Multigrid Methods for Mortar Finite Elements

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Abstract. The framework of mortar methods [3,4] provides a powerful tool to analyze the coupling of different discretizations across subregion boundaries. We present an alternative Lagrange multiplier space without loosing the optimality of the a priori bounds [10]. By means of the biorthogonality between the nodal basis functions of our new Lagrange multiplier space and the finite element trace space, we derive a symmetric positive definite mortar formulation on the unconstrained product space. This new variational problem is the starting point for the application of our multigrid method. Level independent convergence rates for the $\mathcal{W}$–cycle can be established, provided that the number of smoothing steps is large enough.

1 Introduction

Mortar methods, introduced in [3,4], provide a powerful tool to analyze domain decomposition techniques based on the coupling of different discretization schemes or of nonmatching triangulations across interior interfaces. The pointwise continuity of the solution at the interior interfaces is replaced by a weaker one. So far there have been two possibilities to realize these weak continuity conditions. One includes the constraints in the definition of the finite element space resulting in a positive definite formulation on the nonconforming constrained space V_h. An equivalent approach is given in terms of the Lagrange multiplier space M_h, and gives rise to a saddle point formulation on the unconstrained space $X_h \times M_h$. Efficient iterative solvers have been introduced and analyzed in [1,5–9]; see also the literature cited therein.

Working with the alternative Lagrange multiplier space gives diagonal mass matrices on the non–mortar sides. To find the Lagrange multiplier in terms of the solution and the right hand side, we have to invert these mass–matrices. Using the biorthogonality relation, we can locally eliminate the Lagrange multiplier, and obtain a symmetric positive definite variational problem on X_h. We define our multigrid method in terms of level dependent bilinear forms and a special class of smoothing operators. Then, level independent convergence rates for the $\mathcal{W}$–cycle can be shown provided that the number of smoothing steps is large enough.

The rest of this paper is organized as follows: In Section 2, we define the dual basis functions for the Lagrange multiplier space, and present the different equivalent mortar formulations for the scalar elliptic case and linear elasticity problems. Section 3 concerns the introduction of the new positive definite formulation on the unconstrained product space. In Section 4, we specify the level dependent bilinear forms, and introduce a special class of smoothers. Finally, in Section 5 we present numerical examples for the scalar case as well as for a linear elasticity problem.

2 A dual basis as Lagrange multiplier space

In this section, the positive definite nonconforming system and the saddle point problem for a mortar formulation with an alternative Lagrange multiplier, defined by a dual basis, are given. The same qualitative a priori estimates as for the original mortar method can be established for this new Lagrange multiplier space.

Let Ω be a bounded, polygonal domain in $\mathbb{R}^2$. We assume that $\overline{\Omega} = \cup_{k=1}^{K} \overline{\Omega}_k$ is geometrically conforming decomposed into K non–overlapping polyhedral subdomains Ω_k. Each subdomain Ω_k is associated with a family of shape regular simplicial triangulations $\mathcal{T}_{h_k}$, $h_k \leq h_{k;0}$, where h_k is the meshsize parameter of $\mathcal{T}_{h_k}$. We use piecewise linear conforming finite elements $S_1(\Omega_k, \mathcal{T}_{h_k})$ on the individual subdomains, and enforce homogeneous Dirichlet boundary conditions on $\partial\Omega \cap \partial\Omega_k$. A common edge between two subdomains, $\partial\Omega_l \cap \partial\Omega_k$, is called interface and denoted by γ_m, $1 \leq m \leq M$. Each interface γ_m is associated with a one dimensional triangulation, inherited either from $\mathcal{T}_{h_k}$ or from $\mathcal{T}_{h_l}$. Then, the non–mortar side is the one from which the Lagrange multiplier space inherits its triangulation. The opposite side is called mortar side.

Let the unconstrained product space X_h be defined by

$$X_h := \prod_{k=1}^{K} \left(S_1(\Omega_k, \mathcal{T}_{h_k})\right)^d,$$

where $d = 1$ in the scalar case and $d = 2$ for an elasticity problem. We remark that for an element $v \in X_h$, arbitrary jumps at the interfaces are allowed and no constraints are imposed across the interfaces.

As in the standard mortar context, we define the global Lagrange multiplier space as a product space

$$M_h := \prod_{m=1}^{M} \left(M_h(\gamma_m)\right)^d,$$

where $M_h(\gamma_m)$ is spanned by nodal basis functions ψ_i associated with the interior vertices p_i of the non–mortar sides. We do not take the standard hat functions for ψ_i but shifted ones. The dual basis functions, ψ_i, have the same support as the standard hat functions, ϕ_i, and are locally given by $\psi_i := 3\phi_i - 1$ if p_i is not adjacent to an endpoint of γ_m. In the case that p_i is adjacent to one of the two endpoints of γ_m, we have to modify the definition locally such that $\sum_i \psi_i = 1$ on γ_m holds. We refer to [10] for details. It is easy to see that the following biorthogonality relation holds:

$$\int\limits_{\gamma_m} \psi_l \phi_k \, d\sigma = \delta_{lk} \int\limits_{\gamma_m} \phi_k \, d\sigma, \quad 1 \leq l, k \leq \nu_m \; , \tag{1}$$

where ν_m is the number of interior vertices on γ_m. As a consequence, the mass–matrix on the non–mortar side is reduced to a diagonal one.

In a next step, we define the constrained global finite element space V_h in terms of X_h, M_h and the bilinear form $b(\cdot,\cdot)$

$$b(v,\mu) := \sum_{m=1}^{M} \langle [v], \mu \rangle_{\gamma_m}, \quad v \in \prod_{k=1}^{K} \left(H^1(\Omega_k)\right)^d, \; \mu \in \prod_{m=1}^{M} \left(\left(H^{\frac{1}{2}}(\gamma_m)\right)^d \right)' ,$$

where $[\cdot]$ stands for the jump across the interface and $\langle \cdot, \cdot \rangle$ denotes the duality pairing. We can now define our constrained space V_h similar to the standard mortar approach by $V_h := \{ v \in X_h \mid \; b(v,\mu) = 0, \; \mu \in M_h \}$. We remark that this is a nonconforming approach and the finite element spaces V_{2h} and V_h are not nested. The nonconforming mortar formulation can be written as: Find $u_h \in V_h$ such that

$$a(u_h, v) = (f, v)_0 \; , \quad v \in V_h \; , \tag{2}$$

where the bilinear form is $a(\cdot,\cdot)$ is in the scalar case given by $a(u,v) := \sum_{k=1}^{K} \int_{\Omega_k} a \nabla v \, \nabla w \, dx$ for $u, v \in \prod_{k=1}^{K} H^1(\Omega_k)$ and in the case of linear elasticity by

$$a(u,v) := \sum_{k=1}^{K} \sum_{i,j,l,m=1}^{2} \int\limits_{\Omega_k} E_{ijlm} \frac{\partial u_l}{\partial x_m} \frac{\partial v_i}{\partial v_j} \, dx, \quad u, v \in \textstyle\prod_{k=1}^{K} (H^1(\Omega_k))^2 .$$

Here, the coefficient function a and Hooke's tensor E are assumed to be sufficiently smooth, and $f \in (L^2(\Omega))^d$.

Introducing the Lagrange multiplier as an additional unknown, we obtain an equivalent saddle point problem [3]. The weak continuity is not enforced by construction on the space but guaranteed by the second equation of the saddle point problem: Find $(u_h, \lambda_h) \in X_h \times M_h$

$$\begin{array}{lll} a(u,v) + b(v,\lambda) = (f,v)_0, & v \in X_h \; , \\ b(u,\mu) \qquad\qquad\;\; = 0, & \mu \in M_h \; . \end{array} \tag{3}$$

Optimal a priori estimates for the discretization errors in the energy norm, the L^2–norm and a suitable norm for the Lagrange multiplier space have been established in [10] for this new Lagrange multiplier space. In particular, it has been shown that the dual Lagrange multiplier space yields the same order of convergence as the standard multiplier space. Under the assumption of full H^2–regularity, we obtain an order h^2 a priori estimate for the discretization error, see [10].

3 Positive definite formulation

In this section, we introduce the symmetric positive definite variational formulation on the unconstrained product space X_h. The corresponding level dependent bilinear forms are defined, and the algebraic formulation of the positive definite problem is given.

Let us first consider the linear functional $g : X_h \longrightarrow X_h$ by $g(v) := \sum_{m=1}^{M} \sum_{l=1}^{\nu_m} \sum_{k=1}^{d} \alpha_{k;\ell}\phi_{k;\ell}$, where the coefficients $\alpha_{k;\ell}$ are given by $\alpha_{k;\ell} := b(v, \psi_{k;\ell})/\int_{\gamma_m} \phi_{k;\ell}\, d\sigma$. Then, it can be easily verified that $g(v) = 0$, $v \in X_h$ if and only if $v \in V_h$. Moreover, $g(\cdot)$ is a projection and we have $g(g(v)) = g(v)$.

The following lemma can be found in [11] and defines a new equivalent mortar formulation on the unconstrained product space X_h in terms of the projection $g(\cdot)$.

Lemma 1. *Let $u_h \in V_h$ be the unique solution of (2), then u_h is the unique solution of the positive definite symmetric variational problem*

$$\hat{a}_h(u_h, v) := a(u_h - g(u_h), v - g(v)) + a(g(u_h), g(v)) = (f, v - g(v))_0, \quad v \in X_h. \tag{4}$$

The proof is based on a suitable decomposition of $v \in X_h$; $v = (v - g(v)) + g(v)$. We remark that this decomposition is not uniformly stable in the broken H^1–norm. Thus, the new bilinear form $\hat{a}_h(\cdot,\cdot)$ is not uniformly continuous with respect to the broken H^1–norm. However, its condition number is bounded by c/h^2. For the details, we refer to [11].

The algebraic formulation of (4) can be obtained by decomposing the solution u_h into two components $u^T = (u_I^T, u_N^T)$. The first one, u_I, is associated with the interior nodes of the subdomains, all nodes on the mortar sides and the nodes at the endpoints of the non–mortar sides, and the second one, u_N, with the interior nodes on the non–mortar sides. Then, the biorthogonality relation (1) yields that the matrix B associated with the bilinear form $b(\cdot,\cdot)$ has the following structure $B^T = (M^T, D^T)$, where D is a diagonal matrix and M is sparse containing block mass–matrices and its band width depends on the local ratio of the meshsizes on mortar and adjacent non–mortar sides. Introducing $W^T := (0, D^{-1})$, using the explicit representation $\lambda_h = W^T(f + A(WB^T - \mathrm{Id})u_h))$, and setting $\mu = W^T A(WB^T - \mathrm{Id})v$ in the saddle point variational problem (3), we find $A_S u_h = f_S := (\mathrm{Id} - BW^T)f$, where

$$A_S := (\mathrm{Id}, (BW^T - \mathrm{Id})AW) \begin{pmatrix} A & B \\ B^T & 0 \end{pmatrix} \begin{pmatrix} \mathrm{Id} \\ W^T A(WB^T - \mathrm{Id}) \end{pmatrix}. \tag{5}$$

The following lemma has been established in [11].

Lemma 2. *The algebraic form of the variational problem (4) is given by (5).*

4 Multigrid convergence

The proof of the multigrid method will be based on suitable approximation and smoothing properties. A general approximation property for the saddle point formulation can be established in the case of the standard Lagrange multiplier spaces which are nested. This is not the case of our new mortar discretization, $M_{2h} \not\subset M_h$. For this special M_h a weaker approximation property has been shown in [9], where the following lemma can be found.

Lemma 3. *A suitable approximation property holds if the standard restriction is replaced a modified one, $(I_{\mathrm{mod}})_h^{2h}$, and the smoother satisfies $W_h^T d_h = 0$, where d_h is the residuum after $m \geq 1$ smoothing steps.*

The definition of $(I_{\mathrm{mod}})_h^{2h}$ is motivated by the following observation: Applying the standard restriction $I_h^{2h}: X_h \longrightarrow X_{2h}$ on d_h does not, in general, yields $W_{2h}^T I_h^{2h} d_h = 0$, even if $W_h^T d_h = 0$. We now define

$$(I_{\mathrm{mod}})_h^{2h} := (\mathrm{Id} - B_{2h} W_{2h}^T)\, I_h^{2h}, \tag{6}$$

and find by construction $W_{2h}^T (I_{\mathrm{mod}})_h^{2h} d_h = 0$.

To satisfy $W_h^T d_h = 0$, we consider a special class of smoothing operators. Observing that the condition $W_h^T d_h = 0$ is equivalent to $z_h \in V_h$, where z_h is the iterate in the mth–smoothing step, it is easy to construct suitable smoothers. The implementation is not based on A_S but on A_{num}, where A_{num} is obtained from A_S by multiplying the second block line with A_{NN}^{-1}, and the observation that u_h satisfies $A_{\mathrm{num}} u_h = f_S$. A closer look at this line yields that $(u_h)_N = -D^{-1} M^T (u_h)_I$. Thus a suitable smoother has to satisfy the second block line of the system exactly. Since the block diagonal matrix is the identity, this can be easily achieved. In particular, a GaußSeidel smoother where the unknowns are ordered blockwise like $(u_I^T, u_N^T)^T$ guarantees $W_h^T d_h = 0$, and thus satisfies the assumptions of the approximation lemma. We remark that other smoothing operators, e.g., ILU–type smoother, can also be used, if they are modified by one postprocessing step. Additionally, one has to solve a scalar equation for each unknown on the interior of the non–mortar sides. Our multigrid method will be now defined in terms of A_{num}, the modified restriction $(I_{\mathrm{mod}})_h^{2h}$, the special smoother and the standard prolongation I_{2h}^h. A symmetrized version can be obtained by replacing I_{2h}^h by $(\mathrm{Id} - W_h B_h^T) I_{2h}^h$, and using a symmetric smoother.

The following theorem is based on the smoothing and approximation properties and can be found in [9]. We remark that the implementation of A_{num} is based on static condensation, which can be carried out locally, and the saddle point problem.

Theorem 4. *The convergence rates for the $\mathcal{W}$–cycle are independent of the number of refinement levels provided that the number of smoothing steps is large enough.*

5 Numerical examples

The method described above has been implemented in the framework of the finite element toolbox UG [2]. In particular, the subroutines for computing A_{num} and the modified defect restriction have been implemented. We present numerical results for the scalar case as well as for linear elasticity on a L–shaped domain. In both examples, linear elements on triangles and bilinear elements on quadrilaterals are used. The coarse grid used for both computations is shown in the left of Figure 1. Standard uniform refinement techniques are used. For our numerical experiments, we used a symmetric GaußSeidel smoother.

Let us first consider the linear elastic case. The domain is given by $\Omega = [0,1]\times[0,1]\setminus[0.5,1]\times[0.5,1]$. Hooke's tensor is resulting from plane strain assumption and the material parameters are $\lambda = 121154$ and $\mu = 161538$. Dirichlet conditions are imposed on the upper $u = (0,-0.01)$ and right $u = (-0.01,0)$ part of the boundary as well as on the lower left part of the boundary. Here, we have $u_1 = 0$ for $x = 0,\ y \leq 0.5$ and $u_2 = 0$ for $x \leq 0.5,\ y = 0.0$. The resulting deformed grid scaled by a factor of ten is shown in the left part of Figure 1. The asymptotic convergence rates for the

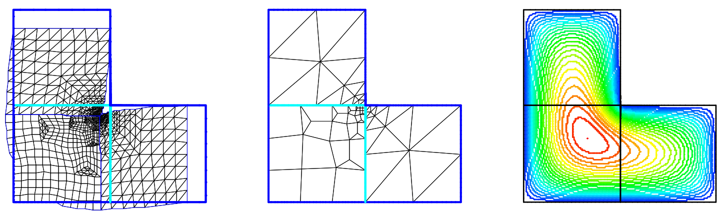
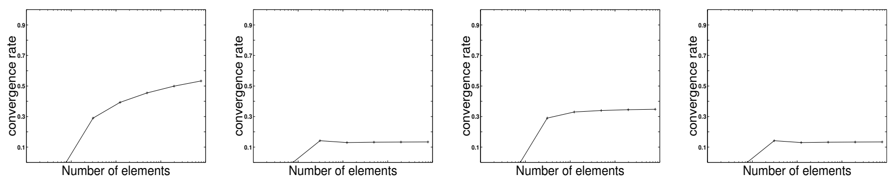


Fig. 1. Distorted grid (elasticity) (left), coarse grid (middle) and isolines (scalar) (right)

$\mathcal{W}$–cycle are depicted in Figure 3 and confirm our theoretical findings as they get constant for small h. Here, for the $\mathcal{V}$–cycle we used a different coarse grid consisting of nonmatching quadrilaterals. The results are shown on the left of Figure 3. The $\mathcal{V}$–cycle does not behave as well as the $\mathcal{W}$–cycle. However, for small h the convergence rates might be independent of the refinement level.

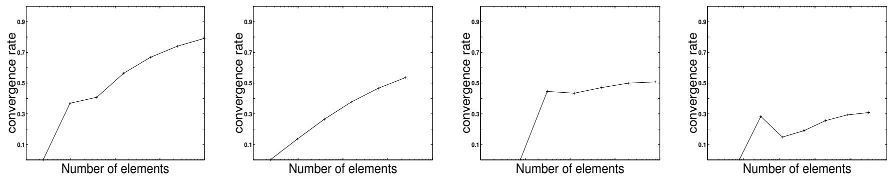
As second and scalar elliptic example we consider the problem $-\Delta u \equiv 1$



$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

Fig. 2. Convergence rates for a Gauß–Seidel smoother (scalar elliptic)

and homogeneous boundary conditions on the same L–shaped domain as described above. Again, the theoretical results for the $\mathcal{W}$–cycle are confirmed by the numerical ones. Furthermore, we obtain level independent convergence rates for the $\mathcal{V}(3,3)$–cycle and the convergence rates of the $\mathcal{V}(1,1)$–cycle seem to get independent of the level for small h. For further scalar elliptic examples showing the robustness of the method for problems with discontinuous coefficients or domains involving crosspoint, we refer to [11]. In addition, even the $\mathcal{V}$–cycle for a problem involving a domain with a slit is shown to converge independent of the refinement level.



$\mathcal{V}(1,1)$–cycle $\mathcal{V}(3,3)$–cycle $\mathcal{W}(1,1)$–cycle $\mathcal{W}(3,3)$–cycle

Fig. 3. Convergence rates for a Gauß–Seidel smoother (linear elasticity)

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