[0,T]$ with time step sizes $\tau_i = t^{i+1} - t^i$, $0 \le i < L$. We apply a discretization in time to the Richards equation which is explicit only for the gravitational term, and implicit for all others. That way, we obtain the spatial problems

$$n(x)\,\theta(x,p^{i+1}) - \tau_i \,\mathrm{div}\left(K_h(x)\,\mathrm{kr}(x,\theta(x,p^{i+1}))\nabla p^{i+1}\right) = \\ n(x)\,\theta(x,p^i) - \tau_i \,\mathrm{div}\left(K_h(x)\,\mathrm{kr}(x,\theta(x,p^i))\nabla z\right)$$

for p^{i+1} at the time step t^{i+1}.

The functions n and K_h on Ω do not play a major role for the investigations in this work. For simplicity, but without loss of generality, we set them to 1 from now on. We also set $\tau_i = 1$, and drop the index i, which leaves us with the spatial problem

$$\theta(x,p) - \mathrm{div}\left(\mathrm{kr}(x,\theta(x,p))\nabla p\right) = f \qquad \text{on } \Omega\,, \tag{2.2}$$

with homogeneous Dirichlet data and an appropriately defined right hand side f.

2.3. Global spatial problem in weak form. For the weak formulation of (2.2) we use the L^2-scalar product $(\cdot,\cdot)_\Omega$ given by

$$(v,w)_\Omega := \int_\Omega v\,w\,dx \qquad \forall v, w \in L^2(\Omega)\,, \tag{2.3}$$

and the bi-form $b(\cdot,\cdot)$ defined as

$$b(p,v) := (\mathrm{kr}(\cdot,\theta(\cdot,p))\nabla p, \nabla v)_\Omega \qquad \forall p, v \in H^1(\Omega) \tag{2.4}$$

on $H^1(\Omega) \times H^1(\Omega)$, where the dots signify the dependence of kr and θ on x. The form $b(\cdot,\cdot)$ is nonlinear in the first component, and linear and continuous in the second component.

By Green's formula the weak form of (2.2) reads

$$p \in H^1_0(\Omega): \qquad (\theta(\cdot,p),v)_\Omega + b(p,v) = (f,v)_\Omega \qquad \forall v \in H^1_0(\Omega)\,. \tag{2.5}$$

Here, we assume $f \in L^2(\Omega)$ for simplicity and without loss of generality. More generally, one should consider $f \in H^{-1}(\Omega) = H^1_0(\Omega)'$ and replace the L^2-scalar product by the duality ${}_{H^{-1}(\Omega)}\langle\,\cdot\,,\,\cdot\,\rangle_{H^1_0(\Omega)}$. We also note that well-definedness of the left hand side in (2.5) is by no means trivial. It follows from Lemma 2.1 below.

2.4. Multi-domain problem: strong form. We rewrite the spatial problem (2.2) in a domain decomposition formulation. For this we assume a nonoverlapping decomposition of Ω into N subdomains $\Omega_1,\dots,\Omega_N$ with Lipschitz boundaries. We consider the problem on each subdomain separately and define

$$\theta^x_k(\cdot) := \theta(x,\cdot)\,, \quad \mathrm{kr}^x_k(\cdot) := \mathrm{kr}(x,\cdot) \qquad \forall x \in \Omega_k\,.$$

The index x in θ^x_k and kr^x_k indicates that these functions may still depend explicitly on $x \in \Omega_k$. With this convention the original equation (2.2) on each Ω_k, $k = 1,\dots,N$, reads

$$\theta^x_k(p_k) - \mathrm{div}(\mathrm{kr}^x_k(\theta^x_k(p_k))\nabla(p_k)) = f \qquad \text{on } \Omega_k \tag{2.6}$$

for $p_k = p_{|\Omega_k}$ with homogeneous Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_k$. We connect these subproblems by imposing the transmission conditions

$$p_k = p_l \qquad \text{on } \Gamma_{kl}\,,\ l \in \mathrm{Nb}(k)\,, \tag{2.7}$$

$$\mathrm{kr}^x_k(\theta^x_k(p_k))\nabla p_k \cdot \mathbf{n}_k = \mathrm{kr}^x_l(\theta^x_l(p_l))\nabla p_l \cdot \mathbf{n}_k \qquad \text{on } \Gamma_{kl}\,,\ l \in \mathrm{Nb}(k)\,, \tag{2.8}$$

on the *local interfaces*

$$\Gamma_{kl} := (\overline{\Omega}_k \cap \overline{\Omega}_l)\backslash\partial\Omega\,, \quad k = 1,\dots,N-1\,,\ l > k\,,$$

that are nontrivial interfaces between neighbouring Ω_k and Ω_l in the sense that

$$l \in \mathrm{Nb}(k) := \{l > k:\ \partial\Omega_k \cap \partial\Omega_l \text{ has positive Hausdorff measure}\}\,.$$

The index set $\mathrm{Nb}(k)$ represents the neighbours of Ω_k, $k = 1,\dots,N-1$, with higher indices $l = k+1,\dots,N$ than k, and we also define $\mathrm{Nb}(N) = \emptyset$. Every local interface is only counted once, i.e., it is uniquely determined. By $\mathbf{n}_k$ we denote the outward normal on $\partial\Omega_k$ that exists almost everywhere with respect to the Hausdorff measure.

The union of all local interfaces gives the *global interface*

$$\Gamma = \bigcup_{k=1}^{N} (\partial\Omega_k\backslash\partial\Omega)\,,$$

which we also call the *skeleton*.

The transmission conditions provide continuity of the pressure and of the time discretized water flux across the local interfaces. These are hydrologically reasonable conditions and have been derived from the global problem in the two-domain case for very general global boundary conditions of Signorini's type [6].

In the following section we discuss the relationship of the decomposed problem (2.6)–(2.8) to the original spatial problem (2.2). We pay particular attention to cross points, i.e., points where more than two subdomains meet.

2.5. Multi-domain problem: weak formulations. For a weak formulation of (2.6)–(2.8) we localize the definitions (2.3) and (2.4) and set

$$(v,w)_{\Omega_k} := \int_{\Omega_k} v\, w\, dx \qquad \forall v, w \in L^2(\Omega_k)\,,$$

as well as

$$b_k(p_k, v_k) := \left(\mathrm{kr}_k^x(\theta_k^x(p_k))\nabla p_k, \nabla v_k\right)_{\Omega_k} \qquad \forall p_k, v_k \in H^1(\Omega_k)$$

for $k = 1, \ldots, N$. By restriction to Ω_k we also apply these forms to functions with a support that contains Ω_k.

LEMMA 2.1. *Let the functions $(x,p) \mapsto \theta_k^x(p)$ and $(x,\theta) \mapsto \mathrm{kr}_k^x(\theta)$, defined on $\Omega_k \times \mathbb{R}$, be Borel measurable and bounded. Then the forms $(\theta_k^x(\cdot),\cdot)_{\Omega_k}$ and $b_k(\cdot,\cdot)$ are well-defined on $H^1(\Omega_k) \times H^1(\Omega_k)$ for $k = 1, \ldots, N$. Moreover, for any $p_k \in H^1(\Omega_k)$ these forms induce functionals $(\theta_k^x(p_k),\cdot)_{\Omega_k},\ b_k(p_k,\cdot) \in H^1(\Omega_k)'$.*

Proof. Since $p_k \in H^1(\Omega_k)$ is Lebesgue measurable on Ω_k and $(x,p) \mapsto \theta_k^x(p)$ is Borel measurable and bounded on $\Omega_k \times \mathbb{R}$, the composition $x \mapsto \theta_k^x(p_k(x))$ is in $L^\infty(\Omega_k)$. With the same argument, since $x \mapsto \theta_k^x(p_k(x))$ is Lebesgue measurable on Ω_k and $(x,\theta) \mapsto \mathrm{kr}_k^x(\theta)$ is Borel measurable and bounded on $\Omega_k \times \mathbb{R}$, the composition $x \mapsto \mathrm{kr}_k^x(\theta_k^x(p_k(x)))$ is in $L^\infty(\Omega_k)$. Therefore, both forms $(\theta_k^x(\cdot),\cdot)_{\Omega_k}$ and $b_k(\cdot,\cdot)$ are well-defined on $H^1(\Omega_k) \times H^1(\Omega_k)$ for $k = 1, \ldots, N$. The second statement follows from the Cauchy–Schwarz inequality. ∎

Now we introduce the spaces

$$V_k := \left\{ v \in H^1(\Omega_k) :\ \mathrm{tr}_{\partial\Omega \cap \partial\Omega_k} v = 0 \right\},$$

where tr_Σ denotes the usual trace operator

$$\mathrm{tr}_\Sigma : H^1(\Omega_k) \to H^{1/2}(\Sigma)$$

for a Lipschitz manifold $\Sigma \subset \partial\Omega_k$ (consult [9, pp. 1.61/65]). In particular, we introduce the trace operators

$$\mathrm{tr}_{\Gamma_{kl}} : H^1(\Omega_k) \to H^{1/2}(\Gamma_{kl}) \qquad \text{and} \qquad \mathrm{tr}_{\Gamma_{lk}} : H^1(\Omega_l) \to H^{1/2}(\Gamma_{kl})$$

from both sides of Γ_{kl} for $k = 1, \ldots, N-1$, $l \in \mathrm{Nb}(k)$. For such situations the following basic gluing result is helpful and easy to prove [4, p. 136].

LEMMA 2.2. *Suppose that $N = 2$. If $p \in H^1(\Omega)$, then we have $p_k := p_{|\Omega_k} \in H^1(\Omega_k)$ for $k = 1,2$ and $\mathrm{tr}_{\Gamma_{12}}(p_1) = \mathrm{tr}_{\Gamma_{21}}(p_2)$. Conversely, if $p_k \in H^1(\Omega_k)$ for $k = 1,2$ and $\mathrm{tr}_{\Gamma_{12}}(p_1) = \mathrm{tr}_{\Gamma_{21}}(p_2)$ holds, then*

$$p := \begin{cases} p_1 & \text{on } \Omega_1 \\ p_2 & \text{on } \Omega_2 \end{cases}$$

is contained in $H^1(\Omega)$.

We define the L^2-space on the skeleton by

$$L^2(\Gamma) := \prod_{\substack{k=1,\ldots,N-1 \\ l \in \mathrm{Nb}(k)}} L^2(\Gamma_{kl})$$

and give an appropriate notion of a trace space on Γ as well as a trace theorem in the following

PROPOSITION 2.3. *The operator* $\mathrm{tr}_\Gamma : H^1_0(\Omega) \to L^2(\Gamma)$ *given by*

$$(\mathrm{tr}_\Gamma\, v)_{|\Gamma_{kl}} := \mathrm{tr}_{\Gamma_{kl}}\left(v_{|\Omega_k}\right) \qquad \text{for } k = 1, \ldots, N-1\,,\ \ l \in \mathrm{Nb}(k)\,,$$

is linear and continuous. Furthermore,

$$\|\mu\| := \inf\left\{\|v\|_{H^1(\Omega)} :\ \mathrm{tr}_\Gamma\, v = \mu\right\}$$

defines a norm on the image space

$$H^{1/2}_{00}(\Gamma) := \mathrm{tr}_\Gamma\left(H^1_0(\Omega)\right). \tag{2.9}$$

This space becomes a Hilbert space for which $\mathrm{tr}_\Gamma : H^1_0(\Omega) \to H^{1/2}_{00}(\Gamma)$ *is a quotient map (i.e., maps the open unit ball onto the open unit ball), and there exists a linear and continuous extension map*

$$R : H^{1/2}_{00}(\Gamma) \to H^1_0(\Omega)$$

satisfying $tr_\Gamma R\mu = \mu$ *for all* $\mu \in H^{1/2}_{00}(\Gamma)$.

Proof. Linearity and continuity of $\mathrm{tr}_\Gamma : H^1_0(\Omega) \to L^2(\Gamma)$ follow from the linearity and continuity of the trace operators $\mathrm{tr}_{\Gamma_{kl}} : H^1(\Omega_k) \to H^{1/2}(\Gamma_{kl})$ for $k = 1, \ldots, N-1$, $l \in \mathrm{Nb}(k)$. We note that by the continuity of tr_Γ, the kernel $\ker(\mathrm{tr}_\Gamma)$ is closed. The norm properties of $\|\cdot\|$ follow from the norm properties of $\|\cdot\|_{H^1(\Omega)}$ and the linearity of tr_Γ.

It follows directly from the definition of the norm $\|\cdot\|$ that $\mathrm{tr}_\Gamma : H^1_0(\Omega) \to H^{1/2}_{00}(\Gamma)$ is a quotient map. Consequently, $H^{1/2}_{00}(\Gamma)$ is isometrically isomorphic to the quotient space $H^1_0(\Omega)/\ker(\mathrm{tr}_\Gamma)$, see [41, pp. 54, 56]. On the other hand, since $H^1_0(\Omega)$ is a Hilbert space, there exists the canonical representation $H^1_0(\Omega) = \ker(\mathrm{tr}_\Gamma) \oplus \ker(\mathrm{tr}_\Gamma)^\perp$, in which $\ker(\mathrm{tr}_\Gamma)^\perp$ is the orthogonal complement of the (closed) kernel $\ker(\mathrm{tr}_\Gamma)$, see [41, p. 221]. Therefore, we have isometric isomorphisms

$$\ker(\mathrm{tr}_\Gamma)^\perp \cong H^1_0(\Omega)/\ker(\mathrm{tr}_\Gamma) \cong H^{1/2}_{00}(\Gamma)\,,$$

in which tr_Γ induces the isomorphism $\ker(\mathrm{tr}_\Gamma)^\perp \cong H^{1/2}_{00}(\Gamma)$. In particular, $H^{1/2}_{00}(\Gamma)$ is also a Hilbert space.

The inverse

$$R : H^{1/2}_{00}(\Gamma) \to \ker(\mathrm{tr}_\Gamma)^\perp \subset H^1_0(\Omega)$$

of tr_Γ restricted to $\ker(\mathrm{tr}_\Gamma)^\perp$ is a continuous linear map with the property $\mathrm{tr}_\Gamma\, R\mu = \mu$ for all $\mu \in H^{1/2}_{00}(\Gamma)$. ∎

REMARK 2.1. *A global linear and continuous extension operator* $R : H^{1/2}_{00}(\Gamma) \to H^1_0(\Omega)$ *can also be defined by all local extension operators* $R_k : \mathrm{tr}_{\partial\Omega_k \setminus \partial\Omega}(V_k) \to V_k$ *for* $k = 1, \ldots, N$, *using the gluing Lemma 2.2 and the local quotient norms in* $\mathrm{tr}_{\partial\Omega_k \setminus \partial\Omega}(V_k)$ *whose sum is equivalent to the norm in* $R : H^{1/2}_{00}(\Gamma)$ *defined above.*

Now we consider the following weak formulation of the multi-domain problem (2.6)–(2.8): Find $p_k \in V_k$ for $k = 1, \ldots, N$ such that we have

$$(\theta^x_k(p_k), v_k)_{\Omega_k} + b_k(p_k, v_k) = (f, v_k)_{\Omega_k} \qquad \forall v_k \in H^1_0(\Omega_k) \tag{2.10}$$

subject to

$$\mathrm{tr}_{\Gamma_{kl}}(p_k) = \mathrm{tr}_{\Gamma_{lk}}(p_l) \qquad \text{in } H^{1/2}(\Gamma_{kl}), \ l \in \mathrm{Nb}(k), \tag{2.11}$$

and

$$\sum_{k=1}^{N} \Big((\theta_k^x(p_k), R\mu)_{\Omega_k} + b_k(p_k, R\mu) \Big) = (f, R\mu)_\Omega \quad \forall \mu \in H_{00}^{1/2}(\Gamma). \tag{2.12}$$

In (2.12), R is an extension operator $H_{00}^{1/2}(\Gamma) \to H_0^1(\Omega)$, as described in Proposition 2.3. Note that in (2.10) we only test with a subspace of the full solution space V_k, in order to accommodate for the continuity conditions (2.11).

The weak forms (2.10) and (2.11) follow directly from the local problems (2.6) and the pressure continuity (2.7), respectively. Condition (2.12) can be regarded as a weak flux transmission condition in a global sense. It contains all local conditions (2.8), but it also combines them across cross points, where simple two-sided conditions are not available. To see this, consider, for sufficiently smooth functions, a test of (2.8) for $k = 1, \dots, N-1$, $l \in \mathrm{Nb}(k)$, with functions $\mu = v_{|\Gamma}$ that are the restrictions of functions $v \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$. Collecting the terms on both sides of any local interface and integrating over all subdomain boundaries provides

$$\sum_{k=1}^{N} \int_{\partial\Omega_k} \mathrm{kr}_k^x(\theta_k^x(p_k)) \nabla p_k \cdot \mathbf{n}_k \, \mu \, d\sigma = 0 \qquad \forall \mu = v_{|\Gamma}, \ v \in C^2(\overline{\Omega}) \cap H_0^1(\Omega), \tag{2.13}$$

in which possible cross points form a nullset on the domain of integration. Note that for these cross points, normal derivatives in the classical sense do not usually exist, since the subdomain boundaries are not smooth there. However, since Green's formula in the classical sense [21, p. 380] also applies to boundaries $\partial\Omega_k$ that are only piecewise smooth, an equivalent formulation of (2.13) is given by

$$\sum_{k=1}^{N} \Big((\theta_k^x(p_k), v)_{\Omega_k} + b_k(p_k, v) - (f, v)_{\Omega_k} \Big) = 0 \qquad \forall v \in C^2(\overline{\Omega}) \cap H_0^1(\Omega).$$

If the functions giving the fluxes (2.8) are smooth enough, this classical variational equation is equivalent to (2.8) except for cross points $\Gamma_{kl} \cap \Gamma_{k'l'} \neq \emptyset$. However, since this variational equation is also meaningful in the general case with cross points, we use its weak variant (2.12) and call it *global flux transmission condition*.

PROPOSITION 2.4. *The multi-domain problem* (2.10)–(2.12) *is equivalent to the global problem* (2.5).

Proof. First, let p be a solution of (2.5). Then we have $p_k := p_{|\Omega_k} \in V_k$ for $k = 1, \dots, N$ and (2.11) due to Lemma 2.2. Furthermore, extending any $v_k \in H_0^1(\Omega_k)$ by 0 on Ω (Lemma 2.2 again) we obtain (2.10) by (2.5). In the same way, since $R\mu \in H_0^1(\Omega)$ for each $\mu \in H_{00}^{1/2}(\Gamma)$, the transmission condition (2.12) follows from (2.5).

Conversely, let p_k, $k = 1, \dots, N$, be solutions of (2.10)–(2.12). Then, due to (2.11) and Lemma 2.2, the function p on Ω defined by $p_{|\Omega_k} = p_k$ for $k = 1, \dots, N$ belongs to $H_0^1(\Omega)$. (Here, in order to treat cross points, the converse statement of Lemma 2.2 needs to be applied successively, using that the restriction of $\mathrm{tr}_{\Gamma_{k_1 l} \cup \Gamma_{k_2 l}}$ on $\Gamma_{k_i l}$, $i = 1, 2$, coincides with $\mathrm{tr}_{\Gamma_{k_i l}}$.) Let $v \in H_0^1(\Omega)$ and $\mu = \mathrm{tr}_\Gamma v \in H_{00}^{1/2}(\Gamma)$.

Then $\tilde v := v - R\mu$ satisfies $\mathrm{tr}_\Gamma\, \tilde v = 0$ by Proposition 2.3, so that $\tilde v_k := \tilde v \in H^1_0(\Omega_k)$ for $k = 1, \dots, N$. We obtain

$$\begin{aligned}
(\theta(\cdot, p), v)_\Omega + b(p, v) &= \sum_{k=1}^N \Big((\theta_k^x(p), \tilde v + R\mu)_{\Omega_k} + b_k(p, \tilde v + R\mu) \Big) \\
&= \sum_{k=1}^N \Big((\theta_k^x(p), \tilde v_k)_{\Omega_k} + b_k(p_k, \tilde v_k) \Big) \\
&\quad + \sum_{k=1}^N \Big((\theta_k^x(p), R\mu)_{\Omega_k} + b_k(p_k, R\mu) \Big) \\
&= \sum_{k=1}^N \Big((f, v_k)_{\Omega_k} + (f, R\mu)_{\Omega_k} \Big) \\
&= (f, v)_\Omega
\end{aligned}$$

by (2.10) and (2.12). This proves (2.5). □

Note that in this proof, neither linearity nor continuity of $R : H^{1/2}_{00}(\Gamma) \to H^1_0(\Omega)$ is needed, i.e., one could have chosen an arbitrary R, which exists by Definition 2.9 of $H^{1/2}_{00}(\Gamma)$.

Besides (2.10)–(2.12) there is a second weak formulation of the multi-domain problem. Using the extension operators

$$R_{kl} : H^{1/2}_{00}(\Gamma_{kl}) \to V_k\,, \qquad R_{lk} : H^{1/2}_{00}(\Gamma_{kl}) \to V_l \qquad \text{for } \; k = 1, \dots, N-1\,, \;\; l \in \mathrm{Nb}(k)\,,$$

from the trace theorem (see [9, pp. 1.61/65]), one can also define the *local flux transmission conditions*

$$\begin{aligned}
&(\theta_k^x(p_k), R_{kl}\eta)_{\Omega_k} + b_k(p_k, R_{kl}\eta) - (f, R_{kl}\eta)_{\Omega_k} = \\
&(\theta_l^x(p_l), R_{lk}\eta)_{\Omega_l} + b_l(p_l, R_{lk}\eta) - (f, R_{lk}\eta)_{\Omega_l} \quad \forall \eta \in H^{1/2}_{00}(\Gamma_{kl})\,, \;\; l \in \mathrm{Nb}(k)\,, \quad (2.14)
\end{aligned}$$

which give a more direct weak formulation of (2.8) for $k = 1, \dots, N-1$ than the global condition (2.12). Indeed, if the functions in (2.8) are smooth enough, the conditions (2.14) are equivalent to (2.8) for $k = 1, \dots, N-1$. What is more, one has the following remarkable result (compare [35, Satz 2.2.2]).

PROPOSITION 2.5. *The multi-domain problem* (2.10), (2.11), (2.14) *is equivalent to the global problem* (2.5).

For the equivalence result it is obviously enough to require that the normal fluxes $\mathrm{kr}_k^x(\theta_k^x(p_k))\nabla p_k \cdot \mathbf{n}_k$ in (2.8) are functionals on $H^{1/2}_{00}(\Gamma_{kl})$ only. In particular, no test with test functions "on cross points" is needed. This fact seems astonishing if we compare it with (2.12) where we explicitly included such test functions. And indeed, (2.14) does not seem to be equivalent to (2.12). However, one can approach test functions "on cross points" by the ones used in (2.14) and in (2.10) in the following sense.

LEMMA 2.6. *For any $v_k \in V_k$, $k = 1, \dots, N$, there is a $v_k^0 \in H^1_0(\Omega_k)$ and a sequence $(v_k^n)_{n\in\mathbb{N}}$ in the space*

$$V_k^{00} := \Big\{ v \in V_k : \mathrm{tr}_{\Gamma_{kl}}\, v \in H^{1/2}_{00}(\Gamma_{kl}),\; l \in \mathrm{Nb}(k) \wedge \mathrm{tr}_{\Gamma_{kl'}}\, v \in H^{1/2}_{00}(\Gamma_{l'k}),\; k \in \mathrm{Nb}(l') \Big\}$$

such that we have the weak convergence $v_k^0 + v_k^n \rightharpoonup v_k$ *in* $H^1(\Omega_k)$ *for* $n \to \infty$.

This result relies on the density of $H_{00}^{1/2}(\Gamma_{kl})$ in $H^{1/2}(\Gamma_{kl})$ and is proved in [35, Lemma 2.2.1] by an application of Hahn–Banach's extension theorem. We use it here to clarify the relation between (2.12) and (2.14) which, as a side-effect, also proves Proposition 2.5.

LEMMA 2.7. *The implications* (2.12) $\Rightarrow$ (2.14) *and* (2.10) $\wedge$ (2.14) $\Rightarrow$ (2.12) *hold.*

Proof. First, let $\eta \in H_{00}^{1/2}(\Gamma_{k'l})$ for a $k' = 1, \dots, N-1$ and an $l \in \mathrm{Nb}(k')$. By definition of $H_{00}^{1/2}(\Gamma_{k'l})$ (see [9, pp. 1.60]), an extension of η with 0 on Γ and an application of local extension operators $R_k : H^{1/2}(\partial\Omega_k) \to H^1(\Omega_k)$ for $k = 1, \dots, N$ (from the trace theorem with $R_k(0) = 0$) provide a $v \in H_0^1(\Omega)$ (Lemma 2.2) with support contained in $\overline{\Omega}_{k'} \cup \overline{\Omega}_l$, such that the support of $\mu = \mathrm{tr}_\Gamma\, v \in H_{00}^{1/2}(\Gamma)$ is contained in $\Gamma_{k'l}$ with $\mu_{|\Gamma_{k'l}} = \eta$. Consequently, (2.12) implies (2.14).

Conversely, let $\mu \in H_{00}^{1/2}(\Gamma)$ and, therefore, a $R\mu \in H_0^1(\Omega)$ be given. Then, by Lemma 2.6, for any $k = 1, \dots, N$, there are $v_k^0 \in H_0^1(\Omega_k)$ and sequences $(v_k^n)_{n\in\mathbb{N}} \subset V_k^{00}$ such that $v_k^0 + v_k^n \rightharpoonup (R\mu)_{|\Omega_k}$ in $H^1(\Omega_k)$ for $n \to \infty$. By definition of V_k^{00} we can define $\eta_{kl}^n := \mathrm{tr}_{\Gamma_{kl}}\, v_k^n \in H_{00}^{1/2}(\Gamma_{kl})$ for $l \in \mathrm{Nb}(k)$ so that

$$v_k^n - \sum_{l\in\mathrm{Nb}(k)} R_{kl}\,(\eta_{kl}^n) - \sum_{\substack{l'=1,\dots,k-1\\ k\in\mathrm{Nb}(l')}} R_{kl'}\,(\eta_{l'k}^n) \quad \in H_0^1(\Omega_k)\,. \tag{2.15}$$

We use in the subsequent argument that running over all $k = 1, \dots, N$, for each local interface Γ_{kl}, $k = 1, \dots, N-1$, $l \in \mathrm{Nb}(k)$, the functions given in (2.15) provide one extension of η_{kl}^n to Ω_k and one to Ω_l. Consequently, since the forms in (2.12) induce functionals in $H^1(\Omega_k)'$ in the second entry (Lemma 2.1), we can conclude

$$\begin{aligned}
&\sum_{k=1}^N \Big((\theta_k^x(p_k), R\mu)_{\Omega_k} + b_k(p_k, R\mu) - (f, R\mu)_{\Omega_k} \Big) \\
&= \sum_{k=1}^N \Big((\theta_k^x(p_k), v_k^0)_{\Omega_k} + b_k(p_k, v_k^0) - (f, v_k^0)_{\Omega_k} \Big) \\
&\quad + \lim_{n\to\infty} \sum_{k=1}^N \Big((\theta_k^x(p_k), v_k^n)_{\Omega_k} + b_k(p_k, v_k^n) - (f, v_k^n)_{\Omega_k} \Big) \\
&\overset{(2.10)}{\underset{(2.15)}{=}} \lim_{n\to\infty} \sum_{k=1}^N \Bigg(\sum_{l\in\mathrm{Nb}(k)} \Big((\theta_k^x(p_k), R_{kl}\,(\eta_{kl}^n))_{\Omega_k} + b_k(p_k, R_{kl}\,(\eta_{kl}^n)) - (f, R_{kl}\,(\eta_{kl}^n))_{\Omega_k} \Big) \\
&\qquad + \sum_{\substack{l'=1,\dots,k-1\\ k\in\mathrm{Nb}(l')}} \Big((\theta_k^x(p_k), R_{kl'}\,(\eta_{l'k}^n))_{\Omega_k} + b_k(p_k, R_{kl'}\,(\eta_{l'k}^n)) - (f, R_{kl'}\,(\eta_{l'k}^n))_{\Omega_k} \Big) \Bigg) \\
&= \lim_{n\to\infty} \sum_{\Gamma_{kl},\, l>k} \Big((\theta_k^x(p_k), R_{kl}\,(\eta_{kl}^n))_{\Omega_k} + b_k(p_k, R_{kl}\,(\eta_{kl}^n)) - (f, R_{kl}\,(\eta_{kl}^n))_{\Omega_k} \\
&\qquad\qquad + (\theta_l^x(p_l), R_{lk}\,(\eta_{kl}^n))_{\Omega_l} + b_l(p_l, R_{lk}\,(\eta_{kl}^n)) - (f, R_{lk}\,(\eta_{kl}^n))_{\Omega_l} \Big) \\
&\overset{(2.14)}{=} 0\,. \qquad \square
\end{aligned}$$

As a matter of fact, v_k^n in Lemma 2.6 can even be constructed in such a way that the function in (2.15) is 0. However, this property is not needed here.

REMARK 2.2. *Although in the light of Proposition 2.5 and Lemma 2.6 the use of local instead of global flux transmission conditions is appealing in the continuous setting, the situation changes in the presence of cross points as soon as we discretize the problem. Then these cross points, which are nullsets in the continuous setting, start to matter since they provide degrees of freedom. More specifically, a finite element discretization of the global flux transmission condition* (2.12) *would naturally contain test functions on cross points. These test functions would neither be represented by discrete versions of* (2.10) *nor of* (2.14) *so that Lemma 2.6 does not have a discrete analogon. Well-posedness results for discretized linear problems with cross points, however, indicate that flux conditions on cross points do have to be taken into account [24].*

3. Kirchhoff transformation. In this chapter we introduce the Kirchhoff transformation and apply it to our multi-domain problem in the strong and in the weak forms. The Kirchhoff transformation is a prerequisite for our special treatment of the Richards equation on the subdomains by convex minimization and numerically by monotone multigrid, see [8]. On an algebraic level, this allows the extension of the 2-Lagrange multiplier method from linear cases to our nonlinear problem. Therefore, we will use the transformed problem formulation in the rest of the paper. In this chapter we also pay special attention on local Robin problems which are an essential ingredient in the 2LM method. Finally we discretize our Kirchhoff transformed problems and give an algebraic formulation for them which is suited for the 2LM method.

The Kirchhoff transformation reads

$$\kappa : p \mapsto u = \int_0^p \mathrm{kr}(\theta(\xi))\, d\xi\,.$$

If θ and kr do not depend on space explicitly, then by the chain rule $\nabla u = kr(\theta(p))\nabla p$, the time-discretized Richards equation(2.2) is transformed into the semilinear equation

$$m(u) - \Delta u = f \qquad \text{on } \Omega\,,$$

for the transformed variable u, where $m := \theta(\kappa^{-1})$. For the rest of this article, we therefore make the following important assumption on the decomposition of Ω into nonoverlapping subdomains Ω_k.

ASSUMPTION 3.1 (Piecewise constant soil parameters). *The nonoverlapping decomposition of Ω into subdomains $\Omega_1, \ldots, \Omega_N$ is such that the parameters θ_m, θ_M, p_b and λ are constant on each subdomain.*

With this assumption, the parameter functions θ_k^x and kr_k^x lose their explicit dependence on x. We therefore abbreviate them by θ_k and kr_k from now on.

3.1. Kirchhoff transformed multi-domain problem: strong form. We apply Kirchhoff transformations

$$\kappa_k : p \mapsto u = \int_0^p \mathrm{kr}_k(\theta_k(\xi))\, d\xi\,, \tag{3.1}$$

separately on each subdomain, to (2.6)–(2.8). They induce superposition operators on Ω_k for $k = 1, \ldots, N$, and lead to the semilinear equations

$$m_k(u_k) - \Delta u_k = f \qquad \text{on } \Omega_k \tag{3.2}$$

in the generalized pressure $u_k := \kappa_k(p_k)$ where the transformed saturation

$$m_k(u_k) := \theta_k(\kappa_k^{-1}(u_k)) \tag{3.3}$$

remains as the nonlinearity. The second nonlinearity in the original coupled equations (2.6)–(2.8) now reoccurs as the inverse Kirchhoff transformation in the transformed transmission conditions (2.7) and (2.8), which read

$$\begin{aligned} \kappa_k^{-1}(u_k) &= \kappa_l^{-1}(u_l) \qquad \text{on } \Gamma_{kl}\,,\ l \in \mathrm{Nb}(k)\,, \\ \nabla u_k \cdot \mathbf{n}_k &= \nabla u_l \cdot \mathbf{n}_k \qquad \text{on } \Gamma_{kl}\,,\ l \in \mathrm{Nb}(k)\,. \end{aligned} \tag{3.4}$$

While the generalized pressure will in general be discontinuous across the interfaces, we obtain continuity of the normal derivatives in the transformed variables. This contrasts the situation for the physical pressure which is continuous across the interfaces, whereas in general its normal derivative is not.

3.2. Kirchhoff transformed multi-domain problem: weak formulations. To obtain a weak formulation we start by defining the well-known bilinear forms

$$a_k(u_k, v_k) := (\nabla u_k, \nabla v_k)_{\Omega_k} \qquad \forall u_k, v_k \in H^1(\Omega_k) \tag{3.5}$$

for $k = 1, \ldots, N$. Although formally it is not hard to give a weak notion to the coupled problem (3.2)–(3.4), the proof of equivalence with the weak form of the untransformed coupled problem (2.10)–(2.12) is not easy. A straightforward Kirchhoff transformed version of (3.2)–(3.4) is as follows: Find $u_k \in V_k$ for $k = 1, \ldots, N$ such that we have

$$(m_k(u_k), v_k)_{\Omega_k} + a_k(u_k, v_k) = (f, v_k)_{\Omega_k} \qquad \forall v_k \in H_0^1(\Omega_k) \tag{3.6}$$

subject to

$$\kappa_k^{-1}\,\mathrm{tr}_{\Gamma_{kl}}\,(u_k) = \kappa_l^{-1}\,\mathrm{tr}_{\Gamma_{lk}}\,(u_l) \qquad \text{in } H^{1/2}(\Gamma_{kl})\,,\ l \in \mathrm{Nb}(k)\,, \tag{3.7}$$

and

$$\sum_{k=1}^{N} \Big((m_k(u_k), R\mu)_{\Omega_k} + a_k(u_k, R\mu) \Big) = (f, R\mu)_{\Omega} \quad \forall \mu \in H_{00}^{1/2}(\Gamma). \tag{3.8}$$

Here, R is again an extension operator $H_{00}^{1/2}(\Gamma) \to H_0^1(\Omega)$, whose existence is ensured by Proposition 2.3.

There are two major issues concerning the equivalence of (2.10)–(2.12) and (3.6)–(3.8). First we need $p_k \in V_k \Leftrightarrow u_k \in V_k$. To see this we first note that $\kappa_k(0) = 0$. Then, by the theory of superposition operators on $H^1(\Omega)$ in[26] one obtains "$\Rightarrow$" easily from $\mathrm{kr}_k \in L^\infty(\mathbb{R})$, since then $\kappa_k : \mathbb{R} \to \mathbb{R}$ is Lipschitz continuous. For "$\Leftarrow$" one needs $\kappa_k^{-1} : \mathbb{R} \to \mathbb{R}$ to be Lipschitz continuous which requires $\mathrm{kr}_k \geq c$ for a $c > 0$. We also point out that the weak chain rule $b_k(p_k, v_k) = a_k(u_k, v_k)\ \forall v_k \in H_0^1(\Omega_k)$ is not straightforward, see [23]. Secondly, the equivalence of (2.11), which reads

$$\mathrm{tr}_{\Gamma_{kl}}\left(\kappa_k^{-1} u_k\right) = \mathrm{tr}_{\Gamma_{lk}}\left(\kappa_l^{-1} u_l\right) \qquad \text{in } H^{1/2}(\Gamma_{kl})\,,\ l \in \mathrm{Nb}(k)\,,$$

with (3.7) requires the commutativity $\mathrm{tr}_{\Gamma_{kl}}\,\kappa_k^{-1} = \kappa_k^{-1}\,\mathrm{tr}_{\Gamma_{kl}}$ of superposition operators κ_k^{-1} acting on V_k and $H^{1/2}(\Gamma_{kl})$, and the trace operators $\mathrm{tr}_{\Gamma_{kl}} : V_k \to H^{1/2}(\Gamma_{kl})$. This

commutativity follows from the continuity of the superposition operators κ_k^{-1} on V_k, shown in [27]. For details we refer to [5].

PROPOSITION 3.1. *If the functions $\theta_k : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and the functions $\mathrm{kr}_k \in L^\infty(\mathbb{R})$ are non-negative for $k = 1, \dots, N$, then* (3.6)–(3.8) *follow from* (2.10)–(2.12) *with $u_k = \kappa_k(p_k)$ and $m_k = \theta_k \circ \kappa^{-1}$. If, in addition, $\mathrm{kr}_k \geq c$ holds for a $c > 0$, then the Kirchhoff transformed coupled problem* (3.6)–(3.8) *is equivalent to the untransformed coupled problem* (2.10)–(2.12).

In a similar fashion to (3.8) it is possible to Kirchhoff-transform the local flux transmission conditions (2.14). If the assumptions of Proposition 3.1 are met, their relationship with the Kirchhoff transformed global flux transmission condition (3.8) can be expressed as in Lemma 2.7. In that case, Propositions 2.4 and 2.5 also hold for the Kirchhoff transformed multi-domain problem.

3.3. Robin problems. Since the 2LM method relies on the solution of local Robin subproblems, we need to consider Robin subproblems for the time-discretized Richards equation on Ω_k, $k = 1, \dots, N$. For details we refer to [4, Sec. 3.4] and [6], where such problems have been studied intensively. Concretely, for a parameter $\gamma > 0$ and Robin boundary values λ_k^c on $\Gamma_k := \partial\Omega_k \cap \Gamma$ we consider the problem to find a p_k on Ω_k with homogeneous Dirichlet values $p_k = 0$ on $\partial\Omega_k \cap \partial\Omega$ such that

$$\theta_k(p_k) - \operatorname{div}\left(\mathrm{kr}_k(\theta_k(p_k))\nabla p_k\right) = f \qquad \text{on } \Omega_k$$

holds subject to the Robin condition

$$\mathrm{kr}_k(p_k)\nabla p_k \cdot \mathbf{n}_k + \gamma p_k = \lambda_k^c \qquad \text{on } \Gamma_k\,.$$

By (3.1) the Kirchhoff-transformed version of this problem is to find a $u_k = \kappa_k^{-1}(p_k)$ on Ω_k with homogeneous Dirichlet values $\kappa_k^{-1}(0) = 0$ on $\partial\Omega_k \cap \partial\Omega$ such that

$$m_k(u_k) - \Delta u_k = f \qquad \text{on } \Omega_k$$

holds with the Robin boundary condition

$$\nabla u_k \cdot \mathbf{n}_k + \gamma\kappa_k^{-1}(u_k) = \lambda_k^c \qquad \text{on } \Gamma_k\,.$$

In order to obtain the weak form of this transformed Robin problem we choose $\lambda_k^c \in H_{00}^{1/2}(\Gamma_k)'$ and, with the duality pairing $\langle\cdot,\cdot\rangle$ in $\left(H_{00}^{1/2}(\Gamma_k)\,,\, H_{00}^{1/2}(\Gamma_k)'\right)$, consider

$$\begin{aligned} u_k \in V_k : \quad (m_k(u_k), v_k)_{\Omega_k} + a_k(u_k, v_k) + \gamma(\kappa_k^{-1}(u_k), \mathrm{tr}_{\Gamma_k} v_k)_{\Gamma_k} \\ = (f, v_k)_{\Omega_k} + \langle\lambda_k^c, \mathrm{tr}_{\Gamma_k} v_k\rangle \qquad \forall v_k \in V_k\,. \end{aligned} \tag{3.9}$$

As the following lemma shows, this problem is well-posed.

LEMMA 3.2. *If the functions $\theta_k : \mathbb{R} \to \mathbb{R}$ are Lipschitz continuous and increasing, and the functions $\mathrm{kr}_k \in L^\infty(\mathbb{R})$ satisfy $\mathrm{kr}_k \geq c$ for a $c > 0$ for $k = 1, \dots, N$, then the Robin problems* (3.9) *are uniquely solvable.*

The proof identifies the Robin problems as uniquely solvable convex minimization problems, see [4, Thm. 3.4.2 and Rem. 3.4.28] or [6, Thm. 2]. The Lipschitz continuity of θ_k can be weakened to Hölder continuity or replaced by continuity and boundedness.

3.4. Discretization in space. Assume that Ω and all the subdomains Ω_k, $k = 1, \dots, N$, are polygons (for $d = 2$) or polyhedra (for $d = 3$). Let $\mathcal{G}$ be a grid of Ω. We assume that the grid is conforming and that the subdomain boundaries are resolved by $\mathcal{G}$. We call $\mathcal{G}_k$ the subgrid corresponding to Ω_k.

For the space discretization of (3.6)–(3.8) we choose piecewise linear finite elements. The spaces of piecewise linear finite elements generated by $\mathcal{G}$ and $\mathcal{G}_k$ preserving homogeneous Dirichlet data on $\partial\Omega$ shall be denoted by $\mathcal{S}$ and $\mathcal{S}_k$, respectively. The subspace of $\mathcal{S}_k$ preserving homogeneous Dirichlet data on $\partial\Omega_k$ is called $\mathcal{S}_k^0$. The set of all nodes of $\mathcal{G} \setminus \partial\Omega$ is denoted by $\mathcal{N}$ with $n := \#\mathcal{N}$. The set of all nodes of $\mathcal{G}_k \setminus \partial\Omega$ is denoted by $\mathcal{N}_k$ with $n_k := \#\mathcal{N}_k$. For any $k = 1, \dots, N$ we assume that an enumeration $q_{i,k}$, $i = 1, \dots, n_k$, of the nodes $q_{i,k} \in \mathcal{N}_k$ is given. If a vertex is a cross point of s subdomains, it has s representations as a node. The Lagrange basis function on Ω_k corresponding to a node $q_{i,k}$ is called $\phi_{i,k}$. The finite element space on the skeleton is defined by $\mathcal{S}_\Gamma := \mathrm{tr}_\Gamma(\mathcal{S})$, and for $k = 1, \dots, N$ we also define $\mathcal{S}_{\Gamma_k} := \mathrm{tr}_{\Gamma_k}(\mathcal{S}_k)$.

We discretize the subproblems (3.6) by considering them in the finite element subspaces $\mathcal{S}_k^0 \subset H_0^1(\Omega_k)$ while treating the nonlinearity in the first integral by piecewise linear interpolation of the integrand. Then with the definition

$$h_{i,k} := \int_{\Omega_k} \phi_{i,k}\, dx \qquad \text{for } q_{i,k} \in \mathcal{N}_k \tag{3.10}$$

the discretized subproblems read

$$u_k \in \mathcal{S}_k : \quad \sum_{q_{i,k} \in \mathcal{N}_k} m_k\left(u_k(q_{i,k})\right) v(q_{i,k})\, h_{i,k} + a_k(u_k, v) = (f, v)_{\Omega_k} \qquad \forall v \in \mathcal{S}_k^0\,. \tag{3.11}$$

These are underdetermined because the test function space does not include any degrees of freedom on $\partial\Omega_k$. The continuity condition (3.7) is discretized in $\mathcal{S}_k$ and $\mathcal{S}_l$ by piecewise linear interpolation in p, so that it reduces to

$$\kappa_k^{-1}(u_k(q)) = \kappa_l^{-1}(u_l(q)) \qquad \forall q \in \mathcal{N}_k \cap \mathcal{N}_l\,. \tag{3.12}$$

The global flux transmission condition (3.8) is discretized analogously as the subproblems (3.6). The discrete form reads

$$\sum_{k=1}^{N} \sum_{q_{i,k} \in \mathcal{N}_k} m_k\left(u_k(q_{i,k})\right) \hat{E}\mu(q_{i,k})\, h_{i,k} + a_k(u_k, \hat{E}\mu) - (f, \hat{E}\mu)_{\Omega_k} = 0 \quad \forall \mu \in \mathcal{S}_\Gamma, \tag{3.13}$$

where $\hat{E} : \mathcal{S}_\Gamma \to \mathcal{S}$ is an extension operator, i.e., satisfying $\mathrm{tr}_\Gamma\, \hat{E} = \mathrm{id}$ on $\mathcal{S}_\Gamma$.

For further use we also note how to discretize the Robin subproblems (3.9). Similar to (3.10) we define

$$h_{i,k}^\Gamma := \int_{\partial\Omega_k \cap \Gamma} \phi_{i,k}\, dx \qquad \text{for } q_{i,k} \in \mathcal{N}_k \cap \Gamma\,,$$

and treat the nonlinearity in the second integral in (3.9) again by piecewise linear interpolation of the integrand. Then, with a discretization $\lambda_k^d \in \mathcal{S}'_{\Gamma_k}$ of λ_k^c, the finite element discretization of (3.9) reads

$$\begin{aligned} u_k \in \mathcal{S}_k : \quad & \sum_{q_{i,k} \in \mathcal{N}_k} m_k\left(u_k(q_{i,k})\right) v(q_{i,k})\, h_{i,k} \;+ \sum_{q_{i,k} \in \mathcal{N}_k \cap \Gamma} \gamma \kappa_k^{-1}\left(u_k(q_{i,k})\right) v(q_{i,k})\, h_{i,k}^\Gamma \\ & + a_k(u_k, v) = (f, v)_{\Omega_k} + \langle \lambda_k^d, \mathrm{tr}_{\Gamma_k}\, v \rangle \qquad \forall v \in \mathcal{S}_k\,. \end{aligned} \tag{3.14}$$

Since the $\mathcal{S}_k$ are finite-dimensional we obtain well-posed problems even if κ_k^{-1} has a singularity.

LEMMA 3.3. *With the conditions of Lemma 3.2 the discrete Robin problems* (3.14) *are uniquely solvable even for* $c = 0$.

For the proof we refer to [4, Thm. 3.4.2, Prop. 2.3.11] or [6, Prop. 6]. Again the proof relies on the reformulation of the Robin problems as convex minimization problems. Well-posedness for singular κ_k^{-1} is useful because it allows to treat, e.g., the Brooks–Corey functions that we use in the numerical example in Section 7, see [4, p. 194].

3.5. Algebraic formulation. The algebraic version of the discretized multidomain problem (3.11)–(3.13) is obtained by inserting the nodal basis functions $\phi_{i,k} \in \mathcal{S}_k$, $i = 1, \dots, n_k$, into (3.11) and all $\operatorname{tr}_\Gamma(\phi_{i,k}) \in \mathcal{S}_\Gamma \backslash \{0\}$ into (3.13), while considering the trivial extension operator $\hat{E}$ satisfying $\hat{E} \operatorname{tr}_\Gamma(\phi_{i,k}) = \phi_{i,k}$ for all relevant $i \in \{1, \dots, n_k\}$. We call $\hat{u}_k$, $k = 1, \dots, N$, the vector in $\mathbb{R}^{n_k}$ consisting of the entries $u_k(q_{i,k})$, $i = 1, \dots, n_k$. Concatenation of the $\hat{u}_k$ for $k = 1, \dots, N$, gives the vector $\hat{u} = (\hat{u}_1^T \ \dots \ \hat{u}_N^T)^T \in \mathbb{R}^n$. Recall that the vector $\hat{u}$ does not represent a continuous function on Ω. Instead, for every vertex of the grid $\mathcal{G}$ situated on the skeleton Γ there are as many entries in $\hat{u}$ as there are subdomains Ω_k touching that vertex. These entries correspond to the values of the functions u_k defined on each Ω_k of these subdomains in the node $q_{i,k}$ associated to that grid vertex.

In the following we drop the $\hat{}$ for better readability. We also consider the vectors $u_{Ik} \in \mathbb{R}^{n_{Ik}}$ for a $n_{Ik} < n_k$ consisting only of the entries $u_k(q_{i,k})$ for the interior nodes $q_{i,k} \in \Omega_k \setminus \Gamma$, and the vectors $u_{\Gamma k} \in \mathbb{R}^{n_{\Gamma k}}$, $n_{\Gamma k} = n_k - n_{Ik}$, containing the entries $u_k(q_{i,k})$ for $q_{i,k} \in \Gamma$ only. We suppose the vertices of each subdomain are enumerated such that we can decompose each u_k as $u_k = (u_{Ik}^T \ u_{\Gamma k}^T)^T$.

For each subdomain Ω_k we define the stiffness matrix A_k whose (i,j)-th entry is given by $a_k(\phi_i, \phi_j)$. It can be written as

$$A_k = \begin{bmatrix} A_{IIk} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k} \end{bmatrix},$$

with the square submatrices $A_{IIk} \in \mathbb{R}^{n_{Ik} \times n_{Ik}}$ and $A_{\Gamma\Gamma k} \in \mathbb{R}^{n_{\Gamma k} \times n_{\Gamma k}}$ and corresponding off-diagonal blocks.

We define the nonlinear operator $\hat{m}_k : \mathbb{R}^{n_k} \to \mathbb{R}^{n_k}$ by

$$\hat{m}_k(v_1, \dots, v_{n_k})^T := (m_k(v_1)\, h_{1,k}, \dots, m_k(v_{n_k})\, h_{n_k,k})^T$$

for all $(v_1, \dots, v_{n_k})^T \in \mathbb{R}^{n_k}$, where m_k is the scalar nonlinearity (3.3), and the $h_{i,k}$ are the weights defined in (3.10). Together, they amount to a discretization of the form $(m_k(\cdot), \cdot)_{\Omega_k}$ with a lumped mass matrix. In an abuse of notation we again drop the $\hat{}$. Analogously to the other quantities, the operator m_k splits into two operators m_{Ik} and $m_{\Gamma k}$. Then, interpreting A_{IIk} and $A_{\Gamma\Gamma k}$ as operators on $\mathbb{R}^{n_{Ik}}$ and $\mathbb{R}^{n_{\Gamma k}}$, respectively, writing the sums $A_{IIk} + m_{Ik}$ and $A_{\Gamma\Gamma k} + m_{\Gamma k}$ makes sense so that

$$\begin{bmatrix} A_{IIk} + m_{Ik} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k} + m_{\Gamma k} \end{bmatrix}$$

can be understood as a nonlinear operator on $\mathbb{R}^{n_k}$.

For the right hand side we define the vectors $f_{Ik} \in \mathbb{R}^{n_{Ik}}$ and $f_{\Gamma k} \in \mathbb{R}^{n_{\Gamma k}}$ with the entries $(f, \phi_{i,k})_{\Omega_k}$ for $1 \le i \le n_{Ik}$ and $n_{Ik} < i \le n_k$, respectively. Finally, we define

$f_k = \left(f_{Ik}^T \; f_{\Gamma k}^T\right)^T$ and $f = \left(f_1^T \; \dots \; f_N^T\right)^T$. With these preparations we can immediately deduce the algebraic subdomain problem.

LEMMA 3.4. *The algebraic equation*

$$\begin{bmatrix} A_{II k} + m_{Ik} & A_{I\Gamma k} \end{bmatrix} \begin{bmatrix} u_{Ik} \\ u_{\Gamma k} \end{bmatrix} = f_{Ik} \tag{3.15}$$

is an equivalent formulation of the discrete problem (3.11).

As (3.11), this problem does not have a unique solution. Indeed we have n_k variables but only n_{Ik} equations. This corresponds to the fact that in (3.11) the solution is searched in $\mathcal{S}_k$, whereas the test space is only $\mathcal{S}_k^0$ (V_k and $H_0^1(\Omega_k)$ in the continuous setting). Well-posedness for the complete problem is obtained by taking the continuity and flux conditions into account. The individual subdomain problems are made well-posed later by adding Robin boundary conditions. Still, if the $u_{\Gamma k}$ are considered fixed, then we obtain a well-posed problem for the u_{Ik}.

LEMMA 3.5. *With the conditions as in Lemma 3.3 $A_{II k} + m_{Ik}$ is invertible and* (3.15) *is a well-posed problem in u_{Ik} with respect to given Dirichlet values $u_{\Gamma k}$ and right hand side f_{Ik}.*

For the proof of Lemma 3.5 we refer to [4, Sec. 2.5] or [8] where well-posedness of discretized boundary value problems for the Richards equation with much more general boundary conditions is proved.

The algebraic formulation of the continuity condition (3.12) is straightforward. For each vertex q contained in at least two subdomains Ω_k and Ω_l we require that

$$\kappa_k^{-1}(u_{q,k}) = \kappa_l^{-1}(u_{q,l}). \tag{3.16}$$

For the flux condition we obtain directly

$$\sum_k E_k \begin{bmatrix} A_{\Gamma I k} & A_{\Gamma\Gamma k} + m_{\Gamma k} \end{bmatrix} \begin{bmatrix} u_{Ik} \\ u_{\Gamma k} \end{bmatrix} = \sum_k E_k f_{\Gamma k},$$

where $E_k : \mathbb{R}^{n_{\Gamma k}} \to \mathbb{R}^n$ is the algebraic form of the extension operator $\hat{E}$, restricted to Ω_k.

For later reference we additionally define a restriction operator, which restricts a single valued function on Ω to a subdomain function on Ω_k. The matrix representation $R_k \in \mathbb{R}^{n_k \times n}$ consists of "rows of the identity": the entries of R_k are all either 0 or 1. An entry 1 of R_k at (i, j) indicates that the vertex q_j of the "global grid" on Ω is the same vertex as $q_{k,i}$ of the "local grid" on Ω_k.

For each $k = 1, \dots, N$, we partition R_k into an interior and interface part

$$R_k = \begin{bmatrix} R_{Ik} \\ R_{\Gamma k} \end{bmatrix}.$$

Then, $E_k = R_{\Gamma k}^T$ for all k and the algebraic form of the global flux transmission condition (3.8) reads

$$\sum_k R_{\Gamma k}^T \begin{bmatrix} A_{\Gamma I k} & A_{\Gamma\Gamma k} + m_{\Gamma k} \end{bmatrix} \begin{bmatrix} u_{Ik} \\ u_{\Gamma k} \end{bmatrix} = \sum_k R_{\Gamma k}^T f_{\Gamma k}. \tag{3.17}$$

Finally, we note the algebraic form of the discrete Robin problems (3.14). We define the matrix

$$B_k := \gamma \operatorname{diag}\left(h^{\Gamma}_{n_{Ik}+1,k}, \dots, h^{\Gamma}_{n_k,k}\right), \tag{3.18}$$

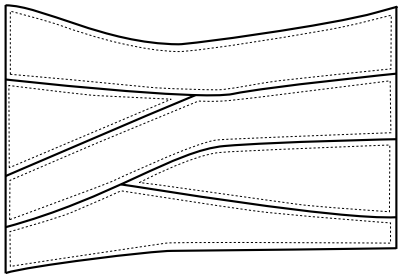

FIG. 4.1. *Many sided traces live separately on each subdomain boundary (dashed lines)*

which is to be interpreted as a scaled lumped mass matrix on $\partial\Omega_k$. With the help of this matrix and with a representation $\lambda_k \in \mathbb{R}^{n_{\Gamma k}}$ of the discrete functional $\lambda_k^d \in \mathcal{S}'_{\Gamma_k}$ the algebraic version of the discrete Robin problem (3.14) reads

$$\begin{bmatrix} A_{Ik} + m_{Ik} & A_{I\Gamma k} \\ A_{\Gamma Ik} & A_{\Gamma\Gamma k} + m_{\Gamma k} + B_k \kappa_k^{-1} \end{bmatrix} \begin{bmatrix} u_{Ik} \\ u_{\Gamma k} \end{bmatrix} = \begin{bmatrix} f_{Ik} \\ f_{\Gamma k} + \lambda_k \end{bmatrix}. \tag{3.19}$$

It follows directly from Lemma 3.3 that this is a well-posed problem.

4. The 2-Lagrange multiplier formulation. In this chapter we bring the algebraic problem (3.15)–(3.17) in a 2-Lagrange multiplier form. Let $u_G = (u_1, \ldots, u_N)$ be a set of subdomain functions, possibly discontinuous at the interfaces. We define the corresponding Robin boundary values $\lambda_k = \nabla u_k \cdot \mathbf{n}_k + \gamma \kappa_k^{-1}(u_k)$. The 2LM method reformulates the original problem as a problem for the λ_k. Hence the variables only live on the skeleton, but unlike in the usual Steklov–Poincaré formulation there are as many values for an interface vertex as there are subdomains bordering the vertex. We call such a function defined on the disjoint union $\bigsqcup_{k=1}^N \partial\Omega_k$ a many-sided trace (Figure 4.1).

4.1. Schur complement formulations of the transmission conditions. We begin by rewriting the continuity conditions (3.16) in a different way. For any vertex q_i of the grid $\mathcal{G}$ let l_i be the number of subdomains that touch q_i. Let $\check{M}_i$ be the $l_i \times l_i$ matrix

$$\check{M}_i = \begin{bmatrix} 1 & -\frac{1}{l_i-1} & \cdots & -\frac{1}{l_i-1} \\ -\frac{1}{l_i-1} & 1 & \cdots & -\frac{1}{l_i-1} \\ \vdots & \vdots & \ddots & \vdots \\ -\frac{1}{l_i-1} & -\frac{1}{l_i-1} & \cdots & 1 \end{bmatrix}, \tag{4.1}$$

and note that its kernel is spanned by the vector $(1, \ldots, 1) \in \mathbb{R}^{l_i}$. Let $\check{\Pi}$ be the permutation matrix that reorders the entries of the many-sided trace u_G so that all degrees of freedom associated with the first interface vertex q_1 appear first, followed by the degrees of freedom associated with the second interface vertex q_2, and so on. We define the continuity matrix

$$M := \check{\Pi}^{-1} \check{M} \check{\Pi},$$

where $\check{M}$ is a block-diagonal matrix whose i-th block $\check{M}_i$, of size $l_i \times l_i$, is given by (4.1).

With this definition the ensemble of continuity conditions (3.16) can be cast into a global form.

LEMMA 4.1. *The constraint* (3.16) *that the physical pressures must agree across interface boundaries can be written as*

$$M\kappa^{-1}u_G = 0\,, \tag{4.2}$$

where

$$\kappa^{-1}u_G := \begin{bmatrix} \kappa_1^{-1}(u_{\Gamma 1}) \\ \kappa_2^{-1}(u_{\Gamma 2}) \\ \vdots \\ \kappa_N^{-1}(u_{\Gamma N}) \end{bmatrix},$$

and where the application of the nonlinearity κ_k^{-1} on a vector $u_{\Gamma k}$, $k = 1, \ldots, N$, is to be understood component-wise.

Since the operator $A_{IIk} + m_{Ik}$ is invertible (cf. Lemma 3.5), we can eliminate u_{Ik} from every subproblem (3.19) to obtain a nonlinear Schur complement equation as follows. The top row of (3.19) gives

$$u_{Ik} = (A_{IIk} + m_{Ik})^{-1}(f_{Ik} - A_{I\Gamma k}u_{\Gamma k}). \tag{4.3}$$

Substituting (4.3) into (3.19) gives

$$(S_k + B_k\kappa_k^{-1})u_{\Gamma k} = f_{\Gamma k} + \lambda_k, \tag{4.4}$$

where the nonlinear Schur complement S_k is given by

$$S_k u_{\Gamma k} = A_{\Gamma Ik}(A_{IIk} + m_{Ik})^{-1}(f_{Ik} - A_{I\Gamma k}u_{\Gamma k}) + (A_{\Gamma\Gamma k} + m_{\Gamma k})u_{\Gamma k}. \tag{4.5}$$

Inserting the solution operator of (4.4) into (4.2) we obtain

$$M \begin{bmatrix} \kappa_1^{-1}\big((S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1)\big) \\ \vdots \\ \kappa_N^{-1}\big((S_N + B_N\kappa_N^{-1})^{-1}(f_{\Gamma N} + \lambda_N)\big) \end{bmatrix} = 0, \tag{4.6}$$

which are the continuity conditions written purely in terms of the Robin data λ_k.

We note that the expression $(S_N + B_N\kappa_N^{-1})^{-1}$ is the inverse of a nonlinear operator; it is the nonlinear Robin-to-Dirichlet map and this inverse is well-defined by Lemma 3.3. Hence (4.4) is equivalent to (3.19).

To treat the flux transmission conditions we first give a simple relation between the Robin data vector λ and the many-sided trace vector u_G, based on the following insight: if one averages Robin data across an interface, the Dirichlet data accumulate and the fluxes cancel.

LEMMA 4.2. *Let $\lambda_1, \ldots, \lambda_N$ be Robin data, and define local solutions to the Robin problems by* (3.19). *If the continuity condition* (4.2) *is satisfied, then the flux continuity condition* (3.17) *is equivalent to*

$$\sum_{k=1}^{N} R_{\Gamma k}^T \lambda_k = \sum_{k=1}^{N} R_{\Gamma k}^T B_k \kappa_k^{-1} u_{\Gamma k}, \tag{4.7}$$

where B_k is given by (3.18).

Proof. We prolong and sum over the subproblems (3.19), and use the smoothness (3.17) to obtain

$$\sum_{k=1}^{N} R_k^T \left(\begin{bmatrix} f_{Ik} \\ f_{\Gamma k} \end{bmatrix} + \begin{bmatrix} 0 \\ \lambda_k - B_k \kappa_k^{-1} u_{\Gamma k} \end{bmatrix} \right) = f.$$

Since

$$\sum_{k=1}^{N} R_k^T \begin{bmatrix} f_{Ik} \\ f_{\Gamma k} \end{bmatrix} = f,$$

we get (4.7). ∎

Combining (4.4) and (4.7) gives

$$\sum_k R_{\Gamma k}^T \lambda_k = \sum_k R_{\Gamma k}^T B_k \kappa_k^{-1} (S_k + B_k \kappa_k^{-1})^{-1} (f_{\Gamma k} + \lambda_k). \tag{4.8}$$

4.2. The 2-Lagrange multiplier system. In the previous section we have reformulated both the continuity and the flux conditions in a Schur complement way ((4.6) and (4.8), respectively). The 2-Lagrange multiplier method operates on Robin data, which is a linear combination of Dirichlet data and fluxes. We therefore combine (4.6) and (4.8) by taking the linear combination

$$B \cdot (4.6) + \begin{bmatrix} R_{\Gamma 1} \\ \vdots \\ R_{\Gamma N} \end{bmatrix} (4.8), \tag{4.9}$$

where $B := \mathrm{diag}(B_1, \ldots, B_N)$ denotes the block diagonal matrix of local lumped mass matrices B_k. The first addend of (4.9) is simply

$$BM \begin{bmatrix} \kappa_1^{-1}\big((S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1)\big) \\ \vdots \\ \kappa_N^{-1}\big((S_N + B_N\kappa_N^{-1})^{-1}(f_{\Gamma N} + \lambda_N)\big) \end{bmatrix} = 0,$$

while the second one is

$$\begin{bmatrix} R_{\Gamma 1} \\ \vdots \\ R_{\Gamma N} \end{bmatrix} \sum_k R_{\Gamma k}^T \lambda_k = \begin{bmatrix} R_{\Gamma 1} \\ \vdots \\ R_{\Gamma N} \end{bmatrix} \sum_k R_{\Gamma k}^T B_k \kappa_k^{-1} (S_k + B_k \kappa_k^{-1})^{-1} (f_{\Gamma k} + \lambda_k),$$

which we rewrite as

$$G \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_N \end{bmatrix} - GB \begin{bmatrix} \kappa_1^{-1}\big((S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1)\big) \\ \vdots \\ \kappa_N^{-1}\big((S_N + B_N\kappa_N^{-1})^{-1}(f_{\Gamma_N} + \lambda_N)\big) \end{bmatrix} = 0,$$

with the Gram matrix

$$G := \begin{bmatrix} R_{\Gamma 1} \\ \vdots \\ R_{\Gamma N} \end{bmatrix} \begin{bmatrix} R_{\Gamma 1}^T & \ldots & R_{\Gamma N}^T \end{bmatrix}.$$

This linear combination (4.9) of the continuity and flux conditions constitutes the 2LM formulation.

DEFINITION 4.3. *Let S be the block-diagonal matrix of nonlinear Schur complements, and κ the block-diagonal matrix of Kirchhoff transformation operators*

$$S = \operatorname{diag}(S_1, \ldots, S_N), \qquad \kappa = \operatorname{diag}(\kappa_1, \ldots, \kappa_N).$$

Let λ be the column vector of Robin data and g the column vector of "trace forces"

$$\lambda = (\lambda_1 \ \ldots \ \lambda_N)^T, \quad g = (f_{\Gamma 1} \ \ldots \ f_{\Gamma N})^T.$$

The 2-Lagrange multiplier formulation for the spatial multi-domain Richards problem is given by the nonlinear operator equation

$$\mathcal{R}_{2LM}(\lambda) = 0, \tag{4.10}$$

where

$$\mathcal{R}_{2LM}(\lambda) := (BM - GB)\kappa^{-1}\big[(S + B\kappa^{-1})^{-1}\big](g + \lambda) + G\lambda. \tag{4.11}$$

We can show equivalence of this formulation to the algebraic multi-domain Richards problem under minor additional assumptions on B.

THEOREM 4.4. *Assume that $BM = MB$. The algebraic multi-domain Richards problem* (3.15)–(3.17) *has a solution $(u_1, \ldots, u_N)$ if and only if* (4.10) *has a solution $(\lambda_1, \ldots, \lambda_N)$. The relationships between the λ_k and the u_k are given by* (3.19).

Proof. If (3.15)–(3.17) has a solution then we may apply Lemma 4.2 to find that the solution $(u_1, \ldots, u_N)$ is obtained by solving (3.15), (4.2), (4.7). We then obtain (4.6) and (4.8) by eliminating the interior degrees of freedom, and we obtain (4.10) via the linear combination (4.9).

For the converse, we show how to recover the equations (4.6) and (4.8) by manipulating (4.10). This shows that no information has been lost when we did the linear combination (4.9).

We begin by recovering (4.6) from (4.9). Recall that the domain of $R_{\Gamma k}$ is the set of single-valued finite element functions on Ω, and hence the range of $R_{\Gamma k}^T$ consists of continuous functions. Therefore, if we left-multiply (4.8) by M, we obtain the trivial equation $0 = 0$ by definition of M. We left-multiply (4.10) by M, which is equivalent to left-multiplying (4.9) by M to obtain

$$M^2 \begin{bmatrix} \kappa_1^{-1}\big((S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1)\big) \\ \vdots \\ \kappa_N^{-1}\big((S_N + B_N\kappa_N^{-1})^{-1}(f_{\Gamma N} + \lambda_N)\big) \end{bmatrix} = 0.$$

Because M is symmetric, we have that $\ker M^2 = \ker M$ and hence this gives (4.6). Thus, a solution $\lambda_1, \ldots, \lambda_q$ of (4.10) yields continuous physical pressures.

We now show how to recover (4.8). We subtract B(4.6) from (4.9) and obtain

$$\begin{bmatrix} R_{\Gamma 1} \\ \vdots \\ R_{\Gamma N} \end{bmatrix} (4.8),$$

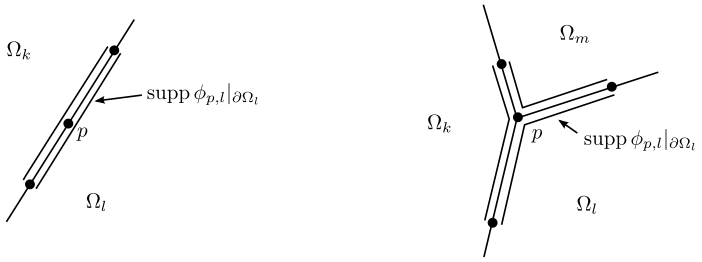

FIG. 4.2. *Left: Vertex p is shared by two subdomains only. The support of the shape functions* $\phi_{p,k}|_{\partial\Omega_k}$ *and* $\phi_{p,l}|_{\partial\Omega_l}$ *is identical, and hence so are their integrals* $h^\Gamma_{p,k}$ *and* $h^\Gamma_{p,l}$*. Right: p is shared by three subdomains. Here the supports of the three shape functions* $\phi_{p,k}|_{\partial\Omega_k}$*,* $\phi_{p,l}|_{\partial\Omega_l}$*, and* $\phi_{p,m}|_{\partial\Omega_m}$ *differ, and so do their integrals.*

which is equivalent to (4.8) because $\left[R^T_{\Gamma 1}\cdots R^T_{\Gamma N}\right]^T$ is injective on the sum of the ranges of the $R^T_{\Gamma k}$. ∎

REMARK 4.1. *The requirement that B and M must commute forces us to modify the matrices* B_k *at cross points. Indeed, let p be a vertex that is shared by precisely two subdomains* Ω_k *and* Ω_l*. Then* $\check{M}_p = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$*, and the* 2×2*-matrix* $\check{B}_p$ *of entries of the* B_k *for the vertex p has the form* $\check{B}_p = \operatorname{diag}(\gamma h^\Gamma_{p,k}, \gamma h^\Gamma_{p,l})$*. Since the basis functions* $\phi_{p,k}|_{\partial\Omega_k}$ *and* $\phi_{p,l}|_{\partial\Omega_l}$ *have the same support we get* $h^\Gamma_{p,k} = h^\Gamma_{p,l}$ *(Figure 4.2, left), and hence* $\check{M}_p$ *and* $\check{B}_q$ *commute.*

The situation is not as favorable at cross points. If p is shared by three subdomains Ω_k*,* Ω_l*,* Ω_m*, then the basis function supports will generally not be equal, and the values* $h^\Gamma_{p,k}$*,* $h^\Gamma_{p,l}$*,* $h^\Gamma_{p,m}$ *will generally not be equal. Consequently,* $\check{B}_p$ *will not commute with* $\check{M}_p$*. This is illustrated in Figure 4.2.*

To force commutativity we modify the matrices B_k *at the cross points. For each cross point p of order* l_p *we replace the matrix* $\check{B}_p = \operatorname{diag}(\gamma h^\Gamma_{p,k_1}, \ldots, \gamma h^\Gamma_{p,k_{l_p}})$ *by the average of its values times the identity matrix.*

5. Solving the 2-Lagrange multiplier system. In Chapter 4 we have reformulated the spatial problem of the time-discretized Richards equation as an operator equation

$$\mathcal{R}_{\mathrm{2LM}}(\lambda) = 0 \tag{5.1}$$

for the multivalued trace variable λ. In the first two subsections of this chapter we show how (5.1) can be solved using a Richardson iteration with a Minimal Polynomial Extrapolation (MPE) acceleration. This relies on an efficient way to evaluate the $\mathcal{R}_{\mathrm{2LM}}$ operator, which we cover in Section 5.3.

5.1. Solving the nonlinear 2-Lagrange multiplier system. Equation (5.1) is an equation for the operator $\mathcal{R}_{\mathrm{2LM}} : \mathbb{R}^{n_\Gamma} \to \mathbb{R}^{n_\Gamma}$, $n_\Gamma = \sum_{k=1}^N n_{\Gamma k}$, defined in (4.11). This operator is continuous by Lemma 3.5, but it is not the optimality condition for a minimization problem. Hence methods based on energy minimization cannot be used to solve it.[1]

[1] One may be tempted to use a (nonsmooth) Newton iteration to solve system (5.1), but it is unclear how the derivatives of $\mathcal{R}_{\mathrm{2LM}}$ can be computed.

The simplest way to solve (5.1) is to use a nonlinear Richardson iteration

$$\lambda_{\nu+1} = \lambda_\nu + \omega r_\nu \text{ where } r_\nu = -\mathcal{R}_{\text{2LM}}(\lambda_\nu), \tag{5.2}$$

with ν the iteration number, and $\omega \in (0, \infty)$ a damping parameter. The quantity $r_\nu = 0 - \mathcal{R}_{\text{2LM}}(\lambda)$ is the residual at iteration ν. The Richardson iteration was used implicitly in [6] for the two-domain case (see Chapter 6). Convergence rates between 0.3 and 0.9 were obtained, depending on the position of the saturation front in relation to the subdomain boundary. However, no theoretical guarantee for convergence was given.

Theoretical convergence analysis is available [13, 24, 25] if $\mathcal{R}_{\text{2LM}}$ is the linear 2LM operator

$$\mathcal{R}^{\text{lin}}_{\text{2LM}}(\lambda) = (BM - GB)\big[(S+B)^{-1}\big](g + \lambda) + G\lambda.$$

with B the identity matrix scaled with γ and the mesh size. In that setting, it can be shown that the optimal scalar Robin parameter γ is the geometric average of the extremal eigenvalues of the Schur complement matrix S. For the nonlinear setting this suggests to scale the Robin parameter γ with the factor in front of the Laplace operator.

5.2. Convergence acceleration. For solving linear problems, the Richardson iteration is usually too slow. One way to accelerate its convergence is Minimal Polynomial Extrapolation (MPE) [36], which we now outline briefly. While MPE can in principle be used to accelerate the convergence of any iterative scheme we focus on the case of the Richardson iteration.

Let $\lambda^0, \ldots, \lambda^\nu \in \mathbb{R}^{n_\Gamma}$ be a sequence of iterates, produced by Richardson iteration. The Minimal Polynomial Extrapolation algorithm produces a second sequence $\mu^0, \mu^2, \ldots \in \mathbb{R}^{n_\Gamma}$ such that each μ^i is a linear combination of the $\lambda^0, \ldots, \lambda^i$. This sequence is expected to converge faster to the limit λ of the original sequence than the original sequence itself.

Minimal polynomial extrapolation is motivated by the idea that for a linear iteration method there is an error propagation matrix $E \in \mathbb{R}^{n_\Gamma \times n_\Gamma}$ such that $\lambda^{\nu+1} - \lambda = E(\lambda^\nu - \lambda)$ for all iterations ν. In the case of the linear Richardson iteration $\lambda^{\nu+1} = \lambda^\nu + \omega(b - A\lambda^\nu)$ we have $E = (I - \omega A)$. If we could construct the minimal polynomial

$$P(E) = E^k + c_{k-1}E^{k-1} + \cdots + c_0$$

of E, then we would get $P(E)(\lambda^0 - \lambda) = 0$, and hence the exact solution after k steps, with k the order of the minimal polynomial. MPE tries to approximate the coefficients c_i of P from the iterates λ^ν. The resulting method has been found to be helpful even for nonlinear methods, which do not have a fixed error propagation matrix.

To compute the ν-th MPE iterate μ^ν let $\lambda^0, \ldots, \lambda^\nu$ be the previous iterates of the Richardson iteration. We construct coefficients $c_i, i = 0, \ldots, \nu$ with $c_\nu = 1$. First setting up the matrix of increments

$$U := \begin{bmatrix}(\lambda^1 - \lambda^0) & \ldots & (\lambda^{\nu-1} - \lambda^{\nu-2})\end{bmatrix} \in \mathbb{R}^{n_\Gamma \times (\nu-1)},$$

we then solve the problem

$$Uc = \lambda^\nu - \lambda^{\nu-1}$$

for the unknowns $c = (c_1, \ldots, c_{\nu-1}) \in \mathbb{R}^{\nu-1}$ in the least-squares sense. If $U^T U$ is nonsingular, this is equivalent to solving the $(\nu-1) \times (\nu-1)$ linear system

$$U^T U c = -U^T(\lambda^\nu - \lambda^{\nu-1}). \tag{5.3}$$

With $c_\nu = 1$, the new iterate is then given by

$$\mu^\nu = \frac{1}{\sum_{i=1}^{\nu} c_i} \begin{bmatrix} \lambda_1^1 & \cdots & \lambda_1^\nu \\ \vdots & \ddots & \vdots \\ \lambda_{n_\Gamma}^1 & \cdots & \lambda_{n_\Gamma}^\nu \end{bmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_\nu \end{pmatrix}.$$

When the λ^ν are produced by the Richardson iteration for a linear problem, then MPE is equivalent to a Krylov-space method [37]. Hence in a sense MPE is a nonlinear Krylov-space method. It is well known that the performance of such methods tends to increase if they get restarted at regular intervals. By restarting MPE at an iterate y^ν we mean erasing the history $\lambda^0, \ldots, \lambda^\nu$ and continuing the Richardson iteration with $\lambda^0 = \mu^\nu$. This technique also keeps the size of the dense linear system (5.3) small.

5.3. Evaluating the 2-Lagrange operator. The Richardson iteration both with and without MPE acceleration uses the operator $\mathcal{R}_{2\mathrm{LM}}$ only by evaluating it at iterates λ^ν. It hence needs to be discussed how $\mathcal{R}_{2\mathrm{LM}}(\lambda)$ can be computed efficiently for given λ. This has been described in [24] for the linear case. The procedure for the Richards equation remains essentially the same, the only difference being that the local subdomain problems are now nonlinear.

Remember Definition (4.11) of the $\mathcal{R}_{2\mathrm{LM}}$ operator

$$\mathcal{R}_{2\mathrm{LM}}(\lambda) = (BM - GB)\kappa^{-1}\left[(S + B\kappa^{-1})^{-1}\right](g + \lambda) + G\lambda$$

for a many-sided trace function $\lambda = (\lambda_1, \ldots, \lambda_N)$. The evaluation of the operator $\mathcal{R}_{2\mathrm{LM}}(\lambda)$ for a given λ can be broken down into two steps.

1. *Subdomain problems*

Begin by computing

$$\lambda_H := \kappa^{-1}[(S + B\kappa^{-1})^{-1}](g + \lambda).$$

Since $\kappa^{-1}[(S + B\kappa^{-1})^{-1}]$ is a block diagonal operator, this calculation decouples into individual calculations for the subdomains. For each subdomain Ω_k, the expression

$$\lambda_{H,k} = \kappa_k^{-1}[(S_k + B_k\kappa_k^{-1})^{-1}](g_k + \lambda_k)$$

has to be evaluated, which is equivalent to solving the local Robin problem (3.19) for $u_k = (u_{Ik}^T \; u_{\Gamma k}^T)^T$, and then setting $\lambda_{H,k} = \kappa_k^{-1}(u_{\Gamma k})$. It was shown in [8] that the local Robin problems can be solved efficiently and robustly with a monotone multigrid method. Describing that method is beyond the scope of this article, and the interested reader should consult [8] for details.

2. *Communication*

The second step computes

$$\mathcal{R}_{2\mathrm{LM}}(\lambda) = (BM - GB)\lambda_H + G\lambda.$$

All of these matrix–vector products are cheap in the sense that there are no Schur complements or inverses to compute. However, the matrices M and G are not block diagonal, and consequently this step involves data transfer between adjacent subdomains. In particular, a multiplication with G sums up the multi-valued trace on the skeleton, while a multiplication with M computes the jump across the interface.

One should note that the second step is identical to the corresponding one in the linear case. Indeed, to go from the Richardson method for the linear 2LM formulation to the same method for the Richards equation, all one has to do is to exchange the subdomain solver. The same holds for other problems of the same structure, such as Stefan problems.

In terms of computational cost, only the first step is expensive, consisting of solving local Richards problems. The second step does only simple arithmetic such as sums and averages. However, Step 1 involves no inter-subdomain communication at all, and in Step 2 only information exchange between subdomains sharing at least one vertex occurs. Hence, unless the subdomains are very small we expect the overall cost to be dominated by the computation cost. This makes our approach well-suited for a parallel implementation. The communication patterns are usually available in finite element software such as, e.g., DUNE [2], which we used for our implementation. Of course we do not expect our method to scale well with increasing numbers of subdomains. A coarse grid correction step that ensures proper scaling will be the subject of a separate article.

6. Relation to the Robin iteration in the case of two subdomains. When there are only two subdomains, solving the 2LM system (4.10) with an undamped ($\omega = 1$) Richardson iteration (5.2) turns out to be equivalent to the nonlinear Robin method proposed in [4, 6]. The proof given in this chapter generalizes a corresponding result for the linear 2LM formulation given in [19].

We briefly introduce the Robin method of [6]. Let Ω be divided in two subdomains Ω_1 and Ω_2 such that Assumption 3.1 holds. To solve the coupled problem (3.2)–(3.4), the Robin method updates the local subdomain solutions by solving local Robin problems of the form

$$\begin{aligned} m_1(u_1) - \Delta u_1^{\nu+1} &= f && \text{in } \Omega_1, \\ \gamma_1 \kappa_1^{-1} u_1^{\nu+1} + \mathbf{n}_1 \cdot \nabla u_1^{\nu+1} &= \gamma_1 \kappa_2^{-1} u_2^{\nu} + \mathbf{n}_1 \cdot \nabla u_2^{\nu} && \text{on } \Gamma, \end{aligned}$$

$$\begin{aligned} m_2(u_2) - \Delta u_2^{\nu+1} &= f && \text{in } \Omega_2, \\ \gamma_2 \kappa_2^{-1} u_2^{\nu+1} + \mathbf{n}_2 \cdot \nabla u_2^{\nu+1} &= \gamma_2 \kappa_1^{-1} u_1^{\nu+1} + \mathbf{n}_2 \cdot \nabla u_1^{\nu+1} && \text{on } \Gamma. \end{aligned}$$

Weak formulations for the two subdomain problems are obtained in the usual way (cf. Section 3.2). Discretizing the weak formulation with finite elements (on matching grids), we arrive at the following algebraic formulation

$$\begin{bmatrix} A_{II1} + m_{I1} & A_{I\Gamma 1} \\ A_{\Gamma I1} & A_{\Gamma\Gamma 1} + m_{\Gamma 1} + B_1 \kappa_1^{-1} \end{bmatrix} \begin{bmatrix} u_{I1}^{\nu+1} \\ u_{\Gamma 1}^{\nu+1} \end{bmatrix} = \begin{bmatrix} f_{I1} \\ \underbrace{f_{\Gamma 1} + f_{\Gamma 2}}_{f_\Gamma} \end{bmatrix} - \begin{bmatrix} 0 \\ R_2^T A_2 u_2^{\nu} - B_1 \kappa_2^{-1} u_{\Gamma 2}^{\nu} \end{bmatrix} \quad (6.1)$$

$$\begin{bmatrix} A_{II2} + m_{I2} & A_{I\Gamma 2} \\ A_{\Gamma I2} & A_{\Gamma\Gamma 2} + m_{\Gamma 2} + B_2 \kappa_2^{-1} \end{bmatrix} \begin{bmatrix} u_{I2}^{\nu+1} \\ u_{\Gamma 2}^{\nu+1} \end{bmatrix} = \begin{bmatrix} f_{I2} \\ \underbrace{f_{\Gamma 1} + f_{\Gamma 2}}_{f_\Gamma} \end{bmatrix} - \begin{bmatrix} 0 \\ R_1^T A_1 u_1^{\nu+1} - B_2 \kappa_1^{-1} u_{\Gamma 1}^{\nu+1} \end{bmatrix}. \quad (6.2)$$

In the remainder of this chapter we will show how the iterates u_1^ν, u_2^ν produced by this method are the same as the sequence of subdomain solutions corresponding to the iterates λ^ν of the undamped Richardson iteration for the 2LM problem (4.10), if proper initial iterates are chosen.

THEOREM 6.1. *The undamped Richardson iteration*

$$\lambda^{\nu+1} = \lambda^\nu + r^\nu(\lambda^\nu) = \lambda^\nu - \mathcal{R}_{2LM}(\lambda^\nu) \tag{6.3}$$

for the operator $\mathcal{R}_{2LM}$ is equivalent to the Robin iteration (6.1)–(6.2).

Proof. We begin by specializing the 2-Lagrange multiplier problem for the case of two subdomains. In this case there are no cross points, and the continuity matrix is

$$M = \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}.$$

The 2-Lagrange multiplier operator is given by

$$\mathcal{R}_{\text{2LM}} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} \lambda_1 + \lambda_2 - (B_1 + B_2)\kappa_2^{-1}\big[(S_2 + B_2\kappa_2^{-1})^{-1}(f_{\Gamma 2} + \lambda_2)\big] \\ \lambda_1 + \lambda_2 - (B_1 + B_2)\kappa_1^{-1}\big[(S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1)\big] \end{pmatrix}. \tag{6.4}$$

We write out the Richardson iteration (6.3) for the operator (6.4) and obtain

$$\begin{aligned} \lambda_1^{\nu+1} &= (B_1 + B_2)\kappa_2^{-1}\big[(S_2 + B_2\kappa_2^{-1})^{-1}(f_{\Gamma 2} + \lambda_2^\nu)\big] - \lambda_2^\nu \\ \lambda_2^{\nu+1} &= (B_1 + B_2)\kappa_1^{-1}\big[(S_1 + B_1\kappa_1^{-1})^{-1}(f_{\Gamma 1} + \lambda_1^\nu)\big] - \lambda_1^\nu. \end{aligned} \tag{6.5}$$

Inspired by (4.4) we introduce the new variables $u_{\Gamma 1}^{\nu+1}$ and $u_{\Gamma 2}^\nu$ defined implicitly by

$$(S_1 + B_1\kappa_1^{-1})u_{\Gamma 1}^{\nu+1} = f_{\Gamma 1} + \lambda_1^{\nu+1} \tag{6.6}$$

$$(S_2 + B_2\kappa_2^{-1})u_{\Gamma 2}^{\nu} = f_{\Gamma 2} + \lambda_2^{\nu}. \tag{6.7}$$

Using (6.7) in (6.5) we arrive at

$$\lambda_1^{\nu+1} = (B_1 + B_2)\kappa_2^{-1}(u_{\Gamma 2}^\nu) - \lambda_2^\nu. \tag{6.8}$$

We now solve (6.6) and (6.7) for $\lambda_1^{\nu+1}$ and λ_2^ν, respectively, and insert the results into (6.8), to obtain

$$(S_1 + B_1\kappa_1^{-1})u_{\Gamma 1}^{\nu+1} - f_{\Gamma 1} = (B_1 + B_2)\kappa_2^{-1}(u_{\Gamma 2}^\nu) - (S_2 + B_2\kappa_2^{-1})u_{\Gamma 2}^\nu + f_{\Gamma 2}. \tag{6.9}$$

Equation (6.9) is a formulation in terms of the Dirichlet traces $u_{\Gamma 1}$ and $u_{\Gamma 2}$. The next step is to introduce variables u_{I1}, u_{I2} for the interior vertices. We define a new variable u_{I1} by

$$(A_{II1} + m_1)u_{I1}^{\nu+1} + A_{I\Gamma 1}u_{\Gamma 1}^{\nu+1} = f_{I1}. \tag{6.10}$$

With this definition, $u_{I1}^{\nu+1}$ satisfies the top row of (6.1). We now check that the bottom row of (6.1) is also satisfied. Substituting (6.10) into the definition of S_1 given in (4.5), we get

$$S_1 u_{\Gamma 1}^{\nu+1} = A_{\Gamma I1}u_{I1}^{\nu+1} + (A_{\Gamma\Gamma 1} + m_{\Gamma 1})u_{\Gamma 1}^{\nu+1}. \tag{6.11}$$

By an analogous reasoning one also obtains

$$S_2 u_{\Gamma 1}^{\nu} = A_{\Gamma I2}u_{I2}^{\nu} + (A_{\Gamma\Gamma 2} + m_{\Gamma 2})u_{\Gamma 2}^{\nu+2}. \tag{6.12}$$

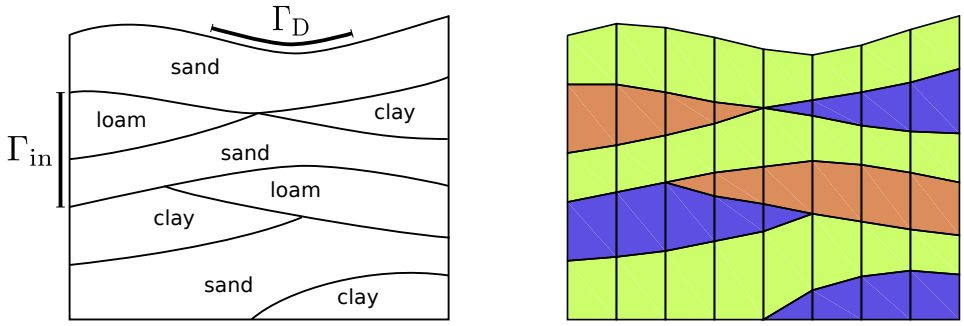

FIG. 7.1. *Left: domain with subdomain decomposition and boundary conditions. Right: the corresponding coarse grid*

	n	θ_m	θ_M	λ	p_b [m]	K_h [m/s]
sand	0.437	0.0458	1	0.694	-0.0726	$6.54 \cdot 10^{-5}$
loam	0.463	0.027	1	0.252	-0.1115	$3.67 \cdot 10^{-6}$
clay	0.475	0.1895	1	0.165	-0.373	$1.67 \cdot 10^{-7}$

TABLE 7.1
Soil parameters

The formulas (6.11) and (6.12) can be inserted into (6.9) to obtain

$$\begin{aligned} &A_{\Gamma I1} u_{I1}^{\nu+1} + (A_{\Gamma\Gamma 1} + m_{\Gamma 1}) u_{\Gamma 1}^{\nu+1} + B_1 \kappa_1^{-1} u_{\Gamma 1}^{\nu+1} - f_{\Gamma 1} \\ &\qquad = (B_1 + B_2)\kappa_2^{-1}(u_{\Gamma 2}^{\nu}) - A_{\Gamma I2} u_{I2}^{\nu} - (A_{\Gamma\Gamma 2} + m_{\Gamma 2}) u_{\Gamma 2}^{\nu+2} - B_2 \kappa_2^{-1} u_{\Gamma 2}^{\nu} + f_{\Gamma 2}. \end{aligned}$$

Collecting and cancelling terms yields the bottom row of (6.1). The relation (6.2) is verified in an analogous manner, which completes the proof. □

7. Numerical experiments. In this section we illustrate the solver performance with a numerical example. We solve the Richards equation on a two-dimensional domain with a subdomain partition containing several cross points and realistic soil types. Note again that we cannot expect our method to scale well for large numbers of subdomains, and hence we consider an example with few subdomains only. Our implementation is based on the DUNE libraries [2], with the auxiliary module DUNE-MULTIDOMAINGRID [28] handling the subdomain decomposition.

We use a two-dimensional domain that is intended to represent a vertical cut through a stretch of soil. The domain, with an extension of $2\,\mathrm{m} \times 1.55\,\mathrm{m}$, is pictured in Figure 7.1. It consists of eight layers, some of which taper off and do not span the entire width of the domain. The resulting decomposition has three cross points, two of third order and one of fourth order. We mark each subdomain to be either sand, loam, or clay. The soil parameters are taken from [30], and we list them in Table 7.1.

For the boundary conditions we choose constant Dirichlet conditions of $p = 9.81\,\mathrm{kPa}$ at a part Γ_D of the top boundary, modeling a surface water of uniform depth $d = p/(g\rho) = 1\,\mathrm{m}$. On a part Γ_{in} of the left side we set a constant inflow condition of $5 \cdot 10^{-5}\,\mathrm{m/s}$. For the remaining boundary we prescribe no-flow conditions. We start the evolution from a constant pressure of $-26\,\mathrm{kPa}$. By the Brooks–Corey relation (2.1) this corresponds to saturations 12 %, 45 %, and 76 % in sand, loam, and clay, respectively. Under the influence of gravity and the boundary conditions, we expect the domain to slowly fill with water.

refinement steps	0	1	2	3	4	5
vertices	59	206	767	2 957	11 609	46 001
elements	47	188	752	3 008	12 032	48 128
multivalued trace	88	166	322	634	1 258	2 506
vertices on skeleton	42	81	159	315	627	1 251

TABLE 7.2
Grid sizes

We discretize the domain using the grid shown in Figure 7.1, right. It consists of 59 vertices and 47 elements. Both triangular and quadrilateral elements occur. The vertices are stacked vertically as is customary in geoscience simulations. This allows to use an upwind discretization for the gravity term as described in [4, Sec. 4.2.1, Eq. (4.2.9)]. From the coarse grid we create a sequence of test grids by up to 5 steps of uniform refinement. The corresponding grid sizes are listed in Table 7.2. We set the time step size to $\tau = 50\,\mathrm{s}$.

Figure 7.2 shows the evolution of the pressure until $t = 25\,000\,\mathrm{s}$, when the system is fully saturated and reaches a stable state. As expected, water enters sand more readily than the loam and clay. In the stationary state one can observe that while the pressure field is indeed continuous across domain boundaries, the normal derivative is not. This is the consequence of the jumping material parameters as they appear in the flux condition (2.8).

We measured the convergence rate of both the simple Richards iteration (5.2) and the Richards iteration with MPE acceleration. For the Richardson iteration both with and without MPE acceleration we set the damping parameter ω to 1. This corresponds to what was used implicitly in [6]. Also note that Gander and Kwok [19] showed convergence for this choice of parameter in the linear setting with a modification of the matrix B. This modification, which is not the one explained in Remark 4.1, did not appear to have any effect in our nonlinear setting, and we therefore disregarded it. The question of whether a different choice of ω will lead to better results is left for later research. To account for the scaling of the problem we set the global Robin parameter γ to $1.74 \cdot 10^{-6}$. This is the average of the factors $K_h p_b$ for the three soil types, which appear in the bilinear form a_k, if written with all constants.

To measure the convergence rate at a given time step we first iterated until the absolute norm of the correction of the physical pressures $p_k^\nu = \kappa_k^{-1}(u_k^\nu)$ dropped below 10^{-12}. We then restarted the solver and recorded the errors $\|p - p^\nu\|$, where $\|\cdot\|^2 = \sum_{k=1}^N a_k(\cdot,\cdot)$ is the energy norm of the problem, and the bilinear forms a_k have been defined in (3.5). The average convergence rate ρ was then computed as the geometric mean of the individual rates

$$\rho^\nu := \frac{\|p^\nu - p\|}{\|p^{\nu-1} - p\|}.$$

At each time step we started the Richardson iteration at $\lambda^0 = 0$. We restarted the MPE procedure after each fifth iteration step. We solved each subdomain problem to machine precision using the monotone multigrid method of [8]. This may mean doing a lot more work than actually necessary. However for the time being it minimizes the influence of the subdomain solver on the outer nonlinear solver, which is what we are interested in.

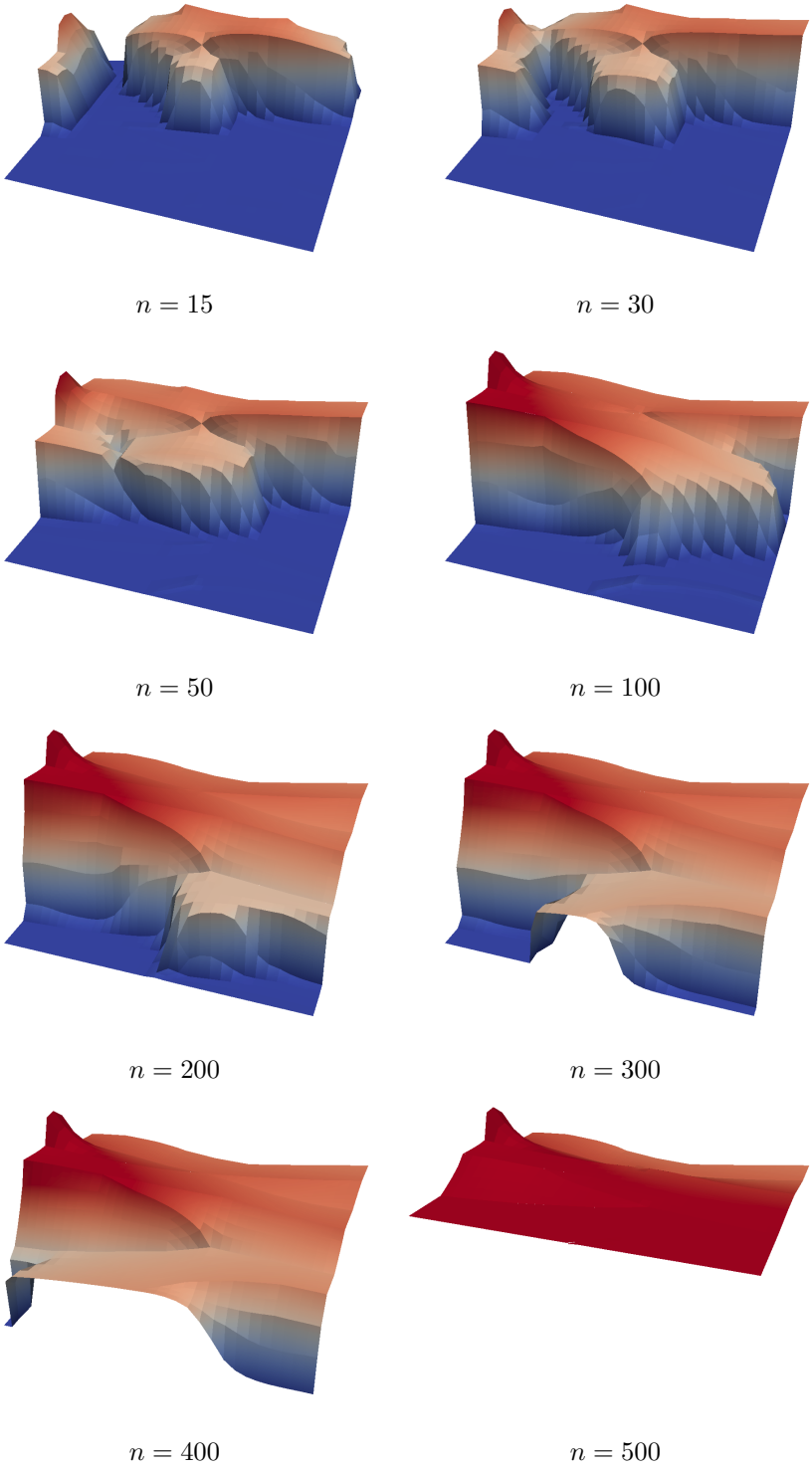

FIG. 7.2. *Time evolution of the physical pressure p*

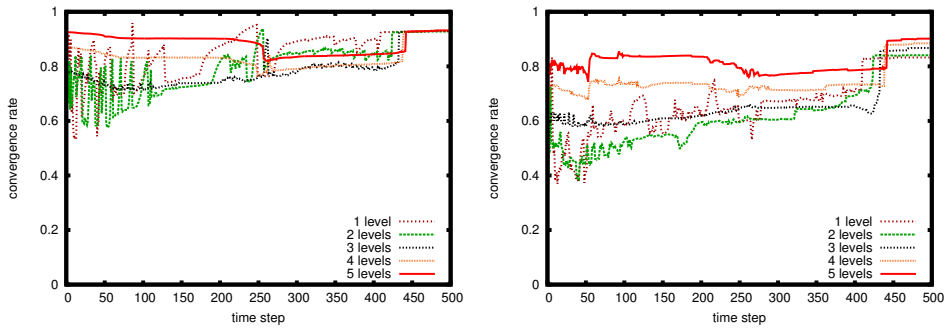

FIG. 7.3. *Convergence rate per time step. Left: Richardson iteration, right: Richardson iteration with MPE acceleration*

Figure 7.3 shows the convergence rates for each time step. The left graph shows the convergence rates for the simple Richardson iteration, and the right graph shows the rates when MPE acceleration is used. One can observe that even the simple Richardson iteration converges, but MPE leads to a definite improvement of the convergence rates, which then stay in the range of 0.5–0.9. This makes the method a competitive choice for solving the Richards equation on decompositions with piecewise space-independent parameter functions.

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