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An axiomatic approach to scalar data interpolation on surfaces

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Abstract We discuss possible algorithms for interpolating data given on a set of curves in a surface of $I\!R^3$. We propose a set of basic assumptions to be satisfied by the interpolation algorithms which lead to a set of models in terms of possibly degenerate elliptic partial differential equations. The Absolutely Minimizing Lipschitz Extension model (AMLE) is singled out and studied in more detail. We study the correctness of our numerical approach and we show experiments illustrating the interpolation of data on some simple test surfaces like the sphere and the torus.

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1 Introduction

Our purpose in this paper will be to discuss possible algorithms for interpolating data on surfaces embedded in $I\!R^3$ starting from data given on a set of curves contained in the surface. Following the approach in [14], where interpolation operators

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in the plane were considered, our approach to the problem will be based on a set of axioms or formal requirements that an interpolation operator should satisfy. Then we deduce that any operator which interpolates continuous data given on a set of curves on the surface can be given as the viscosity solution of an elliptic partial differential equation on the manifold, extending the corresponding results obtained in the plane [14]. In the case of the plane, the main motivation comes from image processing, where interpolation techniques have been proposed in the literature for 'perceptually motivated' coding applications [13], or from data interpolation for the encoding of DEM models from a sample of its level curves [19],[7]. An additional motivation in the case of surfaces in $\mathbb{R}^3$ comes from the interpolation of climate maps of the surface of the sphere [28]. In that case the algorithms currently used for interpolation are based on explicit representations of the surface based on triangulations or grid points [28]. Our numerical approach will be based on an implicit representation of the surface as a level set of the distance function to it and the numerical methods developed in [10] for computing the solutions of partial differential equations on implicit surfaces. Preliminary results were reported in [26].

Let us briefly describe the plan of the paper. In Section 2 we introduce a formal set of axioms which should be satisfied by any interpolation operator on a surface in $\mathbb{R}^3$ (of the type considered here) and derive the associated partial differential equation. In Section 3 we briefly discuss the so called AMLE interpolation, an example particularly relevant in applications. In Section 4 we explain the numerical approach used to solve the AMLE model. In Section 5 we display some numerical results.

2 Axiomatic analysis of interpolation operators

Let $(\mathcal{M}, g)$ be a compact, connected smooth two-dimensional surface in $\mathbb{R}^3$. As usual, given a point $\xi \in \mathcal{M}$, we denote by $T_\xi\mathcal{M}$ the tangent plane to $\mathcal{M}$ at the point ξ. Let $\mathcal{C}$ denote the family of continuous curves $\Gamma : [a, b] \to \mathcal{M}$ which are one-to-one in (a, b) and $\Gamma(a) = \Gamma(b)$. Let $\mathcal{D}$ denote the family of open subsets Ω of $\mathcal{M}$ such that the boundary of Ω, denoted by $\partial\Omega$ consists of a finite union of curves in $\mathcal{C}$. For each $\Omega \in \mathcal{D}$, let $C(\partial\Omega)$ be the set of continuous functions defined on $\partial\Omega$.

$\mathcal{D}$ represents the set of domains Ω where we interpolate the data given on $\partial\Omega$. This set of domains may be too general to be able to interpolate any data $\varphi \in C(\partial\Omega)$ to a continuous function inside Ω, and sometimes it is necessary to assume that the domain Ω has a smooth boundary. Since this does not play an essential role in the present section to identify the interpolation operators, and to avoid any unnecessary complication, we shall use the set $\mathcal{D}$ defined above. When given any explicit example of interpolation operator, the regularity of the domains in $\mathcal{D}$ will be explicitly stated.

We shall consider an interpolation operator as a transformation E which associates with each $\Omega \in \mathcal{D}$ and each $\varphi \in C(\partial\Omega)$ a unique function $u = E(\varphi, \partial\Omega)$ defined on Ω satisfying the following set of assumptions:

- (A1) **Boundary Values:**

$$E(\varphi, \partial\Omega)|_{\partial\Omega} = \varphi \quad \text{for any } \Omega \in \mathcal{D} \text{ and } \varphi \in C(\partial\Omega).$$

In other words, $E(\varphi, \partial\Omega)$ represents an interpolation or extension of φ into Ω.

– (A2) **Comparison Principle:**

$$E(\varphi, \partial\Omega) \leq E(\tilde{\varphi}, \partial\Omega)$$

for any $\Omega \in \mathcal{D}$ and any $\varphi, \tilde{\varphi} \in C(\partial\Omega)$ with $\varphi \leq \tilde{\varphi}$.

– (A3) **Stability Principle:** Let $\Omega, \Omega' \in \mathcal{D}$, $\Omega' \subseteq \Omega$. Then

$$E(E(\varphi, \partial\Omega)|_{\partial\Omega'}, \partial\Omega') = E(\varphi, \partial\Omega)|_{\Omega'}$$

holds for any $\varphi \in C(\partial\Omega)$. This principle means that no new application of the interpolation can improve a given interpolant.

– (A4) **Regularity Principle:** Let ξ be a point on $\mathcal{M}$, $U \subseteq I\!R^2$ an open set containing 0, and $\psi : U \to \mathcal{M}$ be any coordinate system such that $\psi(0) = \xi$. Let $g_{ij}(x)$ and $\Gamma^k_{ij}(x)$ (indices i, j, k run from 1 to 2) denote, respectively, the coefficients of the first fundamental form of $\mathcal{M}$ and the Christoffel symbol computed in the coordinate system ψ. For simplicity we shall denote by G the (symmetric) matrix $(g_{ij}(0))$ and by Γ^k the matrix formed by the coefficients $(\Gamma^k_{ij}(0))$, $i, j, k = 1, 2$. Let us denote by B_r the geodesic ball of radius r around ξ. Let $SM(2)$ be the set of symmetric 2×2 matrices. Let $A = (A^i_j)$ be a matrix such that $GA \in SM(2)$, and $p = (p^i) \in I\!R^2$. We shall use Einstein's convention that repeated indices are summed, and we denote by $(a, b) = a_i b^i$.

We can now state the regularity principle. For any quadratic polynomial $Q : U \to I\!R$ given by

$$\begin{aligned} Q(x) &= \frac{1}{2} g_{ij}(0) A^i_l x^l x^j + g_{ij}(0) p^i x^j + c \\ &= \frac{1}{2}(GAx, x) + (Gp, x) + c, \end{aligned} \tag{1}$$

the operator E should satisfy

$$\lim_{r \to 0} \frac{E(Q \circ \psi^{-1}|_{\partial B_r}, \partial B_r)(\xi) - Q \circ \psi^{-1}(\xi)}{r^2/2} = F(A, p, c, \xi, G, \Gamma^k) \tag{2}$$

where F is a continuous function of its first argument. This requirement embodies several properties. First, it expresses that the interpolant of a quadratic polynomial near ξ may be locally expressed in terms of its elements A, p, c, the point ξ, and the metric tensor and Christoffel symbols. Since any smooth function u on U is given locally as a quadratic polynomial, this (together with the comparison principle) implies that the operator depends only on the first and second derivatives of u. Moreover, when combined with the comparison principle it permits to prove that the interpolation operator is intrinsic and the regularity principle also gives the transformation properties of F when we change coordinates (see Theorem 1).

We could have written that the limit in (2) converges to a function $F(B, q, c, \xi, G, \Gamma^k)$ where $B = GA$, $q = Gp$, but this would mean only an equivalent change of notation. We prefer to use our notation since it will be more convenient in the proof of Theorem 1. Some further clarifying remarks will be given below.

- (A5) **Grey Scale Shift Invariance:**

$$E(\varphi + c, \partial\Omega) = E(\varphi, \partial\Omega) + c$$

for any $\Omega \in \mathcal{D}$, $\varphi \in C(\partial\Omega)$, $c \in I\!R$.
- (A6) **Linear Grey Scale Invariance:**

$$E(\lambda\varphi, \partial\Omega) = \lambda E(\varphi, \partial\Omega)$$

for any $\Omega \in \mathcal{D}$, $\varphi \in C(\partial\Omega)$, and any $\lambda \in I\!R$.

Let us describe the operators satisfying the above set of assumptions. For that let us introduce some more notation which will clarify the notation used in the regularity principle. For any $\xi \in \mathcal{M}$, we denote by $T_\xi\mathcal{M}$ the tangent space to $\mathcal{M}$ at the point ξ. By $T_\xi^*\mathcal{M}$ we denote its dual space. The scalar product of two vectors $v, w \in T_\xi\mathcal{M}$ will be denoted by $\langle v, w\rangle$, and the action of a covector $p^* \in T_\xi^*\mathcal{M}$, on a vector $v \in T_\xi\mathcal{M}$, will be denoted by (p^*, v). If $\psi : U \to \mathcal{M}$ is a coordinate system such that $\psi(0) = \xi$, and $g_{ij}(x)$ are the coefficients of the first fundamental form of $\mathcal{M}$ in ψ, we shall often write $g_{ij}(\xi')$ instead of $g_{ij}(x)$ where $x \in U$ is such that $\psi(x) = \xi'$. Then, if $v, w \in T_\xi\mathcal{M}$, we have $\langle v, w\rangle_\xi = g_{ij}(\xi)v^i w^j$, where v^i, w^i are the coordinates of v, w in the basis $\frac{\partial}{\partial x^i}|_\xi$ of $T_\xi\mathcal{M}$. Using this basis for $T_\xi\mathcal{M}$ and the dual basis on $T_\xi^*\mathcal{M}$, if $p^* \in T_\xi^*\mathcal{M}$, and $v \in T_\xi\mathcal{M}$, we have $(p^*, v) = p_i v^i$. Notice that we may write $(p^*, v) = g_{ij}(\xi)p^i v^j$ where p^i are the coordinates of the vector p associated to the covector p^*. The relation between both coordinates is given by

$$p_i = g_{ij}(\xi)p^j, \quad \text{or} \quad p^i = g^{ij}(\xi)p_j, \tag{3}$$

where $g^{ij}(\xi)$ denotes the coefficients of the inverse matrix of $g_{ij}(\xi)$. By a slight abuse of notation, we shall write (3) as

$$p^* = Gp \quad \text{or} \quad p = G^{-1}p^*.$$

In this way $G : T_\xi\mathcal{M} \to T_\xi^*\mathcal{M}$. In the case that ψ is a geodesic coordinate system, the matrix G is the identity matrix $I = (\delta_{ij})$, and I maps vectors to covectors, i.e., $I : T_\xi\mathcal{M} \to T_\xi^*\mathcal{M}$. We shall denote by I^{-1} the inverse of I, mapping covectors to vectors.

Let us now clarify the notation used in (1). If $U \subseteq I\!R^2$, and $\psi : U \to \mathcal{M}$ is a coordinate system with $\psi(0) = \xi$, then $\psi \circ d\psi(0)^{-1} : U' \subseteq T_\xi\mathcal{M} \to \mathcal{M}$ is a new coordinate system. If we identify T_0U with $I\!R^2$ and $\{e_i\}$ denotes its canonical basis, then $e'_i = d\psi(0)e_i$ satisfy $\langle e'_i, e'_j\rangle = g_{ij}(\xi)$. From now on, we shall use this identification, thus we shall interpret that any coordinate system around a point $\xi \in \mathcal{M}$ is defined on a neighborhood of 0 in the tangent space $T_\xi\mathcal{M}$.

We shall also use this coordinate system to express a bilinear map $\hat{A} : T_\xi\mathcal{M} \times T_\xi\mathcal{M} \to I\!R$. Indeed, if (A_{ij}) is the matrix of $\hat{A}$ in this basis, and $v, w \in T_\xi\mathcal{M}$, we may write $\hat{A}(v, w) = A_{ij}v^j w^i$. If $A^i_{\ j} = g^{ik}(\xi)A_{kj}$, then $A^i_{\ j}$ determines a map, called $A : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$ such that $\hat{A}(v, w) = \langle Av, w\rangle = (GAv, w)$. Observe that $GA : T_\xi\mathcal{M} \to T_\xi^*\mathcal{M}$. Observe also that our notation $A^i_{\ j}$ already indicates that $A = (A^i_{\ j})$ maps vectors to vectors, and this is the interpretation of the matrix argument A in (2). We shall identify matrices with maps.

As usual, we say that a linear map $C : T_\xi\mathcal{M} \to T_\xi^*\mathcal{M}$ is symmetric if $(Cv, w) = (Cw, v)$ for any $v \in T_\xi\mathcal{M}$, $w \in T_\xi\mathcal{M}$. From now on, we shall use the notation

$$SM(2) := \{A : T_\xi\mathcal{M} \to T_\xi^*\mathcal{M},\ A \text{ is symmetric}\}.$$

The above notation explains the use of $g_{ij}(0)$ in formula (1). Thus we may write

$$Q(x) = \frac{1}{2}\langle Ax, x\rangle + \langle p, x\rangle + c = \frac{1}{2}(GAx, x) + (Gp, x) + c.$$

Given a function u on $\mathcal{M}$, let us denote by $D_\mathcal{M}u$ and $D^2_\mathcal{M}u$ the gradient and Hessian of u, respectively. In a coordinate system $D_\mathcal{M}u$ is the covector $\frac{\partial u}{\partial x^i}$, and $D^2_\mathcal{M}u$ is the matrix $\frac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}\frac{\partial u}{\partial x^k}$ which acts on tangent vectors. Thus, with this notation $D^2_\mathcal{M}u(\xi) : T_\xi\mathcal{M} \times T_\xi\mathcal{M} \to \mathbb{R}$ is a bilinear map, $\xi \in \mathcal{M}$. Let us write $\nabla_\mathcal{M}u$ the vector of coordinates $g^{ij}\frac{\partial u}{\partial x^j}$. Then $|\nabla_\mathcal{M}u(\xi)|^2_\xi = \langle\nabla_\mathcal{M}u(\xi), \nabla_\mathcal{M}u(\xi)\rangle_\xi$. To simplify our notation we shall write Du and ∇u instead of $D_\mathcal{M}u$, and $\nabla_\mathcal{M}u$. The vector field ∇u satisfies $\langle\nabla u, v\rangle_\xi = du(v)$, $v \in T_\xi\mathcal{M}$, du being the differential of u.

As usual, $O(f)$ will denote any expression which is bounded by $C|f|$ for some constant $C > 0$.

The following results represent an extension of the results in [14], which, in turn, were inspired by [2].

Proposition 1 *Assume that E is an interpolation operator that satisfies* (A2) *and* (A4) *. Let φ be a smooth function on a neighborhood of $\xi \in \mathcal{M}$. Let $\psi : U \to \mathcal{M}$ be a coordinate system around ξ. Let B_r be the geodesic ball of radius r around ξ. Then*

$$\frac{E(\varphi|_{\partial B_r}, \partial B_r)(\xi) - \varphi(\xi)}{r^2/2} \to F(G^{-1}D^2(\varphi\circ\psi)(0), G^{-1}D(\varphi\circ\psi)(0), \varphi(\xi), \xi, G, \Gamma^k) \tag{4}$$

as $r \to 0$.

Proof Let us consider the Taylor expansion of $\varphi \circ \psi$ around 0

$$\varphi\circ\psi(x) = \varphi\circ\psi(0) + \frac{\partial(\varphi\circ\psi)}{\partial x^i}(0)x^i + \frac{1}{2}\frac{\partial^2(\varphi\circ\psi)}{\partial x^i \partial x^j}(0)x^i x^j + O(|x|^3).$$

For each $\epsilon \in \mathbb{R}$ we define the quadratic polynomial

$$Q_\epsilon(x) = \varphi\circ\psi(0) + \frac{\partial(\varphi\circ\psi)}{\partial x^i}(0)x^i + \frac{1}{2}\frac{\partial^2(\varphi\circ\psi)}{\partial x^i \partial x^j}(0)x^i x^j + \frac{\epsilon}{2}|x|^2.$$

Let $\epsilon > 0$. Then, in a neighborhood of 0, we have

$$Q_{-\epsilon}(x) \le \varphi\circ\psi(x) \le Q_\epsilon(x).$$

By choosing r small enough we have

$$Q_{-\epsilon}\circ\psi^{-1}(\zeta) \le \varphi(\zeta) \le Q_\epsilon\circ\psi^{-1}(\zeta) \tag{5}$$

for all $\zeta \in B_r$.

Call $G_\epsilon(r) : B_r \to I\!R$ the function obtained by interpolating Q_ϵ restricted to ∂B_r,

$$G_\epsilon(r) = E(Q_\epsilon \circ \psi^{-1}|_{\partial B_r}, \partial B_r).$$

By the comparison principle it follows that

$$G_{-\epsilon}(r) \leq E(\varphi|_{\partial B_r}, \partial B_r) \leq G_\epsilon(r),$$

and, in particular, this holds at ξ,

$$G_{-\epsilon}(r)(\xi) \leq E(\varphi|_{\partial B_r}, \partial B_r)(\xi) \leq G_\epsilon(r)(\xi).$$

Since $Q_{-\epsilon}(\xi) = Q_\epsilon(\xi) = \varphi(\xi)$, we subtract this quantity to obtain

$$G_{-\epsilon}(r)(\xi) - Q_{-\epsilon}(\xi) \leq E(\varphi|_{\partial B_r}, \partial B_r)(\xi) - \varphi(\xi) \leq G_\epsilon(r)(\xi) - Q_\epsilon(\xi). \quad (6)$$

Now, applying the regularity principle to the quadratic polynomials Q_ϵ and $Q_{-\epsilon}$, we obtain

$$\lim_{r\to 0} \frac{G_{-\epsilon}(r)(\xi) - Q_{-\epsilon}(\xi)}{r^2/2}$$
$$= F(G^{-1}D^2(\varphi \circ \psi)(0) - \epsilon I, G^{-1}D(\varphi \circ \psi)(0), \varphi(\xi), \xi, G, \Gamma^k), \quad (7)$$

and

$$\lim_{r\to 0} \frac{G_{\epsilon}(r)(\xi) - Q_{\epsilon}(\xi)}{r^2/2}$$
$$= F(G^{-1}D^2(\varphi \circ \psi)(0) + \epsilon I, G^{-1}D(\varphi \circ \psi)(0), \varphi(\xi), \xi, G, \Gamma^k). \quad (8)$$

We can therefore divide each member of inequalities (6) by $r^2/2$, use (7), (8), to obtain

$$F(G^{-1}D^2(\varphi \circ \psi)(0) - \epsilon I, G^{-1}D(\varphi \circ \psi)(0), \varphi(\xi), \xi, G, \Gamma^k)$$
$$\leq \liminf_{r\to 0} \frac{E(\varphi|_{\partial B_r}, \partial B_r)(\xi) - \varphi(\xi)}{r^2/2}$$
$$\leq \limsup_{r\to 0} \frac{E(\varphi|_{\partial B_r}, \partial B_r)(\xi) - \varphi(\xi)}{r^2/2}$$
$$\leq F(G^{-1}D^2(\varphi \circ \psi)(0) + \epsilon I, G^{-1}D(\varphi \circ \psi)(0), \varphi(\xi), \xi, G, \Gamma^k).$$

Letting $\epsilon \to 0$, and using the continuity of F in its first argument, we obtain (4). □

Remark 1 We clearly see from the proof of Proposition 1 that, due to the interplay between assumptions (A2) and (A4), (A4) could be weakened. Indeed, if we assume that for any function Q given by its Taylor expansion in a neighborhood of $x = 0$, the limit in (2) is a function which depends on $D^i Q(0)$, $i \geq 0$, i.e., is $F(\{D^i Q(0) : i \geq 0\}, \xi, G, \Gamma^k)$, then the argument of Proposition 1 proves that F only depends on $Q(0)$, $\nabla Q(0)$ and $D^2 Q(0)$, as in our statement of (A4). Thus, (A4) is an expression of the property that the interpolant of a smooth function depends regularly on the coefficients of its Taylor expansion. Its interplay with (A2), already explained in the paragraph after (2), permits to prove that the underlying operator is at most of second order.

Lemma 1 *Let $\xi \in \mathcal{M}$, and $\psi : U \to \mathcal{M}$ be a coordinate system around ξ. Let G, Γ^k be the metric coefficients and the Christoffel symbols of $\mathcal{M}$ in the coordinate system ψ at the point ξ. Let $A_1, A_2 : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$ be two matrices such that GA_1, GA_2 are symmetric, $p \in T_\xi\mathcal{M}$, $c \in I\!R$. If $GA_1 \leq GA_2$, then*

$$F(A_1, p, c, \xi, G, \Gamma^k) \leq F(A_2, p, c, \xi, G, \Gamma^k).$$

Proof Consider the quadratic polynomials $Q_i : T_\xi M \to I\!R$ defined in the coordinate system ψ by

$$Q_i(x) = \frac{1}{2}(GA_i x, x) + (Gp, x) + c, \quad i = 1, 2.$$

Observe that $Q_1(x) \leq Q_2(x)$ for all $x \in T_\xi\mathcal{M}$ and $Q_1(0) = Q_2(0)$, hence, by the comparison principle (A2) we have

$$E(Q_1 \circ \psi^{-1}|_{\partial B_r}, \partial B_r) - Q_1 \circ \psi^{-1}(\xi) \leq E(Q_2 \circ \psi^{-1}|_{\partial B_r}, \partial B_r) - Q_2 \circ \psi^{-1}(\xi)$$

for $r > 0$ small enough. Dividing by $r^2/2$, letting $r \to 0$, and using the regularity principle we get

$$F(A_1, p, c, \xi, G, \Gamma^k) \leq F(A_2, p, c, \xi, G, \Gamma^k).$$

□

Lemma 2 *Let U^1, U^2 be two neighborhoods of 0 in $I\!R^2$ and let $\psi_i : U^i \to \mathcal{M}$ be two coordinate systems around the point $\xi \in \mathcal{M}$, i.e., $\psi_i(0) = \xi$. Assume that the change of coordinates $\Psi = \psi_1^{-1} \circ \psi_2 : U^2 \to U^1$ is a diffeomorphism. Let G, Γ (resp. $\overline{G}, \overline{\Gamma}$) be the metric coefficients and the Christoffel symbols of $\mathcal{M}$ in the coordinate system ψ_1 (resp. ψ_2) at the point ξ. Let $Q : U^1 \to I\!R$ be the quadratic polynomial*

$$Q(v) = \frac{1}{2}(GAv, v) + (Gp, v) + c \tag{9}$$

Let $\overline{Q}(\bar{v}) := (Q \circ \Psi)(\bar{v})$. Then $\overline{Q}(\bar{v}) = Q'(\bar{v}) + O(|\bar{v}|^3)$ in a neighborhood of 0, where Q' is the quadratic polynomial

$$\begin{aligned} Q'(\bar{v}) = {} & \frac{1}{2}(\overline{G}B^{-1}AB\bar{v}, \bar{v}) + \frac{1}{2}(\overline{\Gamma}(B^tGp)(\bar{v}), \bar{v}) \\ & -\frac{1}{2}(B^t\Gamma(Gp)(B\bar{v}), \bar{v}) + (B^tGp, \bar{v}) + c, \end{aligned} \tag{10}$$

and $B = D\Psi(0)$.

Proof We assumed that U^1, U^2 are such that the change of coordinates $\Psi = \psi_1^{-1} \circ \psi_2 : U^2 \to U^1$ is a diffeomorphism. Since

$$\overline{Q}(\bar{v}) := (Q \circ \Psi)(\bar{v}) = \frac{1}{2}(GA\Psi(\bar{v}), \Psi(\bar{v})) + (Gp, \Psi(\bar{v})) + c, \tag{11}$$

expanding $\Psi(\bar{v})$ as a Taylor series around the point 0

$$\Psi(\bar{v}) = D\Psi(0)\bar{v} + \frac{1}{2}D^2\Psi(0)(\bar{v}, \bar{v}) + O(|\bar{v}|^3),$$

(where $D^2\Psi(0)(\bar{v},\bar{v})$ denotes the vector whose coordinates are $D^2\Psi_i(0)(\bar{v},\bar{v})$, Ψ_i being the coordinates of Ψ, $i=1,2$) and plugging it into (11) we obtain

$$\overline{Q}(\bar{v}) = \frac{1}{2}(GAD\Psi(0)\bar{v}, D\Psi(0)\bar{v}) + (Gp, D\Psi(0)\bar{v}) + \frac{1}{2}(Gp, D^2\Psi(0)(\bar{v},\bar{v})) + c + O(|\bar{v}|^3). \tag{12}$$

Let $Q'(\bar{v})$ be the quadratic polynomial appearing at the right hand side of (12). If we denote $B = D\Psi(0)$, we may write

$$Q'(\bar{v}) = \frac{1}{2}(GAB\bar{v}, B\bar{v}) + (Gp, B\bar{v}) + \frac{1}{2}(Gp, D^2\Psi(0)(\bar{v},\bar{v})) + c.$$

Let us write the map Ψ in coordinates as $u^i = u^i(\bar{u}^k)$. We recall the following transformation formulas for the metric coefficients [24]

$$g_{\alpha\beta}\frac{\partial u^\beta}{\partial \bar{u}^k} = \bar{g}_{lk}\frac{\partial \bar{u}^l}{\partial u^\alpha}, \tag{13}$$

or, in more compact notation,

$$B^t G = \overline{G}B^{-1}, \tag{14}$$

and the Christoffel symbols

$$\frac{\partial^2 u^\beta}{\partial \bar{u}^i \partial \bar{u}^j} = \overline{\Gamma}_{ij}^k \frac{\partial u^\beta}{\partial \bar{u}^k} - \Gamma_{\alpha'\beta'}^\beta \frac{\partial u^{\alpha'}}{\partial \bar{u}^i}\frac{\partial u^{\beta'}}{\partial \bar{u}^j}. \tag{15}$$

Observe that as a map $B : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$, while $B^t : T_\xi^*\mathcal{M} \to T_\xi^*\mathcal{M}$.

If q is a covector of coordinates q_k, let us denote by $\Gamma(q)$ the bilinear map (or matrix) whose coordinates are $\Gamma_{ij}^k q_k$. Now, we may compute

$$\begin{aligned}(Gp, D^2\Psi(0)(\bar{v},\bar{v})) &= g_{\alpha\beta}p^\alpha \frac{\partial^2 u^\beta}{\partial \bar{u}^i \partial \bar{u}^j}\bar{v}^i\bar{v}^j \\ &= g_{\alpha\beta}\frac{\partial u^\beta}{\partial \bar{u}^k}p^\alpha \overline{\Gamma}_{ij}^k \bar{v}^i\bar{v}^j - g_{\alpha\beta}p^\alpha\Gamma_{\alpha'\beta'}^\beta \frac{\partial u^{\alpha'}}{\partial \bar{u}^i}\frac{\partial u^{\beta'}}{\partial \bar{u}^j}\bar{v}^i\bar{v}^j.\end{aligned}$$

Let us transform each of these expressions. The first term is

$$\overline{\Gamma}_{ij}^k \frac{\partial u^\beta}{\partial \bar{u}^k} g_{\alpha\beta}p^\alpha \bar{v}^i\bar{v}^j = \overline{\Gamma}_{ij}(B^t Gp)\bar{v}^i\bar{v}^j = (\overline{\Gamma}(B^t Gp)(\bar{v}), \bar{v}).$$

Now, the second term is

$$\begin{aligned}g_{\alpha\beta}p^\alpha\Gamma_{\alpha'\beta'}^\beta \frac{\partial u^{\alpha'}}{\partial \bar{u}^i}\frac{\partial u^{\beta'}}{\partial \bar{u}^j}\bar{v}^i\bar{v}^j &= \Gamma_{\alpha'\beta'}(Gp)\frac{\partial u^{\alpha'}}{\partial \bar{u}^i}\frac{\partial u^{\beta'}}{\partial \bar{u}^j}\bar{v}^i\bar{v}^j \\ &= \Gamma_{\alpha'\beta'}(Gp)(B\bar{v})^\alpha(B\bar{v})^\beta \\ &= (\Gamma(Gp)(B\bar{v}), B\bar{v}) = (B^t\Gamma(Gp)(B\bar{v}), \bar{v}).\end{aligned}$$

Collecting both expressions we have

$$(Gp, D^2\Psi(0)(\bar{v},\bar{v})) = (\overline{\Gamma}(B^t Gp)(\bar{v}), \bar{v}) - (B^t\Gamma(Gp)(B\bar{v}), \bar{v}). \tag{16}$$

Finally, let us observe that

$$(GAB\bar{v}, B\bar{v}) = (B^t GAB\bar{v}, \bar{v}) = (\overline{G}B^{-1}AB\bar{v}, \bar{v}) \tag{17}$$

and

$$(Gp, B\bar{v}) = (B^t Gp, \bar{v}). \tag{18}$$

Thus, we may write

$$Q'(\bar{v}) = \frac{1}{2}(\overline{G}B^{-1}AB\bar{v}, \bar{v}) + \frac{1}{2}(\overline{\Gamma}(B^t Gp)(\bar{v}), \bar{v})$$
$$-\frac{1}{2}(B^t\Gamma(Gp)(B\bar{v}), \bar{v}) + (B^t Gp, \bar{v}) + c.$$

□

Proposition 2 *Let E be an interpolation operator on $\mathcal{M}$ satisfying the comparison principle. Let $\psi_1 : U^1 \to \mathcal{M}$ be a coordinate systems around $\xi \in \mathcal{M}$. Let G, Γ be the metric coefficients and the Christoffel symbols of $\mathcal{M}$ in the coordinate system ψ_1 at the point ξ. For any symmetric matrix $X = (X_{ij})$, $q \in T^*_\xi\mathcal{M}$, and $a \in \mathbb{R}$, let us define the function*

$$H(X, q, a, \xi) = F(I^{-1}X, I^{-1}q, a, \xi, I, 0), \tag{19}$$

that is, H is the function F obtained when using a geodesic coordinate system. Then

$$F(A, p, a, \xi, G, \Gamma^k) = H(B^t(GA - \Gamma(Gp))B, B^t Gp, c, \xi) \tag{20}$$

for any matrix A such that $GA \in SM(2)$, and any vector p, where $BB^t = G^{-1}$. Moreover the function H satisfies

$$H(A', p', c, \xi) = H(R^t A' R, R^t p', c, \xi). \tag{21}$$

for any matrix $A' \in SM(2)$, any covector p', and any $R : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$ rotation matrix.

Our notation $BB^t = G^{-1}$ contains a slight abuse of notation, since $B : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$ and $B^t : T^*_\xi\mathcal{M} \to T^*_\xi\mathcal{M}$. The correct notation should be $BI^{-1}B^t$.

Proof We shall use the notation of Lemma 2. For any symmetric matrix $X = (X_{ij})$, any $q \in T^*_\xi\mathcal{M}$, and $a \in \mathbb{R}$, let us define the function F_1 by the identity

$$F_1(X, q, a, \xi, G, \Gamma^k) = F(G^{-1}X, G^{-1}q, a, \xi, G, \Gamma^k). \tag{22}$$

Since $Q \circ \psi_1^{-1} = \overline{Q} \circ \psi_2^{-1}$ in $U^1 \cap U^2$,

$$\lim_{r\to 0} \frac{E(Q \circ \psi_1^{-1}|_{\partial B_r}, \partial B_r)(\xi) - Q \circ \psi_1^{-1}(\xi)}{r^2/2} = F_1(GA, Gp, c, \xi, G, \Gamma^k), \tag{23}$$

and

$$\lim_{r\to 0} \frac{E(\overline{Q} \circ \psi_2^{-1}|_{\partial B_r},\ \partial B_r)(\xi) - \overline{Q} \circ \psi_2^{-1}(\xi)}{r^2/2}$$
$$= F_1(\overline{G}B^{-1}AB + \overline{\Gamma}(B^t Gp) - B^t \Gamma(Gp)B,\ B^t Gp, c, \xi, \overline{G}, \overline{\Gamma}^k) \quad (24)$$

we have

$$F_1(GA, Gp, c, \xi, G, \Gamma^k)$$
$$= F_1(\overline{G}B^{-1}AB + \overline{\Gamma}(B^t Gp) - B^t \Gamma(Gp)B,\ B^t Gp, c, \xi, \overline{G}, \overline{\Gamma}^k) \quad (25)$$

or, using (14),

$$F_1(GA, Gp, c, \xi, G, \Gamma^k)$$
$$= F_1(B^t(GA - \Gamma(Gp))B + \overline{\Gamma}(B^t Gp),\ B^t Gp, c, \xi, \overline{G}, \overline{\Gamma}^k). \quad (26)$$

Now, for any symmetric matrix $X = (X_{ij})$, any $q \in T^*_\xi \mathcal{M}$, and $a \in I\!R$, let us define the function F_2 by the identity

$$F_2(X, q, a, \xi, G, \Gamma^k) = F_1(X + \Gamma(q), q, a, \xi, G, \Gamma^k). \quad (27)$$

In terms of F_2, (26) can be written as

$$F_2(GA - \Gamma(Gp), Gp, c, \xi, G, \Gamma^k)$$
$$= F_2(B^t(GA - \Gamma(Gp))B,\ B^t Gp, c, \xi, \overline{G}, \overline{\Gamma}^k). \quad (28)$$

By varying the quadratic polynomials, the above equation holds for any matrix $A = (A^i{}_j)$ such that GA is symmetric, any invertible matrix $B : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$, and any $p \in T_\xi\mathcal{M}$.

Now, we choose ψ_1 as a geodesic coordinate system around ξ for which $G = I$, and $\Gamma^k = 0$. In this case, (14) can be written as $\overline{G} = B^t I B = B^t B$. We may write (28) as

$$F_2(IA, Ip, c, \xi, I, 0) = F_2(B^t IAB,\ B^t p, c, \xi, B^t B, \overline{\Gamma}^k), \quad (29)$$

and this identity holds for any symmetric matrix IA, any vector $p \in T_\xi\mathcal{M}$, and any invertible matrix B. Once again, we change variables and write $A' = B^t IAB$, $p' = B^t Ip$, $B' = B^{-1}$. Then we write (29) as

$$F_2(A', p', c, \xi, \overline{G}, \overline{\Gamma}^k) = F_2(B'^t A' B',\ B'^t p', c, \xi, I, 0), \quad (30)$$

and this identity holds for any symmetric matrix $A' : T_\xi\mathcal{M} \to T^*_\xi\mathcal{M}$, any $p' \in T^*_\xi\mathcal{M}$, and any invertible matrix $B' : T_\xi\mathcal{M} \to T_\xi\mathcal{M}$, where $\overline{G} = (B'^t)^{-1} B'^{-1}$. Now, for any symmetric matrix $X = (X_{ij})$, any $q \in T^*_\xi\mathcal{M}$, and scalar a, let us define the function H by the identity

$$H(X, q, a, \xi) = F_2(X, q, a, \xi, I, 0). \quad (31)$$

Note that by (22), (27), and (31), we have

$$H(X, q, a, \xi) = F(I^{-1}X, I^{-1}q, a, \xi, I, 0)$$

that is, H is the function F obtained when using a geodesic coordinate system. Hence, (30) can be written as

$$F_2(A', p', c, \xi, \overline{G}, \overline{\Gamma}^k) = H(B'^t A' B', B'^t p', c, \xi), \tag{32}$$

and using (22), (27), (31), we have formula (20). In particular, if we take ψ_1 as a geodesic coordinate system around ξ, and ψ_2 to be a rotation R with respect to ψ_1 so that $B' = R$, $\overline{G} = I$, and $\overline{\Gamma}^k = 0$ at the point ξ, then we may write (32) as

$$H(A', p', c, \xi) = H(R^t A' R, R^t p', c, \xi). \tag{33}$$

where A' is any matrix in $SM(2)$, $p' \in T^*_\xi \mathcal{M}$, and R is a rotation in $T_\xi \mathcal{M}$. □

Before continuing, let us prepare some more notation. Let $\xi \in \mathcal{M}$, and let $\{e_1^*, e_2^*\}$ denote a fixed orthonormal basis of $T^*_\xi \mathcal{M}$. Given $q \in T_\xi \mathcal{M}^*$, $q \neq 0$, let $|q|^2 = (I^{-1}q, q)$, and let R_q be the rotation in $T_\xi \mathcal{M}$ such that $R_q^t q = |q| e_1^*$. Let $\nu = \frac{I^{-1}q}{|q|}$, and $\nu^\perp$ the counterclockwise rotation of ν of $\frac{\pi}{2}$ degrees. Then we may write $R_q^t(x) = (\nu \otimes e_1^*)(x) + (\nu^\perp \otimes e_2^*)(x)$ (where $(a \otimes b)(x) = (a, x)b, a \in T_\xi \mathcal{M}$, $b, x \in T^*_\xi \mathcal{M}$), and for any $A : T_\xi \mathcal{M} \to T^*_\xi \mathcal{M}$ the matrix of $R_q^t A R_q$ in the basis $\{\nu, \nu^\perp\}$ is

$$R_q^t A R_q = \begin{pmatrix} A(\nu, \nu) & A(\nu, \nu^\perp) \\ A(\nu^\perp, \nu) & A(\nu^\perp, \nu^\perp) \end{pmatrix}.$$

Since $A(\nu, \nu^\perp) = A(\nu^\perp, \nu)$ we see that $R_q^t A R_q$ only depends on the three scalars $A(\nu, \nu)$, $A(\nu, \nu^\perp)$, $A(\nu^\perp, \nu^\perp)$. Note that these quantities are intrinsic, i.e., they do not depend on the coordinate system.

Proposition 3 *Assume that the interpolation operator E satisfies* (A2)*,* (A4)*,* (A5)*. Then there is a function $\mathcal{H} : SM(2) \times I\!R \times \mathcal{M} \to I\!R$ such that $H(A, q, c, \xi) = \mathcal{H}(R_q^t A R_q, |q|, \xi)$ for any $A \in SM(2)$, $q \in T^*_\xi \mathcal{M}$, $q \neq 0$, $c \in I\!R$, $\xi \in \mathcal{M}$. Moreover, $\mathcal{H}$ is a continuous and nondecreasing function of A.*

Proof Let $\xi \in \mathcal{M}$ and let $Q(x) = \frac{1}{2}(GAx, x) + (Gp, x) + c$ in the coordinate system ψ around ξ. Let $\alpha \in I\!R$. Since, by (A5), $E(Q \circ \psi^{-1} + \alpha, \partial B_r) - \alpha = E(Q \circ \psi^{-1}, \partial B_r)$, substracting $Q(\xi)$ at both sides, dividing by $\frac{r^2}{2}$, and letting $r \to 0+$, using Proposition 2 we obtain

$$F(A, p, c + \alpha, \xi, G, \Gamma^k) = F(A, p, c, \xi, G, \Gamma^k).$$

Since this holds for any $\alpha \in I\!R$ we deduce that F does not depend on c. This implies that H also does not depend on c. From now on, we shall write $H(A, q, \xi)$.

Let $A \in SM(2)$, $q \in I\!R^2$, $q \neq 0$, representing a covector, and $\xi \in \mathcal{M}$. Using (21) we have

$$\begin{aligned} H(A, q, \xi) &= H(R_q^t A R_q, R_q^t q, \xi) = H(R_q^t A R_q, |q| e_1^*, \xi) \\ &=: \mathcal{H}(R_q^t A R_q, |q|, \xi) \end{aligned}$$

The last assertion of the proposition is a consequence of the continuity of F in its first argument stated in (A4) and of Lemma 1. □

Remark 2 As an alternative, we could have restricted the regularity principle to geodesic coordinate systems, and, by means of the comparison principle and Lemma 2, we could have deduced the existence of the limit in (2) and proved it to be equal to the right hand side of (2). In any case, the computations in Proposition 2 connecting the functions F and H are unavoidable since Proposition 1, written in terms of F, is used in the proof of Theorem 1.

From now on we shall write $\mathcal{H}(A(\nu,\nu), A(\nu,\nu^\perp), A(\nu^\perp,\nu^\perp), |p|, \xi)$ instead of $\mathcal{H}(R_p^t A R_p, |p|, \xi)$.

Definition 1 *Let $\mathcal{H} : SM(2) \times I\!R \times \mathcal{M} \to I\!R$. We shall say that $\mathcal{H}(A, s, \xi)$ is elliptic if $\mathcal{H}$ is a nondecreasing function of A. If $A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$ and we define $\mathcal{H}(a, b, c, s, \xi) = \mathcal{H}(A, s, \xi)$, we shall also say that $\mathcal{H}(a, b, c, s, \xi)$ is elliptic.*

We now come to the central theorem of this section. It shows that the interpolation functions $u = E(\varphi, \partial\Omega)$ are solutions of a partial differential equation. For simplicity of notation let us introduce the terms:

$$\Lambda_1(u,\xi) = D^2_{\mathcal{M}} u\left(\frac{\nabla u}{|\nabla u|_\xi}, \frac{\nabla u}{|\nabla u|_\xi}\right)(\xi)$$
$$\Lambda_2(u,\xi) = D^2_{\mathcal{M}} u\left(\frac{\nabla u}{|\nabla u|_\xi}, \frac{\nabla u^\perp}{|\nabla u|_\xi}\right)(\xi)$$
$$\Lambda_3(u,\xi) = D^2_{\mathcal{M}} u\left(\frac{\nabla u^\perp}{|\nabla u|_\xi}, \frac{\nabla u^\perp}{|\nabla u|_\xi}\right)(\xi)$$

Theorem 1 *Assume that the interpolation operator E satisfies* (A1), (A2), (A3), (A4), *and* (A5). *Let $\mathcal{H}$ be the elliptic function given in Proposition 3. Set $\Omega \in \mathcal{D}$, $\theta \in C(\partial\Omega)$, and $u = E(\theta, \partial\Omega)$. Then u is a viscosity solution of*

$$\mathcal{H}\left(\Lambda_1(u,\xi), \Lambda_2(u,\xi), \Lambda_3(u,\xi), |\nabla u|_\xi, \xi\right) = 0 \quad \text{in } \Omega, \tag{34}$$

satisfying the boundary data $u|_{\partial\Omega} = \theta$, that is, for any $\varphi \in C^\infty(\Omega)$ with bounded derivatives such that $u-\varphi$ has a local maximum (minimum) at ξ_0, and $\nabla\varphi(\xi_0) \neq 0$, we get

$$\mathcal{H}\left(\Lambda_1(\varphi,\xi_0), \Lambda_2(\varphi,\xi_0), \Lambda_3(\varphi,\xi_0), |\nabla\varphi|_{\xi_0}, \xi_0\right) \geq 0 \tag{35}$$

(respectively, ≤ 0).

Proof By (A1), $u = E(\theta, \partial\Omega)$ assumes the prescribed boundary values. Let $\varphi \in C^\infty(\Omega)$ and suppose that $u - \varphi$ has a local maximum at ξ_0, and $\nabla\varphi(\xi_0) \neq 0$. As usual, B_r will be the geodesic ball of radius r around ξ_0. Then for some $r > 0$

$$u(\xi) \leq \varphi(\xi) + u(\xi_0) - \varphi(\xi_0) \quad \text{in } \partial B_r.$$

Using the comparison principle, we have

$$E(u|_{\partial B_r}, \partial B_r) \leq E((\varphi + u(\xi_0) - \varphi(\xi_0))|_{\partial B_r}, \partial B_r)$$

and, subtracting $u(\xi_0)$ on both sides,

$$\begin{aligned} &E(u|_{\partial B_r}, \partial B_r)(\xi_0) - u(\xi_0) \\ &\quad \le E((\varphi + u(\xi_0) - \varphi(\xi_0))|_{\partial B_r}, \partial B_r)(\xi_0) - u(\xi_0). \end{aligned}$$

By the stability principle (A3), the left hand side above is 0. We have

$$0 \le E\big((\varphi + u(\xi_0) - \varphi(\xi_0))|_{\partial B_r},\ \partial B_r\big)(\xi_0) - u(\xi_0).$$

Grey scale shift invariance (A5) enables us write this as

$$0 \le E(\varphi|_{\partial B_r},\ \partial B_r)(\xi_0) - \varphi(\xi_0).$$

Dividing by $r^2/2$ and letting $r \to 0$, using Proposition 1, we get

$$0 \le F(G^{-1}D^2(\varphi \circ \psi^{-1})(0), G^{-1}D(\varphi \circ \psi^{-1})(0), \varphi(\xi), \xi, G, \Gamma^k). \tag{36}$$

Now, using Proposition 2, and writing $D\varphi$ instead of $D(\varphi \circ \psi^{-1})(0)$ to simplify the notation, we may write

$$F(G^{-1}D^2\varphi, G^{-1}D\varphi, \varphi(\xi), \xi, G, \Gamma^k) = H(B^t(D^2\varphi - \Gamma(D\varphi))B, B^tD\varphi, \varphi(\xi), \xi),$$

where B is defined by the identity $G^{-1} = BI^{-1}B^t$. Now, we compute

$$\begin{aligned} |B^tD\varphi|^2 &= (I^{-1}B^tD\varphi, B^tD\varphi) = (BI^{-1}B^tG\nabla\varphi, G\nabla\varphi) \\ &= (G\nabla\varphi, \nabla\varphi) = |\nabla\varphi|_\xi^2 \end{aligned}$$

and

$$\begin{aligned} &B^t(D^2\varphi - \Gamma(D\varphi))B\Big(\frac{I^{-1}B^tD\varphi}{|B^tD\varphi|}, \frac{I^{-1}B^tD\varphi}{|B^tD\varphi|}\Big) \\ &= \frac{1}{|\nabla\varphi|_\xi^2}(D^2\varphi - \Gamma(D\varphi))(BI^{-1}B^tD\varphi, BI^{-1}B^tD\varphi) \\ &= (D^2\varphi - \Gamma(D\varphi))\left(\frac{\nabla\varphi}{|\nabla\varphi|_\xi}, \frac{\nabla\varphi}{|\nabla\varphi|_\xi}\right) \\ &= D^2_{\mathcal{M}}\varphi\left(\frac{\nabla\varphi}{|\nabla\varphi|_\xi}, \frac{\nabla\varphi}{|\nabla\varphi|_\xi}\right) \end{aligned}$$

and similarly for Λ_2 and Λ_3. Collecting all these facts we obtain (35). Similarly, if $u - \varphi$ has a local minimum at ξ_0, and $\nabla\varphi(\xi_0) \neq 0$, we obtain (the quantities are computed at $\xi = \xi_0$)

$$\mathcal{H}\left(\Lambda_1(\varphi, \xi_0), \Lambda_2(\varphi, \xi_0), \Lambda_3(\varphi, \xi_0), |\nabla\varphi|_{\xi_0}, \xi_0\right) \le 0$$

Since this holds for all $\xi_0 \in \Omega$, u is a viscosity solution of (34). □

Lemma 3 *Assume that the interpolation operator E satisfies* (A1)− (A5). *If, in addition, it satisfies* (A6)*, then*

$$\mathcal{H}(\lambda a, \lambda b, \lambda c, |\lambda p|_\xi, \xi) = \lambda\mathcal{H}(a, b, c, |p|_\xi, \xi), \tag{37}$$

for any $a, b, c \in I\!R$, $\xi \in \mathcal{M}$, $p \in T_\xi\mathcal{M}$, $\lambda > 0$.

Proof Let $\xi \in \mathcal{M}$, ψ be a coordinate system around ξ, and let Q be a quadratic polynomial in the coordinate system ψ

$$Q(x) = \frac{1}{2}(GAx, x) + (Gp, x) + c.$$

Since linear grey scale invariance is assumed to hold, we have

$$E(\lambda Q \circ \psi^{-1}|_{\partial B_r}, \partial B_r)(\xi) - \lambda Q(\xi) = \lambda E(Q \circ \psi^{-1}|_{\partial B_r}, \partial B_r)(\xi) - \lambda Q(\xi)$$

for any $\lambda \in I\!R$. We divide by $r^2/2$ and apply the regularity principle, to obtain (37). □

Let us finish this Section with some complementary information about $\mathcal{H}$. For simplicity, in the next proposition we shall not denote the argument ξ of $\mathcal{H}$.

Proposition 4 *i) If $\mathcal{H}$ does not depend upon its first or its third argument, then it also does not depend on its second argument. In other terms*

$$\text{If } \mathcal{H}(\alpha, \beta, \gamma, s) = \hat{\mathcal{H}}(\beta, \gamma, s), \quad \text{then } \mathcal{H} = \hat{\mathcal{H}}(\gamma, s)$$
$$\text{If } \mathcal{H}(\alpha, \beta, \gamma, s) = \hat{\mathcal{H}}(\alpha, \beta, s), \quad \text{then } \mathcal{H} = \hat{\mathcal{H}}(\alpha, s)$$

where $\alpha, \beta, \gamma, s \in I\!R$.
ii) If $\mathcal{H}$ is differentiable at $(0, 0, 0, 0)$ then $\mathcal{H}$ may be written as $\mathcal{H}(A, s) = \mathrm{tr}(BA) + ds$ where B is a nonnegative matrix with constant coefficients and d is a real constant.

Proof i) Assume that $\mathcal{H}(\alpha, \beta, \gamma, s) = \hat{\mathcal{H}}(\beta, \gamma, s)$. Choose two numbers $\lambda \in I\!R$ and $\epsilon \in I\!R^+$ and a symmetric matrix $A = (a_{ij})_{i,j=1}^2$. Let us define the matrix $B = (b_{ij})_{i,j=1}^2 \in SM(2)$ by

$$B = \begin{pmatrix} a_{11} - \frac{\lambda^2}{\epsilon^2} & a_{12} - \lambda \\ a_{12} - \lambda & a_{22} - \epsilon \end{pmatrix}.$$

We have

$$(A - B)(x, x) = \frac{\lambda^2}{\epsilon^2}(x^1)^2 + 2\lambda x^1 x^2 + \epsilon (x^2)^2 = (\frac{\lambda}{\epsilon} x^1 + \epsilon x^2)^2 \geq 0.$$

Thus, $A \geq B$, which, using the ellipticity of $\mathcal{H}$ implies

$$\begin{aligned}\hat{\mathcal{H}}(b_{12} + \lambda, b_{22} + \epsilon, s) &= \hat{\mathcal{H}}(a_{12}, a_{22}, s) = \mathcal{H}(A, s) \geq \mathcal{H}(B, s)\\ &= \hat{\mathcal{H}}(b_{12}, b_{22}, s)\end{aligned}$$

for all $\epsilon > 0$ and $\lambda \in I\!R$. Letting $\epsilon \to 0$, we obtain

$$\hat{\mathcal{H}}(b_{12} + \lambda, b_{22}, s) \geq \hat{\mathcal{H}}(b_{12}, b_{22}, s), \quad \text{for all } \lambda \in I\!R.$$

Thus $\hat{\mathcal{H}}$ does not depend on its first argument, i.e., $\mathcal{H} = \hat{\mathcal{H}}(\gamma, s)$.
ii) Let $\alpha, \beta, \gamma \in I\!R$ and $\epsilon > 0$. Since $\mathcal{H}$ is differentiable at $(0, 0, 0, 0)$ then

$$\mathcal{H}(\epsilon\alpha, \epsilon\beta, \epsilon\gamma, \epsilon s) = \mathcal{H}(0, 0, 0, 0) + \epsilon \langle \nabla \mathcal{H}(0, 0, 0, 0), (\alpha, \beta, \gamma, s) \rangle + o(\epsilon).$$

Since $\mathcal{H}(0,0,0,0) = 0$ and $\mathcal{H}(\epsilon\alpha, \epsilon\beta, \epsilon\gamma, \epsilon s) = \epsilon\mathcal{H}(\alpha, \beta, \gamma, s)$, dividing the above identity by ϵ and letting $\epsilon \to 0$ leads us to

$$\mathcal{H}(\alpha, \beta, \gamma, s) = a\alpha + 2b\beta + c\gamma + ds$$

where we have set $(a, 2b, c, d) = \nabla\mathcal{H}(0,0,0,0)$. Observe that this expression can be written as $\operatorname{tr}(BA) + ds$ where

$$B = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad \text{and } A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}.$$

Since $\mathcal{H}$ is an increasing function of A, the matrix B must be nonnegative. □

Thus if we assume that $\mathcal{H}$ is differentiable at $(0,0,0,0)$ then we may rewrite Equation (34) as

$$a\Lambda_1(u,\xi) + 2b\Lambda_2(u,\xi) + c\Lambda_3(u,\xi) + d|\nabla u|_\xi = 0, \tag{38}$$

where $a, c \geq 0$ and $ac - b^2 \geq 0$, which is the same as saying that the matrix B above is nonnegative.

If we take $a = c = 1$, $b = d = 0$ we recover the Laplace-Beltrami operator. If we choose $a = 1$, $b = c = d = 0$ we obtain the extension to two-dimensional manifolds of the so-called infinity Laplacian. In next section we shall discuss in detail this particular instance of equation (38), which has been proved to be relevant in some applications to image processing [14],[11],[1],[7].

3 Absolutely minimizing Lipschitz extensions

Let $\Omega \in \mathcal{D}$. For reasons that will become apparent when we discuss the mathematical results available we shall assume that $\overline{\Omega}$ may be mapped diffeomorphically to a subset of the plane. Let $\psi : U \subseteq I\!R^2 \to \Omega$ be such a mapping and $\theta \in C(\partial\Omega)$. Let us consider the problem

$$D^2_{\mathcal{M}} u \left(\frac{\nabla u}{|\nabla u|_\xi}, \frac{\nabla u}{|\nabla u|_\xi} \right) = 0 \quad \text{in } \Omega \tag{39}$$

coupled with the boundary condition

$$u|_{\partial\Omega} = \theta. \tag{40}$$

We consider equation (39) in the viscosity sense. Again, for convenience of notation, let us write

$$D^{2,\psi}_{ij} u(x) = \frac{\partial^2 (u \circ \psi)}{\partial x^i \partial x^j}(x) - \Gamma^k_{ij}(x) \frac{\partial (u \circ \psi)}{\partial x^k}(x),$$
$$\partial^\psi_\alpha u(x) = \frac{\partial (u \circ \psi)}{\partial x^\alpha}(x), \quad x \in U.$$

Definition 2 *An upper (resp. lower) semicontinuous function u in Ω is called a viscosity subsolution (supersolution) of (39) if for any $\varphi \in C^2(\Omega)$ and any ξ_0 local maximum (minimum) of $u - \varphi$ in Ω such that $\nabla\varphi(\xi_0) \neq 0$ we have*

$$\frac{1}{|\nabla\varphi(\xi_0)|^2_{\xi_0}} D^{2,\psi}_{ij}(\varphi \circ \psi)(x_0) g^{i\alpha}(x_0) g^{j\beta}(x_0) \partial^{\psi}_{\alpha}\varphi(x_0) \partial^{\psi}_{\beta}\varphi(x_0) \geq 0 \quad (\leq 0)$$

where $\psi(x_0) = \xi_0$. A function $u \in C(\Omega)$ is a viscosity solution of (39) if u is a viscosity sub- and supersolution.

The above definition immediately contains the fact that viscosity solutions of (39) are equivalent to viscosity solutions of

$$D^2_{\mathcal{M}} u\,(\nabla u, \nabla u) = 0 \quad \text{in } \Omega. \tag{41}$$

Since Ω can be mapped into the plane by ψ, the above definition also contains the fact that we may work in a domain of $\mathbb{R}^2$ and after simplifying our notation by writing u instead of $u \circ \psi$, consider our equation as

$$\left(\tfrac{\partial^2 u}{\partial x^i \partial x^j} - \Gamma^k_{ij}(x) \tfrac{\partial u}{\partial x^k}\right) g^{i\alpha}(x) g^{j\beta}(x) \tfrac{\partial u}{\partial x^\alpha} \tfrac{\partial u}{\partial x^\beta} = 0 \quad \text{in } U. \tag{42}$$

Following the notation in [23], (42) may be written as

$$-\frac{\partial}{\partial x^i}\left(g^{\alpha\beta}(x) \frac{\partial u}{\partial x^\alpha} \frac{\partial u}{\partial x^\beta}\right) g^{ik}(x) \frac{\partial u}{\partial x^k} = 0 \quad \text{in } U, \tag{43}$$

or, in a more compact notation,

$$-\frac{\partial}{\partial x^i}(G^{-1}(x)\nabla u \cdot \nabla u) \cdot G^{-1}(x)\nabla u = 0 \quad \text{in } U. \tag{44}$$

When $\mathcal{M} = \mathbb{R}^N$ this PDE is called the infinity Laplace equation. It was introduced by G. Aronsson in [4,5] as the Euler-Lagrange equation for the problem of Absolutely Minimizing Lipschitz Extensions (or AMLE). It was later studied by Battacharya-DiBenedetto-Manfredi [9] as limit of solutions of p-Laplace equations as $p \to \infty$, and R. Jensen [22] who, basing its proof on the above approximation, proved the uniqueness of viscosity solutions of (41). A more direct proof based on the theory of viscosity solutions is due to Barles-Busca [8]. For a thorough survey with simplified arguments we refer to [6]. The extension to more general equations which include the case of variable coefficients was initiated by P. Juutinen [23] who also considered the case of obstacle problems. Let us consider the existence, uniqueness and comparison principle in this more general formulation.

Theorem 2 ([23], Corollary 4.31) *Suppose that u, v are locally bounded, u is an upper semi-continuous viscosity subsolution, and v is a lower semicontinuous viscosity supersolution of (43) in U. If*

$$\limsup_{x \to z} u(x) \leq \liminf_{x \to z} v(x) \tag{45}$$

for all $z \in \partial U$ and if both sides of (45) are not simultaneously ∞ or $-\infty$, then $u \leq v$ in U.

For simplicity, let us denote

$$F(x, \nabla u) = G^{-1}(x)\nabla u \cdot \nabla u.$$

Given an open and bounded set D, we denote by $C_0(D)$ the space of continuous functions in $\overline{D}$ vanishing at the boundary of D.

Definition 3 *A function $u \in W^{1,\infty}_{loc}(U)$ is called an F-absolute minimizer in U if*

$$\|F(x, \nabla u(x))\|_{L^\infty(D)} \le \|F(x, \nabla v(x))\|_{L^\infty(D)}$$

whenever $D \subset\subset U$ is open and $v \in W^{1,\infty}(D)$ is such that $u - v \in C_0(D)$.

Theorem 3 *([23], Corollary 4.33) If $h \in C(\partial U)$, there exists a unique viscosity solution $u \in C(\overline{U})$ of (43) in U such that $u|_{\partial U} = h$. Furthermore $u \in W^{1,\infty}_{loc}(U)$ and is also an F-absolute minimizer.*

Let us mention that, as it is proved by M.G. Crandall [15], any locally Lipschitz continuous function which is an F-absolute minimizer is a viscosity solution of (43). Since Ω is mapped into U by ψ, Theorem 3 is translated into the following existence result.

Corollary 1 *If $\theta \in C(\partial\Omega)$, then there exists a unique viscosity solution $u \in C(\overline{\Omega})$ of (41) such that $u|_{\partial\Omega} = \theta$.*

This result enables us to define the following interpolation operator. Given $\theta \in C(\partial\Omega)$, let $E(\theta, \partial\Omega)$ be the viscosity solution of (41) with boundary data θ.

The operator E satisfies the requirements set forth in Section 2. This is obvious for (A1), (A5), and (A6). The requirement (A3) follows directly from the definition of AMLE. The comparison principle (A2) is contained in the statement of Theorem 2. The regularity principle (A4) will be proved in our next theorem.

Theorem 4 *The operator E satisfies the regularity principle.*

Proof Let $\xi \in \mathcal{M}$ and $\psi_1 : U \to \mathcal{M}$ be a geodesic coordinate system around ξ. Let

$$Q(x) = \frac{1}{2}(Ax, x) + (p, x) + c$$

be a quadratic polynomial in the coordinate system ψ_1, thus A is a symmetric matrix, $p \in I\!R^2$, and $c \in I\!R$. Assume that $p \neq 0$. Without loss of generality, we may assume in the computations below that $c = 0$.

Let us choose the coordinate system ψ_1 such that p points in the direction of the first basis vector of $T_\xi\mathcal{M}$. Let $\nu = \frac{p}{|p|_\xi}$, where $|p|_\xi = \langle p, p\rangle^{1/2} = (p, p)^{1/2}$ since $G(\xi) = I$ in the coordinate system ψ_1. Since $DQ(0) = p$, in the canonical basis $\{\nu, \nu^\perp\}$, the matrix A can be written

$$A = \begin{pmatrix} a & b \\ b & c \end{pmatrix}, \quad a = D^2Q(0)\left(\frac{DQ}{|DQ|}(0), \frac{DQ}{|DQ|}(0)\right), \text{ and } p = (|p|_\xi, 0).$$

Let us define

$$Q_\epsilon(x) = \frac{(Ax, x) - \epsilon(x^1)^2}{2} + (p, x), \quad x = (x^1, x^2). \tag{46}$$

Observe that on the boundary of $B_r := B(0,r) \subseteq U$, $r > 0$, we may write

$$Q_\epsilon(x) = \frac{a}{2}r^2 - \frac{\varepsilon}{2}(x^1)^2 + bx^1x^2 + \frac{c-a}{2}(x^2)^2 + (p,x). \tag{47}$$

We look for a supersolution S of (41) such that $S \geq Q_\epsilon$ on $\partial B(0,r)$ for $r > 0$ small enough. We claim that

$$S(x) = \frac{a}{2}r^2 - \frac{\epsilon}{2}(x^1)^2 + bx^1x^2 + \frac{c-a}{2}(x^2)^2 + |p|_\xi x^1.$$

is a supersolution of (41). According to (47), $S \geq Q_\epsilon$ on ∂B_r. Since in local coordinates the AMLE equation can be written as

$$D^2_{\mathcal{M}} S(\nabla S, \nabla S) = g^{im} g^{jn} \left(\frac{\partial^2 S}{\partial x^i \partial x^j} - \Gamma^k_{ij} \frac{\partial S}{\partial x^k} \right) \frac{\partial S}{\partial x^m} \frac{\partial S}{\partial x^n},$$

and we have chosen a normal coordinate system for which $\Gamma^k_{ij}(0) = 0$, $g_{ij}(0) = \delta_{ij}$, at the origin the equation simplifies to

$$D^2_{\mathcal{M}} S(0)(\nabla S(0), \nabla S(0)) = \frac{\partial^2 S(0)}{\partial x^i \partial x^j} \frac{\partial S(0)}{\partial x^i} \frac{\partial S(0)}{\partial x^j}.$$

After some computations we obtain

$$D^2_{\mathcal{M}} S(\nabla S, \nabla S)(0) = -\epsilon |p|^2_\xi < 0.$$

Since the coefficients of the PDE and the function S are smooth, we also have

$$D^2_{\mathcal{M}} S(\nabla S, \nabla S) < 0,$$

in B_r for $r > 0$ small enough.

Then, according to Theorem 2, we have

$$E(Q_\epsilon, \partial B_r) \leq S \quad \text{in } B_r$$

and

$$\sup_{B_r} |E(Q_\epsilon, \partial B_r) - E(Q, \partial B_r)| \leq \sup_{B_r} |Q_\epsilon - Q| \leq \frac{\epsilon}{2} r^2.$$

Then, by choosing $r > 0$ even smaller we have

$$\begin{aligned} E(Q, \partial B_r)(0) - Q(0) &\leq \frac{\epsilon}{2} r^2 + E(Q_\epsilon, \partial B_r)(0) - Q(0) \\ &\leq \frac{\epsilon}{2} r^2 + S(0) \frac{\epsilon}{2} r^2 + \frac{a}{2} r^2. \end{aligned}$$

Now, dividing by $r^2/2$ and letting $r \to 0$ and $\epsilon \to 0$ in this order we get

$$\limsup_{r \to 0} \frac{E(Q, \partial B_r)(0) - Q(0)}{r^2/2} \leq a.$$

In the same manner, but working with $Q_{-\epsilon}$ instead of Q_ϵ, we prove that

$$\liminf_{r\to 0} \frac{E(Q, \partial B_r)(0) - Q(0)}{r^2/2} \geq a.$$

Thus, the regularity axiom (A2) holds with

$$F(A, p) = a = A(\nu, \nu),$$

the other coordinates of F being omitted.

Let $\psi_2 : U^2 \to \mathcal{M}$ be an arbitrary coordinate system around the point ξ, and let $Q_{\psi_2} : U^2 \to I\!R$ be the quadratic polynomial in the coordinate system ψ_2 given by

$$Q_{\psi_2}(x) = \frac{1}{2}(G\tilde{A}x, x) + (Gp, x) + c,$$

where $\tilde{A}$ is a 2×2 matrix such that $G\tilde{A}$ is symmetric, $p \in I\!R^2$, $c \in I\!R$. Let $\Psi = \psi_2^{-1} \circ \psi_1$, and $Q(\bar{x}) = Q_{\psi_2}(\Psi(\bar{x}))$, $\bar{x} \in U$. Then, by Lemma 2 (with ψ_2 playing the role of ψ_1 and viceversa) , we may write

$$Q(\bar{x}) = \frac{1}{2}(B^t(G\tilde{A} - \Gamma(Gp))B\bar{x}, \bar{x}) + (B^tGp, \bar{x}) + c + O(|\bar{x}|^3)$$

where we have set $B = D\Psi(0)$. Now, proceeding as in Proposition 1, we deduce that

$$\lim_{r\to 0} \frac{E(Q_{\psi_2} \circ \psi_2^{-1}, \partial B_r)(0) - Q(0)}{r^2/2} = B^t(G\tilde{A} - \Gamma(p))B\Big(\frac{B^tGp}{|B^tGp|}, \frac{B^tGp}{|B^tGp|}\Big)$$

where $|B^tGp|^2 = (B^tGp, B^tGp) = (BB^tGp, Gp) = (G^{-1}Gp, Gp) = (Gp, p) = |p|_\xi^2$ (since $BB^t = G^{-1}$). Hence

$$\lim_{r\to 0} \frac{E(Q_{\psi_2} \circ \psi_2^{-1}, \partial B_r)(0) - Q(0)}{r^2/2} = (G\tilde{A} - \Gamma(p))\Big(\frac{p}{|p|_\xi}, \frac{p}{|p|_\xi}\Big).$$

□

4 Numerical approach

In order to solve the partial differential equation (41) on a two-dimensional manifold we use the method introduced in [10]. In short, the idea is to extend the PDE defined on a hypersurface to an associated PDE defined on a neighborhood of $\mathcal{M}$ in the surrounding space and to solve it there. Now, we are in Euclidean space and the equation can be discretized using standard finite difference schemes on cartesian grids, thus we can avoid the use of approximating the surface by a triangulated surface. Also, even though we are solving an equation in a space with a dimension higher than the original problem, the asymptotic complexity does not change since we only compute the solution on a narrow band around $\mathcal{M}$.

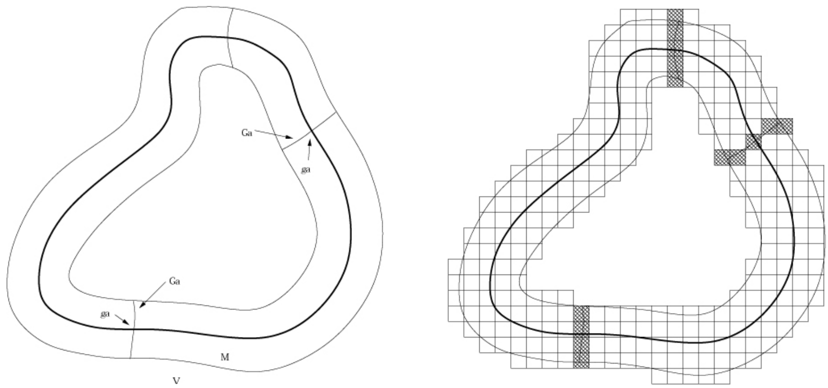

Fig. 1 The AMLE equation on the hypersurface $\mathcal{M}$ is solved using an associated equation defined on a neighborhood V of $\mathcal{M}$. On $V \subset I\!R^3$, this associated equation can be solved using finite differences on a cartesian grid. We have written Γ and $\tilde{\Gamma}$ instead of $\partial\Omega$ and $\widetilde{\partial\Omega}$

To make things precise, let Ω be a domain in a smooth compact two-dimensional manifold $\mathcal{M}$ isometrically embedded in $I\!R^3$, let $\theta \in C(\partial\Omega)$. To guarantee the convergence of our numerical algorithm and to be able to use the results of [18] we shall assume that Ω together with its boundary can be mapped diffeomorphically to a domain of class C^2 in $I\!R^2$. We are looking for solutions of the boundary value problem

$$\begin{array}{ll} D^2_{\mathcal{M}} u\,(\nabla u, \nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \theta & \text{in } \partial\Omega. \end{array} \tag{48}$$

Introducing the signed distance function $\Theta : I\!R^3 \to I\!R$ to $\mathcal{M}$, for $\delta > 0$ small enough, we define an open neighborhood V of $\mathcal{M}$ by setting

$$V = \{x \in I\!R^3 : -\delta < \Theta(x) < \delta\}.$$

Let $\tilde{\Omega}$, respectively $\widetilde{\partial\Omega}$, be the subset of V obtained by prolongation of Ω, respectively $\partial\Omega$, along the gradient vector field of Θ. That is, if, for each $x \in \Omega$ we define $X(t, x)$ to be the solution of

$$\begin{array}{l} X'(t) = \nabla\Theta(X(t)) \\ X(0) = x. \end{array} \tag{49}$$

Since $\frac{d}{dt}\Theta(X(t, x)) = \nabla\Theta(X(t, x)) \cdot X'(t, x) = |\nabla\Theta(X(t, x))|^2 = 1$, we have $\Theta(X(t, x)) = t$, and

$$\begin{array}{l} \tilde{\Omega} = \{X(t, x) : |t| < \delta,\ x \in \Omega\} \\ \widetilde{\partial\Omega} = \{X(t, x) : |t| < \delta,\ x \in \partial\Omega\}. \end{array}$$

We shall refer to $\widetilde{\partial\Omega}$ as the lateral boundary of $\tilde{\Omega}$. We also extend the boundary data θ to $\widetilde{\partial\Omega}$ by setting

$$\tilde{\theta}(y) = \theta(x)$$

where $y = X(t, x)$, $x \in \partial\Omega$, $|t| < \delta$.

For each $x \in V$, we define by $P(x)$ the projection operator from $I\!R^3$ onto the tangent plane to the manifold $\mathcal{M}_t$ defined by the equation $\Theta = t$ at the point x where $t = \Theta(x)$. We may write $P(x)$ as

$$P(x) = I - \nabla\Theta(x) \otimes \nabla\Theta(x).$$

As proposed in [10], the solutions of (48) are computed by solving the associated boundary value problem

$$\begin{aligned} \langle PD(P\nabla U)P\nabla U, P\nabla U\rangle &= 0 \text{ on } \tilde{\Omega}, \\ U &= \tilde{\theta} \text{ on } \widetilde{\partial\Omega}, \end{aligned} \tag{50}$$

This formulation is based on the following well-known facts ([25], p. 38) which we state without proof as a Lemma.

Lemma 4 *(i) If $U : V \to I\!R$, and $u : \mathcal{M} \to I\!R$ are related by $U|_{\mathcal{M}} = u$, then $\nabla u(x) = P\nabla U(x)$ for any $x \in \mathcal{M}$.*

(ii) Let $X, Y : V \to I\!R^3$ be two arbitrary smooth tangent vector fields on $\mathcal{M}$. Then

$$PDX(Y) = \nabla_Y(X)$$

at any point of $\mathcal{M}$.

Statement (ii) can be justified by proving that the map $X \to PDX$ satisfies the axioms that define a connection on $\mathcal{M}$. The following simple Lemma justifies the use of (50). The proof is included for the sake of completeness.

Lemma 5 *Let $\Phi : V \to I\!R$ be twice differentiable and $\phi = \Phi|_{\mathcal{M}}$. Then*

$$\langle PD(P\nabla\Phi)P\nabla\Phi, P\nabla\Phi\rangle(x) = D^2_{\mathcal{M}}\phi(\nabla\phi, \nabla\phi)(x) \tag{51}$$

for all $x \in \mathcal{M}$.

Proof This is merely an application of Lemma 4. In fact, for any point $x \in \mathcal{M} \subset V$ we have

$$\begin{aligned} D^2_{\mathcal{M}}\phi(\nabla\phi, \nabla\phi)(x) &= (D^2_{\mathcal{M}}\phi(\nabla\phi), \nabla\phi)(x) \\ &= \langle \nabla_{\nabla\phi}\nabla\phi, \nabla\phi\rangle(x) \\ &= \langle PD(P\nabla\Phi)(\nabla\phi), \nabla\phi\rangle(x) \quad \text{by Lemma 4.(ii)} \\ &= \langle PD(P\nabla\Phi)(P\nabla\Phi), P\nabla\Phi\rangle(x) \quad \text{by Lemma 4.(i)} \end{aligned}$$

□

Remark 3 As an illustration, let us write the PDE in (50) in the simple case where $\mathcal{M}$ is the plane of $I\!R^3$ given by $x^3 = 0$. Then $P = I - e_3 \otimes e_3$, $e_3 = (0, 0, 1)$, and the PDE in (50) is

$$\sum_{i,j=1}^{2} \frac{\partial^2 u}{\partial x^i \partial x^j} \frac{\partial u}{\partial x^i} \frac{\partial u}{\partial x^j} = 0.$$

In other words, this PDE is the AMLE on each plane $x^3 = \lambda$, $\lambda \in (-\delta, \delta)$.

Definition 4 *Let* $\Omega_\lambda = \{x \in \tilde{\Omega} : \Theta(x) = \lambda\}$, $\lambda \in (-\delta, \delta)$. *Let* $v : \tilde{\Omega} \to I\!R$ *be such that* $v|_{\Omega_\lambda}$ *is upper (lower) semicontinuous on* Ω_λ *for each* λ. *We say that* v *is a viscosity subsolution (supersolution) of (50) if for any* $\varphi \in C^2(\tilde{\Omega})$ *and any* x_0 *local maximum (minimum) of* $v - \varphi$ *in* $\tilde{\Omega}$ *we have*

$$\langle PD(P\nabla\varphi)P\nabla\varphi, P\nabla\varphi\rangle(x_0) \geq 0 \quad (\leq 0).$$

A viscosity solution is a viscosity sub and supersolution.

Let us denote by $\mathcal{USC}(\tilde{\Omega})$ (resp. $\mathcal{LSC}(\tilde{\Omega})$) the set of upper (resp. lower) semicontinuous functions in $\tilde{\Omega}$.

Proposition 5 *Let* $\mathcal{M}_\lambda = \{x \in V : \Theta(x) = \lambda\}$, $\Omega_\lambda = \{x \in \tilde{\Omega} : \Theta(x) = \lambda\}$, $\lambda \in (-\delta, \delta)$. *Let* u_λ *be the viscosity solution of*

$$\begin{aligned} &D^2_{\mathcal{M}_\lambda} u\,(\nabla u, \nabla u) = 0 \quad \text{in } \Omega_\lambda \\ &u|_{\partial\Omega_\lambda} = \tilde{\theta}|_{\partial\Omega_\lambda} \end{aligned} \tag{52}$$

given by Corollary 1. Let $U(x) = u_\lambda(x)$, $x \in \Omega_\lambda$, $|\lambda| < \delta$. *Then* U *is a viscosity solution of (50). Conversely, if* $U \in \mathcal{USC}(\tilde{\Omega})$ *(resp.* $U \in \mathcal{LSC}(\tilde{\Omega})$*) is a viscosity subsolution (supersolution) of (50), then for each* $\lambda \in (-\delta, \delta)$, $u_\lambda = U|_{\Omega_\lambda}$ *is a viscosity subsolution (supersolution) of (52). In particular, viscosity solutions of (50) which are continuous in* $\tilde{\Omega} \cup \widetilde{\partial\Omega}$ *are unique.*

Proof Assume that for each $\lambda \in (-\delta, \delta)$, u_λ is a viscosity solution of (52) and let U be the function defined in the statement. Let $\varphi \in C^2(\tilde{\Omega})$ be such that $U - \varphi$ has a maximum at $x_0 \in \tilde{\Omega}$. Let $\lambda = \Theta(x_0)$. Then x_0 is also a maximum of $u_\lambda - \varphi$ in Ω_λ. Then

$$D^2_{\mathcal{M}_\lambda}\varphi(\nabla\varphi, \nabla\varphi)(x_0) \geq 0.$$

By Lemma 5 we have

$$\langle PD(P\nabla\varphi)P\nabla\varphi, P\nabla\varphi\rangle(x_0) \geq 0.$$

We have proved that U is a viscosity subsolution of (50). Similarly, we prove that U is a viscosity supersolution of (50).

Let $U \in \mathcal{USC}(\tilde{\Omega})$ be a viscosity subsolution of (50). Let us fix $\lambda \in (-\delta, \delta)$, and let $u_\lambda = U|_{\Omega_\lambda}$. Let $\varphi \in C^2(\Omega_\lambda)$ be such that $u_\lambda - \varphi$ has a strict global maximum at the point $x_0 \in \Omega_\lambda$. Let $\Phi \in C^2(\tilde{\Omega})$ be such that $\Phi|_{\Omega_\lambda} = \varphi$. Let $x_\epsilon \in \tilde{\Omega}$ be a maximum of

$$U - \Phi - \frac{1}{\epsilon} d(x, \mathcal{M}_\lambda)^2,$$

where $d(x, \mathcal{M}_\lambda)$ denotes the signed distance from x to $\mathcal{M}_\lambda$. As $\epsilon \to 0$, we have $d(x_\epsilon, \mathcal{M}_\lambda) \to 0$. Let $x_1 \in \mathcal{M}_\lambda$ be such that $x_\epsilon \to x_1$. We deduce that x_1 is a maximum of $u_\lambda - \varphi$, hence $x_1 = x_0$. Since $PDd(x, \mathcal{M}_\lambda)^2 = 0$, we have

$$\langle PD(P\nabla\Phi)P\nabla\Phi, P\nabla\Phi\rangle(x_\epsilon) \geq 0.$$

Letting $\epsilon \to 0$, we obtain

$$\langle PD(P\nabla\Phi)P\nabla\Phi, P\nabla\Phi\rangle(x_0) \geq 0.$$

Now, by Lemma 5 we have

$$D^2_{\mathcal{M}_\lambda}\varphi(\nabla\varphi, \nabla\varphi)(x_0) \geq 0.$$

□

Let us observe that the PDE (50) holds in $\tilde{\Omega}$, that $\partial\tilde{\Omega} = \widetilde{\partial\Omega} \cup \Omega_\delta \cup \Omega_{-\delta}$, and there is no boundary condition in $\Omega_\delta \cup \Omega_{-\delta}$. There is no reason for the solution of (50) to satisfy Neumann boundary conditions neither on Ω_δ or $\Omega_{-\delta}$. If that would be the case, we could argue that these Neumann boundary conditions would be satisfied at any Ω_t for any $t \in (-\delta, \delta)$ and, as a consequence we would have that

$$\nabla U \cdot \nabla\Theta = 0 \tag{53}$$

holds in $\tilde{\Omega}$. Then we would conclude that

$$U(X(t,x)) = u(x) \tag{54}$$

for any $x \in \Omega$, and $t \in (-\delta, \delta)$. But, if U satisfies (54), then on each $\mathcal{M}_\lambda$ the function $u_\lambda = U|_{\mathcal{M}_\lambda}$ satisfies

$$D^2_{\mathcal{M}_\lambda}u_\lambda(\nabla_{\mathcal{M}_\lambda}u_\lambda, \nabla_{\mathcal{M}_\lambda}u_\lambda) = D^2_{\mathcal{M}'}u(\nabla_{\mathcal{M}'}u, \nabla_{\mathcal{M}'}u) \tag{55}$$

where $\mathcal{M}'$ is the manifold $\mathcal{M}$ with the metric induced on $\mathcal{M}$ by the inverse of the map $X(\lambda, \cdot) : \mathcal{M} \to \mathcal{M}_\lambda$. We see that there is no reason why the right hand side of (55) is null in general (it may happen to be null in some case, like the sphere where the metric in $\mathcal{M}'$ is a multiple of the metric in $\mathcal{M}$).

Since boundary conditions are needed at the computational level we approximate (50) by adding a vanishing viscosity term so that the modified equation can be solved numerically. To be able to use available results, we propose to approximate (50) by

$$\begin{aligned} g(P\nabla U)\langle PD(P\nabla U)P\nabla U, P\nabla U\rangle + \epsilon\Delta U = 0 \;\text{ on } \tilde{\Omega}', \\ U = \tilde{\theta}' \text{ on } \partial\tilde{\Omega}', \end{aligned} \tag{56}$$

for some function g. We choose as domain $\tilde{\Omega}'$ a smooth domain contained in $\tilde{\Omega}$ and containing $\{x \in \tilde{\Omega} : |\Theta(x)| \leq \frac{\delta}{2}\}$. We take the function $\tilde{\theta}'$ to be an extension of $\tilde{\theta}$ restricted to $\{x \in \widetilde{\partial\Omega} : |\Theta(x)| \leq \frac{\delta}{2}\}$. The function g has to be chosen so that, as $\epsilon \to 0$, the solution U_ϵ of (56) when restricted to $\mathcal{M}$ converges to the solution of (48). We used Dirichlet boundary conditions to be able to use available results (when passing to the limit as $\epsilon \to 0+$, these boundary conditions are lost at the top and bottom parts of the boundary). If we take $g(P\nabla U) = \frac{1}{1+|P\nabla U|^2}$, then (56) satisfies the assumptions of [20], Theorem 15.18. We obtain the following existence result.

Theorem 5 *There exists a solution* $U_\epsilon \in C(\overline{\tilde{\Omega}'}) \cap C^2(\tilde{\Omega}')$ *of the Dirichlet problem (56).*

Proof Let us observe that, since the constant function $\|\tilde{\theta}'\|_\infty$ is a solution of (56) with boundary data $\|\tilde{\theta}'\|_\infty$, using the comparison principle ([20], Theorem 10.1), we have

$$-\|\tilde{\theta}'\|_\infty \le U_\epsilon \le \|\tilde{\theta}'\|_\infty.$$

Let

$$U^*(x) = \limsup_{\epsilon\to 0, y\to x, y\in\overline{\tilde{\Omega}'}} U_\epsilon(y)$$
$$U_*(x) = \liminf_{\epsilon\to 0, y\to x, y\in\overline{\tilde{\Omega}'}} U_\epsilon(y)$$

Then by the standard theory of viscosity solutions we know that U^* is a viscosity subsolution of

$$\begin{aligned} -\langle PD(P\nabla U)P\nabla U, P\nabla U\rangle &= 0 \text{ on } \tilde{\Omega}', \\ \min(-\langle PD(P\nabla U)P\nabla U, P\nabla U\rangle, U - \tilde{\theta}') &= 0 \text{ on } \partial\tilde{\Omega}', \end{aligned} \tag{57}$$

and U_* is a viscosity supersolution of

$$\begin{aligned} -\langle PD(P\nabla U)P\nabla U, P\nabla U\rangle &= 0 \text{ on } \tilde{\Omega}', \\ \max(-\langle PD(P\nabla U)P\nabla U, P\nabla U\rangle, U - \tilde{\theta}') &= 0 \text{ on } \partial\tilde{\Omega}', \end{aligned} \tag{58}$$

Before continuing, let us write the first equation of (57) as

$$-\frac{\partial P_{ij}(x)}{\partial x_k} u_j P_{ir}(x) u_r P_{ks}(x) u_s - P_{ij}(x)\frac{\partial u_j}{\partial x_k} P_{ir}(x) u_r P_{ks}(x) u_s = 0,$$

where $u_i = \frac{\partial u}{\partial x^i}$. We define

$$F(x, p, X) = -\frac{\partial P_{ij}(x)}{\partial x_k} p_j P_{ir}(x) p_r P_{ks}(x) p_s - P_{ij}(x) X_{kj} P_{ir}(x) p_r P_{ks}(x) p_s.$$

Let us recall the following result proved by F. Da Lio [18]. For that, let $d(y)$ a smooth function agreeing in a neighborhood of $\partial\tilde{\Omega}'$ with the signed distance function to $\partial\tilde{\Omega}'$ which is positive in $\tilde{\Omega}'$ and negative in $\mathbb{R}^3 \setminus \tilde{\Omega}'$, and let $n(y) = -\nabla d(y)$ in a neighborhood of $\partial\tilde{\Omega}'$, i.e., $n(y)$ is an extension of the outer unit normal to $\partial\tilde{\Omega}'$ to a neighborhood of it. Then, Proposition 2.1 in [18] proves that, if $x_0 \in \partial\tilde{\Omega}'$ and $U^*(x) > \tilde{\theta}'(x_0)$, then

$$\liminf_{y\to x_0, \alpha\downarrow 0} F\Big(y, \frac{-n(y) + o(1)}{\alpha}, -\frac{1}{\alpha^2} n(y)\otimes n(y) + \frac{o(1)}{\alpha^2}\Big) \le 0, \tag{59}$$

for some constant $C > 0$, where $o(1) \to 0$ as $\alpha \to 0$.

Now, let $x_0 \in \widetilde{\partial\Omega}$, $|\Theta(x_0)| < \frac{\delta}{2}$. Then $P(x_0)n(x_0) = n(x_0)$, and, for y near x_0, we have $(P(y)n(y), n(y)) \geq \frac{1}{2}$; hence

$$
\begin{aligned}
&F\Big(y, \frac{-n(y) + o(1)}{\alpha}, -\frac{1}{\alpha^2}n(y) \otimes n(y) + \frac{o(1)}{\alpha^2}\Big) \\
&= \frac{\partial P_{ij}}{\partial x_k}\Big(\frac{n_j + o(1)}{\alpha}\Big) P_{ir}\Big(\frac{n_r + o(1)}{\alpha}\Big) P_{ks}\Big(\frac{n_s + o(1)}{\alpha}\Big) \\
&\quad + P_{ij}\Big(\frac{n \otimes n}{\alpha^2} + \frac{o(1)}{\alpha^2}\Big)_{kj} P_{ir}\Big(\frac{n_r + o(1)}{\alpha}\Big) P_{ks}\Big(\frac{n_s + o(1)}{\alpha}\Big) \\
&\geq -\frac{C}{\alpha^3} + \frac{1}{4\alpha^4} + \frac{o(1)}{\alpha^4},
\end{aligned}
$$

where all the above terms are evaluated at the point y. Thus

$$
\liminf_{y \to x_0, \alpha \downarrow 0} F\Big(y, \frac{-n(y) + o(1)}{\alpha}, -\frac{1}{\alpha^2}n(y) \otimes n(y) + \frac{o(1)}{\alpha^2}\Big) > 0, \tag{60}
$$

and, as a consequence of [18], Proposition 2.1, we obtain that $U^*(x_0) \leq \tilde{\theta}'(x_0)$. Similarly, since

$$
\limsup_{y \to x_0, \alpha \downarrow 0} F\Big(y, \frac{n(y) + o(1)}{\alpha}, \frac{1}{\alpha^2}n(y) \otimes n(y) + \frac{o(1)}{\alpha^2}\Big) < 0, \tag{61}
$$

again, by Proposition 2.1 in [18], we obtain that $U_*(x_0) \geq \tilde{\theta}'(x_0)$. Hence

$$
U^*(x_0) \leq \tilde{\theta}'(x_0) \leq U_*(x_0).
$$

It follows that $U^* = U_* = \tilde{\theta}'$ on the set of points $x \in \widetilde{\partial\Omega}$ for which $|\Theta(x)| < \frac{\delta}{2}$. In particular $U^*|_{\mathcal{M}} = U_*|_{\mathcal{M}} = \theta$ on $\partial\Omega$. Since U^* is an upper semicontinuous viscosity subsolution of (50), then, by Proposition 5, we conclude that $U^*|_{\mathcal{M}}$ is an upper semicontinuous subsolution of (48). In the same way $U_*|_{\mathcal{M}}$ is a lower semicontinuous supersolution of (48). Using Theorem 2 (or the uniqueness result of Barles-Busca [8]), we conclude that $U^*|_{\mathcal{M}} = U_*|_{\mathcal{M}}$ on Ω. Let us call $u = U^*|_{\mathcal{M}}$. Then u is the viscosity solution of (48). □

Remark 4 The same argument proves that $U^* = U_*$ on $\{x \in \tilde{\Omega} : |\Theta(x)| \leq \frac{\delta}{2}\}$, hence U_ϵ converges to a continuous function in $\{x \in \tilde{\Omega} : |\Theta(x)| \leq \frac{\delta}{2}\}$.

Remark 5 In a recent paper, J.B. Greer [21] has taken equation (53) as a basic requirement to be satisfied by the extension of the solution of the PDE on the manifold to its neighborhood, and this lead him to the use of a modified projection matrix instead of P.

5 Numerical experiments

In this section we display two experiments showing the potential applications of model (48) to interpolate data given on a set of curves or points on a surface. The first experiment displays the interpolation of an image on a torus from a subset of its level lines. This may be useful in the context of inpainting the appearance of digitized historical monuments or sculptures. Our second experiment displays the reconstruction of some digital elevation data on a sphere knowing them on a family of curves and points. This method is potentially useful in the interpolation of missing parts in digital elevation models given either on a flat terrain or on a sphere [1]; and in the interpolation of climate maps on the earth surface [28]. Even if most of the interpolation (or PDE based) algorithms on surfaces which are currently used are based on triangulated representations of the surface, recently, numerical methods based on implicit representations (in fixed grids) using distance functions have been developed for processing and solving PDEs in surfaces [10],[21]. The algorithms proposed here could be implemented using both approaches, and our implementation follows the second one which is described in Section 4.

Our first example is the interpolation of an image from a subset of its level lines. We take the bitmap of a photo, displayed in Fig. 2.a, quantize it at intervals of thirty and extract the boundary of the level sets. The resulting lines can be seen in Fig. 2.b. We map them canonically on a torus (Fig. 2.c), and perform the interpolation as described in Sec. 4. The results of the interpolation are shown in Fig. 2.d and 2.e, on the torus and pulled-back onto a rectangle, respectively. The quality of the interpolation is better than the quantized image. But, since we did not apply a mirror symmetry before mapping the image to the torus, some black regions appeared at the bottom of the image, due to the interpolation with the corresponding regions at the top of it. This example shows that (48) can be used to interpolate data given on a set of curves.

In our second example we display the reconstruction of a *digital elevation model* given on a sphere. Consider the level lines of a bitmap encoding the terrain elevation of some mountainous rectangular area. We map two copies onto the unit sphere in the following way. Let I be the bitmap considered as a rectangle embedded in $\mathbb{R}^3$. Take two disjoint copies of I and glue them together at the edges. The resulting object is homeomorphic to the unit sphere S^2. We use the technique described in [27] to automatically construct a suitable homeomorphism h between the two spaces and use h to map the bitmaps onto S^2. This yields a continuous set of level lines on the unit sphere (Fig. 3.a). Note that the mountain tops appear as point data. We interpolate again with the absolutely minimizing Lipschitz extension operator which, as shown in [22], can interpolate data given

Fig. 2 From left to right and top to bottom: a) the original image, b) the level lines of the image in a) computed at multiples of 30 with the gray level on it, c) the same level lines mapped on a torus, d) the result of the AMLE interpolation on the torus (applied to the data given in c)), e) the result d) mapped onto a rectangle.

on a set of curves and/or points. The results are visualized as a grey level encoding in Fig. 3.b, and as an actual elevation in Fig. 3.c.

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Fig. 3 Interpolation of a digital elevation model from a set of level lines mapped onto the unit sphere. From top to bottom: a) the data on the sphere, b) its AMLE interpolation, c) the same result displayed as a graph on the sphere.

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