EFFICIENT AND RELIABLE HIERARCHICAL ERROR ESTIMATES FOR THE DISCRETIZATION ERROR OF ELLIPTIC OBSTACLE PROBLEMS

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Abstract. We present and analyze novel hierarchical a posteriori error estimates for self-adjoint elliptic obstacle problems. Our approach differs from straightforward, but non-reliable estimators [9] by an additional extra term accounting for the deviation of the discrete free boundary in the localization step. We prove efficiency and reliability on a saturation assumption and a regularity condition on the underlying grid. Heuristic arguments suggest that the extra term is of higher order and preserves full locality. Numerical computations confirm our theoretical findings.

1. Introduction

Hierarchical a posteriori error estimates are based on the extension of the given finite element space $\mathcal{S}$ by an incremental space $\mathcal{V}$. After discretization of the actual defect problem with respect to the extended space $\mathcal{S}+\mathcal{V}$, the corresponding hierarchical splitting and a subsequent localization step give rise to local defect problems associated with low-dimensional subspaces of $\mathcal{V}$. These local subproblems can be often solved exactly providing local contributions that finally sum up to the desired a posteriori estimate of the error. We refer to the pioneering work of Zienkiewicz et al. [22] and Deuflhard et al. [6] or to the monographs of Verfürth [18] and Ainsworth & Oden [1].

An attractive feature of hierarchical a posteriori error estimates is their robustness. For linear self-adjoint elliptic problems, the local lower bounds and global upper bounds (up to higher order terms) do not involve unknown constants weighting different contributions of different nature, like jumps across the edges and inner residuals. Moreover, the ratio of true and estimated error does not depend on possible jumps of coefficients resolved by the underlying mesh [19, 20, 21]. The upper bound is typically proved on the so-called saturation assumption that the extended space $\mathcal{S}+\mathcal{V}$ provides a more accurate approximation than the original space $\mathcal{S}$ [3, 6]. The saturation assumption holds, if data oscillation is relatively small [8].

Another advantage of hierarchical error estimation is their intriguing simplicity. As a consequence, hierarchical concepts have been applied to various non-smooth nonlinear problems [12], in particular to obstacle problems [9, 11, 13, 17] or two-body contact problems in linear elasticity [16]. Astonishingly good effectivity rates

1991 *Mathematics Subject Classification.* MSC (2000): Primary 65N15, 65N30.

The authors gratefully acknowledge substantial support by Carsten Gräser and Oliver Sander by fruitful discussions and numerical assistance. The second author is supported in part by NSFC under the grant 10601070 and in part by Alexander von Humboldt Foundation hosted by Freie Universität in Berlin.

were observed in all these applications. Moreover, the local contributions as resulting from the local defect problems provided highly effective and fully local error indicators in adaptive refinement.

On the other hand, even for obstacle problems the theoretical analysis of hierarchical error estimates is still in its infancy. Very recently, Siebert and Veeser [17] derived efficient and reliable hierarchical error estimates for the energy functional in obstacle problems which were later improved by Kornhuber et al. [13]. However, straightforward hierarchical error estimates for the discretization error [9] might fail to provide upper bounds for the discretization error, because reliability is lost in the localization step (see, e.g., the counterexample at the end of Section 2). Related versions are more reliable but still no mesh-independent upper bounds are available [11].

In this paper, we present an extension of straightforward hierarchical error estimates [9] by an *additional extra term* accounting for the deviation of the discrete free boundary in course of the localization step. In this way, we are able to prove mesh-independent lower and upper bounds for the discretization error. To our knowledge there are no related results in the literature. The proof is carried out on a convexity condition on the obstacle function, a saturation assumption, and a regularity condition on the underlying grid. More precisely, we assume that the off-diagonal elements of the stiffness matrix are non-positive so that a monotonicity argument can be applied. Numerical computations indicate that this condition is not necessary for mesh-independence.

The novel extra term is a sum of local residuals associated with certain *exceptional nodes*. The exceptional nodes are always contained in the coincidence set. Hence, our a posteriori error estimates reduce to well-known results [3, 6], if no obstacle is present. Heuristic reasoning suggests that for non-degenerate problems the exceptional nodes are concentrated at the discrete free boundary. This explains why previous hierarchical error estimates [9] work well in practise. Indeed, the extra term is of higher order, preserves full locality, and there is no over-estimation of the error inside of the coincidence set in this case. Our theoretical considerations are nicely supported by numerical computations.

Throughout this paper, "$A \lesssim B$" means that A can be bounded by B multiplied with a generic constant depending only on the shape regularity of the actual triangulation $\mathcal{T}$, and "$A \sim B$" stands for "$A \lesssim B$" and "$B \lesssim A$".

2. Hierarchical extensions and local defect problems

Let $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, be a bounded, polygonal or polyhedral domain with Lipschitz-continuous boundary $\partial\Omega$ and denote $H = H_0^1(\Omega)$. We consider the obstacle problem

$$u \in K : \qquad a(u, v - u) \geq \ell(v - u) \qquad \forall v \in K, \tag{2.1}$$

involving the H-elliptic, symmetric bilinear form

$$a(v, w) = \int_\Omega \nabla v \cdot \nabla w \, dx, \qquad v, w \in H, \tag{2.2}$$

with the associated energy norm $\|v\| = a(v, v)^{\frac{1}{2}}$, and a bounded linear functional $\ell \in H'$. The closed, convex subset

$$K = \{v \in H \mid v \geq \psi \text{ a.e. in } \Omega\} \subset H$$

is induced by an obstacle function $\psi \in C(\overline{\Omega})$ satisfying $\psi \le 0$ on $\partial\Omega$. It is well-known [10] that (2.1) admits a unique solution $u \in K$. Hence, the Lagrange multiplier $\sigma \in H'$,

$$\langle \sigma, v\rangle = \ell(v) - a(u,v), \qquad v \in H, \tag{2.3}$$

is well-defined.

Let $\mathcal{T}$ be a *conforming* and *shape regular* triangulation of Ω with $\mathcal{N}$ and $\mathcal{E}$ denoting the set of all interior vertices and edges, respectively. We introduce the space $\mathcal{S} \subset H$ of piecewise linear finite elements on $\mathcal{T}$ spanned by the nodal basis $\{\phi_P \mid P \in \mathcal{N}\}$. Now the finite element discretization of (2.1) reads as

$$u_{\mathcal{S}} \in K_{\mathcal{S}} : \qquad a(u_{\mathcal{S}}, v - u_{\mathcal{S}}) \ge \ell(v - u_{\mathcal{S}}) \qquad \forall v \in K_{\mathcal{S}} \tag{2.4}$$

with discrete constraints

$$K_{\mathcal{S}} = \{v \in \mathcal{S} \mid v(P) \ge \psi(P) \ \ \forall P \in \mathcal{N}\} \subset \mathcal{S}.$$

Note that $K_{\mathcal{S}} \subset K$, if $\psi \in \mathcal{S}$. Of course, (2.4) is also uniquely solvable. In analogy to (2.3), we introduce the discrete Lagrange multiplier $\sigma_{\mathcal{S}} \in H'$,

$$\langle \sigma_{\mathcal{S}}, v\rangle = \ell(v) - a(u_{\mathcal{S}}, v), \qquad v \in H.$$

Note that $\langle \sigma_{\mathcal{S}}, \phi_P\rangle \le 0$ holds for all $P \in \mathcal{N}$. Obviously, the error $e = u - u_{\mathcal{S}}$ is the unique solution of the continuous defect problem

$$e \in D : \qquad a(e, v - e) \ge \langle \sigma_{\mathcal{S}}, v - e\rangle \qquad \forall v \in D \tag{2.5}$$

with defect constrains

$$D = \{v \in H \mid v \ge \psi - u_{\mathcal{S}} \text{ a.e. in } \Omega\} \subset H.$$

In order to derive a computable approximation of $e \in H$, (2.5) is discretized by another finite element space $\mathcal{Q}$ which should be larger than $\mathcal{S}$. To fix the ideas, we select the space $\mathcal{Q} \subset H$ of piecewise quadratic finite elements on $\mathcal{T}$. Note that each function $v \in \mathcal{Q}$ is uniquely determined by its nodal values in $P \in \mathcal{N}_{\mathcal{Q}} = \mathcal{N} \cup \{x_E \mid E \in \mathcal{E}\}$, where x_E stands for the midpoint of $E \in \mathcal{E}$. We emphasize that $\mathcal{Q}$ can be regarded as a hierarchical extension of $\mathcal{S}$, i.e.,

$$\mathcal{Q} = \mathcal{S} + \mathcal{V}, \qquad \mathcal{V} = \text{span}\,\{\phi_E \mid E \in \mathcal{E}\}, \tag{2.6}$$

involving the quadratic bubble functions $\phi_E \in \mathcal{Q}$ characterized by $\phi_E(P) = \delta_{x_E,P}$, $\forall P \in \mathcal{N}_{\mathcal{Q}}$ (Kronecker-δ).

Remark 2.1. *Our subsequent analysis carries over to hierarchical extensions as spanned by other bubble functions. For example, we could as well define ϕ_E as the piecewise linear nodal basis functions associated with the vertices $x_E \in \mathcal{N}'$ of the triangulation $\mathcal{T}'$ resulting from uniform refinement of $\mathcal{T}$ or, equivalently, select $\mathcal{Q}$ to be the space the piecewise linear finite elements on $\mathcal{T}'$.*

We consider the discrete defect problem

$$e_{\mathcal{Q}} \in D_{\mathcal{Q}} : \qquad a(e_{\mathcal{Q}}, v - e_{\mathcal{Q}}) \ge \rho_{\mathcal{S}}(v - e_{\mathcal{Q}}) \qquad \forall v \in D_{\mathcal{Q}} \tag{2.7}$$

with discrete constraints

$$D_{\mathcal{Q}} = \{v \in \mathcal{Q} \mid v(P) \ge \psi(P) - u_{\mathcal{S}}(P) \ \ \forall P \in \mathcal{N}_{\mathcal{Q}}\}.$$

Observe that $u_{\mathcal{Q}} = u_{\mathcal{S}} + e_{\mathcal{Q}} \in \mathcal{Q}$ is just the piecewise quadratic approximation of u. It is well-known [1, 2, 3, 11, 12] that the so-called *saturation assumption*

$$\|u - u_{\mathcal{Q}}\| \le \beta \|u - u_{\mathcal{S}}\|, \qquad \beta < 1, \tag{2.8}$$

implies the a posteriori error estimate

$$(1+\beta)^{-1}\|e_{\mathcal{Q}}\| \le \|u-u_{\mathcal{S}}\| \le (1-\beta)^{-1}\|e_{\mathcal{Q}}\|. \tag{2.9}$$

In the unconstrained case, it has been shown in [8] that small data oscillation implies the saturation assumption (2.8).

Of course, the evaluation of $e_{\mathcal{Q}}$ is still far too costly to be used as an a posteriori error estimate. Using the uniquely determined splitting $v = v_{\mathcal{S}} + v_{\mathcal{V}}$ and $w = w_{\mathcal{S}} + w_{\mathcal{V}}$ of $v, w \in \mathcal{Q}$ into $v_{\mathcal{S}}, w_{\mathcal{S}} \in \mathcal{S}$ and $v_{\mathcal{V}}, w_{\mathcal{V}} \in \mathcal{V}$, we define the bilinear form

$$a_{\mathcal{Q}}(v_{\mathcal{Q}}, w_{\mathcal{Q}}) = a(v_{\mathcal{S}}, w_{\mathcal{S}}) + \sum_{E\in\mathcal{E}} v_{\mathcal{V}}(x_E) w_{\mathcal{V}}(x_E) a(\phi_E, \phi_E) \tag{2.10}$$

and the associated energy norm $\|v\|_{\mathcal{Q}} = a_{\mathcal{Q}}(v,v)^{\frac{1}{2}}$ on $\mathcal{Q}$. Note that $a_{\mathcal{Q}}(\cdot,\cdot)$ is resulting from decoupling of $\mathcal{S}$ and $\mathcal{V}$ and further diagonalization on $\mathcal{V}$. The norm equivalence

$$a_{\mathcal{Q}}(v,v) \sim a(v,v) \qquad \forall v \in \mathcal{Q} \tag{2.11}$$

follows from the estimates

$$\|v_{\mathcal{S}}\| + \|v_{\mathcal{V}}\| \sim \|v\|, \quad \|v_{\mathcal{V}}\|_{\mathcal{Q}} = \left(\sum_{E\in\mathcal{E}} v_{\mathcal{V}}(x_E)^2 a(\phi_E,\phi_E)\right)^{\frac{1}{2}} \sim \|v_{\mathcal{V}}\|, \tag{2.12}$$

as obtained from related local versions [3, 6]

$$\|v_{\mathcal{S}}\|_T + \|v_{\mathcal{V}}\|_T \sim \|v\|_T, \quad \left(\sum_{E\in\mathcal{E}_T} v_{\mathcal{V}}(x_E)^2 a(\phi_E,\phi_E)\right)^{\frac{1}{2}} \sim \|v_{\mathcal{V}}\|_T, \tag{2.13}$$

where $\mathcal{E}_T$ denotes the set of edges of $T \in \mathcal{T}$.

It has been shown in [11] that the unique solution $\varepsilon_{\mathcal{Q}}$ of the associated variational inequality

$$\varepsilon_{\mathcal{Q}} \in D_{\mathcal{Q}}: \qquad a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, v - \varepsilon_{\mathcal{Q}}) \ge \langle \sigma_{\mathcal{S}}, v - \varepsilon_{\mathcal{Q}} \rangle \qquad \forall v \in D_{\mathcal{Q}} \tag{2.14}$$

inherits the norm equivalence (2.11), i.e.,

$$\|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}} \sim \|e_{\mathcal{Q}}\|. \tag{2.15}$$

Due to the remaining coupling of $\mathcal{S}$ and $\mathcal{V}$ by the constraints $D_{\mathcal{Q}}$, the unique solution $\varepsilon_{\mathcal{Q}}$ is still not available in closed form. Hence, we introduce the subset

$$D_{\mathcal{V}} = \{v \in \mathcal{V} \mid v(x_E) \ge \psi(x_E) - u_{\mathcal{S}}(x_E)\ \forall E \in \mathcal{E}\} \subset D_{\mathcal{Q}}$$

and the corresponding approximate discrete defect problem

$$\tilde{\varepsilon}_{\mathcal{V}} \in D_{\mathcal{V}}: \qquad a_{\mathcal{Q}}(\tilde{\varepsilon}_{\mathcal{V}}, v - \tilde{\varepsilon}_{\mathcal{V}}) \ge \langle \sigma_{\mathcal{S}}, v - \tilde{\varepsilon}_{\mathcal{V}} \rangle \qquad \forall v \in D_{\mathcal{V}}. \tag{2.16}$$

The solution $\tilde{\varepsilon}_{\mathcal{V}} \in \mathcal{V}$ is explicitly given by

$$\tilde{\varepsilon}_{\mathcal{V}}(x_E) = \begin{cases} -d_E\|\phi_E\|^{-1} & \forall E \in \mathcal{E}_1 = \{E \in \mathcal{E} \mid \rho_E \le -d_E\} \\ \rho_E\|\phi_E\|^{-1} & \forall E \in \mathcal{E}_2 = \{E \in \mathcal{E} \mid \rho_E > -d_E\} \end{cases}, \tag{2.17}$$

where we have set

$$d_E = (u_{\mathcal{S}}(x_E) - \psi(x_E))\|\phi_E\|, \qquad \rho_E = \langle \sigma_{\mathcal{S}}, \phi_E \rangle \|\phi_E\|^{-1}, \qquad E \in \mathcal{E}.$$

The resulting a posteriori estimate

$$\|\tilde{\varepsilon}_{\mathcal{V}}\|_{\mathcal{Q}}^2 = \sum_{E\in\mathcal{E}} \eta_E^2, \qquad \eta_E = |\tilde{\varepsilon}_{\mathcal{V}}(x_E)|\|\phi_E\|, \quad E \in \mathcal{E}, \tag{2.18}$$

for the discretization error $\|u - u_{\mathcal{S}}\|^2$ has been suggested in [9] where the local contributions η_E have also been used successfully as refinement indicators. In the unconstrained case, it is easily checked that $\varepsilon_{\mathcal{Q}} = \tilde{\varepsilon}_{\mathcal{V}}$ so that, by (2.9) and (2.15) the error estimate (2.18) is efficient and reliable on the saturation assumption (2.8). However, this is no longer true for obstacle problems. The following counterexample shows that in general $\|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}$ can not be bounded by $\|\tilde{\varepsilon}_{\mathcal{V}}\|_{\mathcal{Q}}$ at all.

Let $\Omega = (0,1)$, $a(v,w) = \int_0^1 v'w'\, dx$, $\psi = 0$, and

$$\ell(v) = \int_0^{\frac{1}{4}} (-3)v(x)\, dx + \int_{\frac{1}{4}}^{\frac{3}{4}} v(x)\, dx + \int_{\frac{3}{4}}^{1} (-3)v(x)\, dx.$$

Obviously, the piecewise linear finite element approximation resulting from $\mathcal{S} = \text{span}\,\{\phi_P\}$, $P = \frac{1}{2}$, is $u_{\mathcal{S}} = 0$. The corresponding piecewise quadratic finite element approximation $e_{\mathcal{Q}}$ of the error $u - u_{\mathcal{S}}$ cannot be zero, because $\ell(\phi_P^{\mathcal{Q}}) - a(0, \phi_P^{\mathcal{Q}}) > 0$ holds for the quadratic nodal basis function $\phi_P^{\mathcal{Q}}$. On the other hand, it is easily checked that $\tilde{\varepsilon}_{\mathcal{V}} = 0$.

One might conclude that the hierarchical error estimate (2.18) needs some extension accounting for the deviation in the localization step from (2.14) to (2.16). This will be the subject of the following section.

3. Efficiency and Reliability

For each $P \in \mathcal{N}$ and $E \in \mathcal{E}$, we define

$$\omega_P = \text{supp}\ \phi_P, \quad \gamma_P = \{E' \in \mathcal{E} \mid P \in E'\}, \quad \omega_E = \text{supp}\ \phi_E,$$

and the piecewise quadratic nodal basis function $\phi_P^{\mathcal{Q}} \in \mathcal{Q}$,

$$\phi_P^{\mathcal{Q}} = \phi_P - \sum_{E \in \gamma_P} \phi_P(x_E)\phi_E, \tag{3.1}$$

associated with P. We further introduce the subset of exceptional nodes

$$\mathcal{N}_b = \{P \in \mathcal{N} \mid \rho_P > 0\} \subset \mathcal{N},$$

denoting

$$\rho_P = \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \|\phi_P\|^{-1}, \qquad \tilde{\phi}_P = \phi_P - \sum_{E \in \gamma_P^1} \phi_P(x_E)\phi_E, \quad \gamma_P^1 = \gamma_P \cap \mathcal{E}_1. \tag{3.2}$$

Remark 3.1. *Obviously, $\langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \le 0$ and thus $P \notin \mathcal{N}_b$ holds, if $\gamma_P^1 = \emptyset$ and therefore $\tilde{\phi}_P = \phi_P$. Moreover, for non-degenerate problems the discrete Lagrange multiplier $\sigma_{\mathcal{Q}} = \sigma_{\mathcal{S}} - a(e_{\mathcal{Q}}, \cdot)$ associated with the piecewise quadratic approximation $u_{\mathcal{Q}} = u_{\mathcal{S}} + \varepsilon_{\mathcal{Q}}$ satisfies $\langle \sigma_{\mathcal{Q}}, \tilde{\phi}_P \rangle < 0$, if $u_{\mathcal{Q}}$ is node-wise identical with the obstacle on ω_P and therefore $\tilde{\phi}_P = \phi_P^{\mathcal{Q}}$. Hence, we can also expect $P \notin \mathcal{N}_b$ in this case, as soon as $\sigma_{\mathcal{S}}$ approximates $\sigma_{\mathcal{Q}}$ sufficiently well. As a consequence, for well-behaved problems the set of exceptional nodes $\mathcal{N}_b$ can be expected to concentrate along the continuous free boundary with increasing refinement.*

Now we are ready to formulate the main result of this paper.

Theorem 3.2. *Assume that the obstacle function ψ satisfies the condition*

$$u_{\mathcal{S}}(x_E) \ge \psi(x_E) \qquad \forall E \in \mathcal{E} \tag{3.3}$$

and that $\mathcal{T}$ satisfies the regularity condition

$$a(\phi_P, \phi_{P'}) \leq 0 \qquad \forall P, P' \in \mathcal{N},\ P \neq P'. \tag{3.4}$$

Then the equivalence

$$\|e_{\mathcal{Q}}\|^2_{\mathcal{Q}} \sim \left(\|\tilde{\varepsilon}_{\mathcal{V}}\|^2_{\mathcal{Q}} + \sum_{P \in \mathcal{N}_b} \rho_P^2 \right) \tag{3.5}$$

of hierarchical error estimates holds.

Obviously, (3.3) is valid, if ψ is piecewise convex along the edges $E \in \mathcal{E}$. For general obstacles, say $\psi \in H$, an equivalent problem with zero obstacle could be derived by affine transformation. More regular obstacles could be replaced by piecewise linear approximations which would lead to a corresponding higher order term in the a posteriori error estimates.

It is well-known that (3.4) is satisfied in $d = 2$ space dimensions, if (and only if with some possible rare exceptions near the boundary) $\mathcal{T}$ is a Delaunay triangulation [5]. There are counterexamples [14] showing that this is is not the case for $d = 3$.

Remark 3.3. *In contrast to $\|e_{\mathcal{Q}}\|$, the error estimate $\|\tilde{\varepsilon}_{\mathcal{V}}\|^2_{\mathcal{Q}} + \sum_{P \in \mathcal{N}_b} \rho_P^2$ consists of explicitly computable quantities (cf.* (2.18) *and* (3.2)*). Moreover, in the light of Remark 3.1, the extra term $\sum_{P \in \mathcal{N}_b} \rho_P^2$ can be regarded as a* higher order term*.*

We start the proof of Theorem 3.2 by collecting some local properties of solution $\varepsilon_{\mathcal{Q}}$ of the preconditioned defect problem (2.14).

Lemma 3.4. *The inequality $\varepsilon_{\mathcal{Q}}(x_E) > \psi(x_E) - u_{\mathcal{S}}(x_E)$ implies*

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_E) = \langle \sigma_{\mathcal{S}}, \phi_E \rangle. \tag{3.6}$$

Let $\varepsilon_{\mathcal{Q}} = \varepsilon_{\mathcal{S}} + \varepsilon_{\mathcal{V}}$ denote the hierarchical splitting of $\varepsilon_{\mathcal{Q}}$ into $\varepsilon_{\mathcal{S}} \in \mathcal{S}$ and $\varepsilon_{\mathcal{V}} \in \mathcal{V}$. Then

$$\varepsilon_{\mathcal{V}}(x_E) = \max\{\rho_E \|\phi_E\|^{-1}, \psi(x_E) - (u_{\mathcal{S}} + \varepsilon_{\mathcal{S}})(x_E)\} \tag{3.7}$$

holds for all $E \in \mathcal{E}$.

Proof. Inserting $v = \varepsilon_{\mathcal{Q}} + \phi_E \in \mathcal{D}_{\mathcal{Q}}$ into (2.14), we get

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_E) \geq \langle \sigma_{\mathcal{S}}, \phi_E \rangle, \qquad \forall E \in \mathcal{E}. \tag{3.8}$$

If $\varepsilon_{\mathcal{Q}}(x_E) > \psi(x_E) - u_{\mathcal{S}}(x_E)$, then there is a sufficient small $\alpha > 0$ such that $\varepsilon_{\mathcal{Q}}(x_E) - \alpha\phi_E(x_E) \geq \psi(x_E) - u_{\mathcal{S}}(x_E)$. Inserting $v = (\varepsilon_{\mathcal{Q}}(x_E) - \alpha\phi_E(x_E))\phi_E \in \mathcal{D}_{\mathcal{Q}}$ into (2.14), we get

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, -\alpha\phi_E) \geq \langle \sigma_{\mathcal{S}}, -\alpha\phi_E \rangle.$$

This proves (3.6). Now we use the splitting $\varepsilon_{\mathcal{Q}} = \varepsilon_{\mathcal{S}} + \varepsilon_{\mathcal{V}}$. We write out the definition of $a_{\mathcal{Q}}(\cdot, \cdot)$ to reformulate (3.8) as

$$\varepsilon_{\mathcal{V}}(x_E) \geq \rho_E \|\phi_E\|^{-1},$$

where, as we have have already shown above, equality holds, if $\varepsilon_{\mathcal{Q}}(x_E) > \psi(x_E) - u_{\mathcal{S}}(x_E)$ or, equivalently, $\varepsilon_{\mathcal{V}}(x_E) > \psi(x_E) - (u_{\mathcal{S}} + \varepsilon_{\mathcal{S}})(x_E)$. This concludes the proof. $\square$

Note that the inequality $\varepsilon_{\mathcal{Q}}(P) > \psi(P) - u_{\mathcal{S}}(P)$ does *not* imply

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_P) = \langle \sigma_{\mathcal{S}}, \phi_P \rangle.$$

Indeed, we have $\varepsilon_{\mathcal{Q}} - \alpha\phi_P \notin \mathcal{D}_{\mathcal{Q}}$ for all $\alpha > 0$, if $\varepsilon_{\mathcal{Q}}(x_E) = \psi(x_E) - u_{\mathcal{S}}(x_E)$ holds for some $E \in \gamma_P$.

In the next two lemmata we further analyze the components $\varepsilon_{\mathcal{S}} \in \mathcal{S}$ and $\varepsilon_{\mathcal{V}} \in \mathcal{V}$ of the hierarchical splitting $\varepsilon_{\mathcal{Q}} = \varepsilon_{\mathcal{S}} + \varepsilon_{\mathcal{V}} \in \mathcal{D}_{\mathcal{Q}}$.

Lemma 3.5. *Assume that the regularity condition (3.4) is satisfied. Then*

$$\varepsilon_{\mathcal{S}} \geq 0.$$

Proof. We decompose $\varepsilon_{\mathcal{S}} = \varepsilon_{\mathcal{S}}^+ + \varepsilon_{\mathcal{S}}^-$ into its positive part $\varepsilon_{\mathcal{S}}^+ \in \mathcal{S}$ and its negative part $e_{\mathcal{S}}^- \in \mathcal{S}$ with the nodal values

$$\varepsilon_{\mathcal{S}}^+(P) = \max(0, \varepsilon_{\mathcal{S}}(P)), \qquad \varepsilon_{\mathcal{S}}^-(P) = \min(0, \varepsilon_{\mathcal{S}}(P)), \qquad P \in \mathcal{N}.$$

Obviously, it is sufficient to show $\varepsilon_{\mathcal{S}}^- = 0$. Inserting $v = \varepsilon_{\mathcal{Q}} + \phi_P \in \mathcal{D}_{\mathcal{Q}}$ into (2.14) we get

$$a(\varepsilon_{\mathcal{S}}, \phi_P) = a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_P) \geq \langle \sigma_{\mathcal{S}}, \phi_P \rangle \qquad \forall P \in \mathcal{N}$$

so that $\varepsilon_{\mathcal{S}}^-(P) \leq 0$ yields

$$a(\varepsilon_{\mathcal{S}}, \varepsilon_{\mathcal{S}}^-) \leq \langle \sigma_{\mathcal{S}}, \varepsilon_{\mathcal{S}}^- \rangle.$$

As either $u_{\mathcal{S}}(P) > \psi(P)$ implies $\langle \sigma_{\mathcal{S}}, \phi_P \rangle = 0$ or $u_{\mathcal{S}}(P) = \psi(P)$ leads to $\varepsilon_{\mathcal{S}}^-(P) = 0$, it is easily checked that

$$a(\varepsilon_{\mathcal{S}}, \varepsilon_{\mathcal{S}}^-) \leq \langle \sigma_{\mathcal{S}}, \varepsilon_{\mathcal{S}}^- \rangle = \sum_{P \in \mathcal{N}} e_{\mathcal{S}}^-(P) \langle \sigma_{\mathcal{S}}, \phi_P \rangle = 0. \tag{3.9}$$

Utilizing the regularity condition (3.4), i.e., $a(\phi_{P_1}, \phi_{P_2}) \leq 0$ for $P_1 \neq P_2$ and the identity $\varepsilon_{\mathcal{S}}^+(P_1)\varepsilon_{\mathcal{S}}^-(P_2) = 0$ for $P_1 = P_2$, we directly obtain

$$-a(\varepsilon_{\mathcal{S}}^+, \varepsilon_{\mathcal{S}}^-) = \sum_{P_1, P_2 \in \mathcal{N}} \varepsilon_{\mathcal{S}}^+(P_1)(-\varepsilon_{\mathcal{S}}^-(P_2)) a(\phi_{P_1}, \phi_{P_2}) \leq 0.$$

The above two estimates finally yield

$$a(\varepsilon_{\mathcal{S}}^-, \varepsilon_{\mathcal{S}}^-) = a(\varepsilon_{\mathcal{S}}, \varepsilon_{\mathcal{S}}^-) - a(\varepsilon_{\mathcal{S}}^+, \varepsilon_{\mathcal{S}}^-) \leq 0.$$

This concludes the proof. □

As a direct consequence of the preceding two lemmata, we can now compare the piecewise quadratic components $\varepsilon_{\mathcal{V}}$ and $\tilde{\varepsilon}_{\mathcal{V}}$.

Lemma 3.6. *Assume that the regularity condition (3.4) is satisfied. Then*

$$\rho_E \|\phi_E\|^{-1} \leq \varepsilon_{\mathcal{V}}(x_E) \leq \tilde{\varepsilon}_{\mathcal{V}}(x_E) \qquad \forall E \in \mathcal{E} \tag{3.10}$$

and both inequalities hold with equality for all $E \in \mathcal{E}_2$.

Proof. From Lemma 3.4 it is known that

$$\varepsilon_{\mathcal{V}}(x_E) = \max\{\rho_E \|\phi_E\|^{-1}, \psi(x_E) - (u_{\mathcal{S}} + \varepsilon_{\mathcal{S}})(x_E)\}$$

while

$$\tilde{\varepsilon}_{\mathcal{V}}(x_E) = \max\{\rho_E \|\phi_E\|^{-1}, \psi(x_E) - u_{\mathcal{S}}(x_E)\}$$

holds by definition (2.17). Now the assertion follows directly from Lemma 3.5. □

In the light of Lemma 3.6, $\mathcal{E}_2$ can be regarded as a (sub-)set of non-contact nodes in the sense that

$$\varepsilon_{\mathcal{Q}}(x_E) > \psi(x_E) - u_{\mathcal{S}}(x_E) \qquad \forall E \in \mathcal{E}_2. \tag{3.11}$$

However, $E \in \mathcal{E}_1$ does not imply $\varepsilon_Q(x_E) = \psi(x_E) - u_{\mathcal{S}}(x_E)$.

We are now ready to prove the efficiency of our hierarchical error estimate.

Proposition 3.7. *Assume that the regularity condition (3.4) is satisfied. Then the estimate*

$$\left(\|\tilde{\varepsilon}_{\mathcal{V}}\|_{\mathcal{Q}}^2 + \sum_{P \in \mathcal{N}_b} \rho_P^2 \right) \lesssim \|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}^2 \tag{3.12}$$

holds.

Proof. By Lemma 3.6, we have

$$|\tilde{\varepsilon}_{\mathcal{V}}(x_E)| \le |\varepsilon_{\mathcal{V}}(x_E)| \qquad \forall E \in \mathcal{E}$$

and therefore

$$\|\tilde{\varepsilon}_{\mathcal{V}}\|_{\mathcal{Q}}^2 \le \|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}}^2 \le \|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}^2.$$

It remains to show $\sum_{P \in \mathcal{N}_b} \rho_P^2 \lesssim \|\varepsilon_{\mathcal{Q}}\|^2$. Let $P \in \mathcal{N}_b$. Note that $\tilde{\phi}_P = \phi_P^{\mathcal{Q}} + \sum_{E \in \gamma_P^2} \phi_P(x_E)\phi_E$. Inserting $\varepsilon_{\mathcal{Q}} + \phi_P^{\mathcal{Q}} \in \mathcal{D}_{\mathcal{Q}}$ into (2.14), we get

$$\langle \sigma_{\mathcal{S}}, \phi_P^{\mathcal{Q}} \rangle \le a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_P^{\mathcal{Q}})$$

which, in combination with (3.11) and (3.6), leads to

$$\langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \le a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \tilde{\phi}_P).$$

Now we write out the definitions of $a_{\mathcal{Q}}(\cdot,\cdot)$ and $\tilde{\phi}_P$ to obtain

$$\begin{aligned} 0 < \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle &\le a(\varepsilon_{\mathcal{S}}, \phi_P) - \sum_{E \in \gamma_P^1} \phi_P(x_E)\varepsilon_{\mathcal{V}}(x_E) a(\phi_E, \phi_E) \\ &\le \|\varepsilon_{\mathcal{S}}\|_{\omega_P} \|\phi_P\| + \sum_{E \in \gamma_P^1} |\varepsilon_{\mathcal{V}}(x_E)| \|\phi_E\|^2 \end{aligned}$$

exploiting the Cauchy-Schwarz inequality, the triangle inequality and $|\phi_P(x_E)| \le 1$. Another application of the Cauchy-Schwarz inequality provides

$$\rho_P^2 \lesssim \|\varepsilon_{\mathcal{S}}\|_{\omega_P}^2 + \sum_{E \in \gamma_P^1} |\varepsilon_{\mathcal{V}}(x_E)|^2 \|\phi_E\|^2.$$

We sum up these estimates for all $P \in \mathcal{N}_b$, to get

$$\begin{aligned} \sum_{P \in \mathcal{N}_b} \rho_P^2 &\lesssim \sum_{P \in \mathcal{N}_b} \|\varepsilon_{\mathcal{S}}\|_{\omega_P}^2 + \sum_{P \in \mathcal{N}_b} \sum_{E \in \gamma_P^1} |\varepsilon_{\mathcal{V}}(x_E)|^2 \|\phi_E\|^2 \\ &\lesssim \|\varepsilon_{\mathcal{S}}\|^2 + \|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}}^2 = \|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}^2 \end{aligned}$$

which concludes the proof. □

In preparation of proving reliability we state two further lemmata.

Lemma 3.8. *The inequality $\varepsilon_{\mathcal{Q}}(P) > \psi(P) - u_{\mathcal{S}}(P)$ implies*

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_P^{\mathcal{Q}}) = \langle \sigma_{\mathcal{S}}, \phi_P^{\mathcal{Q}} \rangle, \qquad a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \tilde{\phi}_P) = \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \tag{3.13}$$

with $\phi_P^{\mathcal{Q}}$ and $\tilde{\phi}_P$ defined in (3.1) *and* (3.2)*, respectively.*

Proof. As $\phi_P^{\mathcal{Q}}(P') = \delta_{P,P'}$ for all $P, P' \in \mathcal{N}_{\mathcal{Q}}$, the left equality in (3.13) is shown in a similar way as (3.6). Exploiting (3.11)

$$\begin{aligned} a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \tilde{\phi}_P) &= a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi^{\mathcal{Q}}) + \textstyle\sum_{E\in\gamma_P\cap\mathcal{E}_2} \phi_P(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \phi_E) \\ &= \langle \sigma_{\mathcal{S}}, \phi_P^{\mathcal{Q}}\rangle + \textstyle\sum_{E\in\gamma_P\cap\mathcal{E}_2} \phi_P(x_E)\langle\sigma_{\mathcal{S}}, \phi_E\rangle = \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P\rangle \end{aligned}$$

follows from the left inequality in (3.13) and (3.6). $\square$

Lemma 3.9. *The estimate*

$$|v(P) - v(x_E)| \lesssim \|v\|_{\omega_P} \|\phi_P\|^{-1} \tag{3.14}$$

holds for all $P \in \mathcal{N}$, $E \in \gamma_P$ and $v \in \mathcal{Q}$.

Proof. Let $P \in \mathcal{N}$, $E \in \gamma_P$ and $v = v_{\mathcal{S}} + v_{\mathcal{V}} \in \mathcal{Q}$ with $v_{\mathcal{S}} \in \mathcal{S}$ and $v_{\mathcal{V}} \in \mathcal{V}$. Since

$$v(x_E) = v_{\mathcal{V}}(x_E) + v_{\mathcal{S}}(x_E) = v_{\mathcal{V}}(x_E) + \sum_{P'\in\mathcal{N}} v_{\mathcal{S}}(P')\phi_{P'}(x_E),$$

and $\sum_{P'\in\mathcal{N}} \phi_{P'}(x_E) = 1$, it is clear that

$$v(x_E) - v(P) = v_{\mathcal{V}}(x_E) + \sum_{P'\in\mathcal{N}, P'\neq P} (v_{\mathcal{S}}(P') - v_{\mathcal{S}}(P))\phi_{P'}(x_E). \tag{3.15}$$

Note that $\phi_{P'}(x_E) \neq 0$, if and only if $P' \in \omega_P$. Select $T \in \mathcal{T}$ such that $P, P' \in T \subset \omega_P$ and let $h_T = \operatorname{diam} T$. Then the shape regularity of $\mathcal{T}$ implies

$$|v_{\mathcal{S}}(P') - v_{\mathcal{S}}(P)| \le h_T |\nabla v_{\mathcal{S}}|_T| \lesssim h_T^{1-d/2} \|v_{\mathcal{S}}\|_{\omega_P},$$

because $\nabla v_{\mathcal{S}}|_T$ is constant. It is easily checked that $\|\phi_P\| \sim h_T^{d/2-1}$ giving

$$|v_{\mathcal{S}}(P') - v_{\mathcal{S}}(P)| \lesssim \|\phi_P\|^{-1} \|v_{\mathcal{S}}\|_{\omega_P}.$$

Now choose $T \in \mathcal{T}$ such that $x_E \in T \subset \omega_P$. Then

$$|v_{\mathcal{V}}(x_E)| \le \left(\sum_{E'\in\mathcal{E}_T} v_{\mathcal{V}}(x_{E'})^2 \right)^{\frac{1}{2}} \lesssim h_T^{1-d/2} \|v_{\mathcal{V}}\|_T \lesssim \|\phi_P\|^{-1} \|v_{\mathcal{V}}\|_{\omega_P}$$

follows from $\|\phi_{E'}\| \sim h_T^{d/2-1}$ and the equivalence (2.13) of local norms. Inserting these estimates into (3.15), we get

$$|v(x_E) - v(P)| \lesssim (\|v_{\mathcal{V}}\|_{\omega_P} + \|v_{\mathcal{S}}\|_{\omega_P}) \|\phi_P\|^{-1}.$$

Now the assertion follows from left estimate in (2.13) and the shape regularity of $\mathcal{T}$. $\square$

We are now ready to prove the reliability of our hierarchical error estimate.

Proposition 3.10. *Assume that condition* (3.3) *and the regularity condition (3.4) is satisfied. Then the estimate*

$$\|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}^2 \lesssim \left(\|\tilde{\varepsilon}_{\mathcal{V}}\|_{\mathcal{Q}}^2 + \sum_{P\in\mathcal{N}_b} \rho_P^2 \right). \tag{3.16}$$

holds.

Proof. We write out the definition (2.10) of $a_{\mathcal{Q}}(\cdot,\cdot)$ to obtain

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{Q}}) = a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{S}}) + \sum_{E\in\mathcal{E}} \varepsilon_{\mathcal{V}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E).$$

As a consequence of (3.11) and (3.6) this leads to

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{Q}}) = a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{S}}) + \sum_{E\in\mathcal{E}_1} \varepsilon_{\mathcal{V}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E) + \sum_{E\in\mathcal{E}_2} \rho_E^2.$$

Utilizing the splitting $\mathcal{E}_1 = \mathcal{E}_1^+ \cup \mathcal{E}_1^-$,

$$\mathcal{E}_1^+ = \{E\in\mathcal{E}_1 \mid \varepsilon_{\mathcal{Q}}(x_E) > 0\}, \qquad \mathcal{E}_1^- = \{E\in\mathcal{E}_1 \mid \varepsilon_{\mathcal{Q}}(x_E) \le 0\},$$

and $\varepsilon_{\mathcal{V}}(x_E) = \varepsilon_{\mathcal{Q}}(x_E) - \varepsilon_{\mathcal{S}}(x_E)$, we rewrite this identity according to

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{Q}}) = I_1 + I_2 + I_3 + \sum_{E\in\mathcal{E}_2} \rho_E^2, \tag{3.17}$$

where

$$I_1 = a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{S}}) - \sum_{E\in\mathcal{E}_1} \varepsilon_{\mathcal{S}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E),$$

and

$$I_2 := \sum_{E\in\mathcal{E}_1^+} \varepsilon_{\mathcal{Q}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E), \qquad I_3 = \sum_{E\in\mathcal{E}_1^-} \varepsilon_{\mathcal{Q}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E).$$

Exploiting (3.3), $|\varepsilon_{\mathcal{Q}}(x_E)| = -\varepsilon_{\mathcal{Q}}(x_E) \le u_{\mathcal{S}}(x_E) - \psi(x_E) = d_E\|\phi_E\|^{-1}$ holds for all $E\in\mathcal{E}_1^-$. Hence, the Cauchy-Schwarz inequality, the identity $a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E) = \varepsilon_{\mathcal{V}}(x_E)a(\phi_E,\phi_E)$, and the right norm equivalence in (2.12) yield

$$I_3 \le \sum_{E\in\mathcal{E}_1^-} d_E|\varepsilon_{\mathcal{V}}(x_E)|\|\phi_E\| \le \left(\sum_{E\in\mathcal{E}_1} d_E^2\right)^{\frac{1}{2}} \|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}}. \tag{3.18}$$

In the next step, we consider the term I_1. Let

$$\mathcal{N}_1 = \{P\in\mathcal{N} \mid \gamma_P^1 \ne \emptyset\}.$$

Note that $\mathcal{N}_b \subset \mathcal{N}_1$, because $P\in\mathcal{N}\backslash\mathcal{N}_1$ implies $\tilde{\phi}_P = \phi_P$ and thus $P\in\mathcal{N}\backslash\mathcal{N}_b$. Let us consider some $P\in\mathcal{N}\backslash\mathcal{N}_1$ and assume that $\varepsilon_{\mathcal{Q}}(P) = \varepsilon_{\mathcal{S}}(P) > 0 \ge \psi(P) - u_{\mathcal{S}}(P)$. Then Lemma 3.8 provides

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_P) = \langle\sigma_{\mathcal{S}},\phi_P\rangle \le 0.$$

As, by Lemma 3.5, $\varepsilon_{\mathcal{S}}(P) < 0$ does not occur, we have shown

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\varepsilon_{\mathcal{S}}) = \sum_{P\in\mathcal{N}} \varepsilon_{\mathcal{S}}(P) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_P) \le \sum_{P\in\mathcal{N}_1} \varepsilon_{\mathcal{S}}(P) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_P).$$

Let us consider the second term of I_1. We insert the nodal representation $\varepsilon_{\mathcal{S}}(x_E) = \sum_{P\in\mathcal{N}_E} \varepsilon_{\mathcal{S}}(P)\phi_P(x_E)$ with $\mathcal{N}_E = \{P\in\mathcal{N} \mid \phi_P(x_E)\ne 0\}$ and rearrange terms to obtain

$$\begin{aligned}
\sum_{E\in\mathcal{E}_1} \varepsilon_{\mathcal{S}}(x_E) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E) &= \sum_{E\in\mathcal{E}_1} \Big(\sum_{P\in\mathcal{N}_E} \varepsilon_{\mathcal{S}}(P)\phi_P(x_E)\Big) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}},\phi_E) \\
&= \sum_{P\in\mathcal{N}_1} \varepsilon_{\mathcal{S}}(P) a_{\mathcal{Q}}\Big(\varepsilon_{\mathcal{Q}}, \sum_{E\in\gamma_P^1} \phi_P(x_E)\phi_E\Big).
\end{aligned}$$

Hence, we have shown

$$I_1 \quad \leq \quad \sum_{P\in\mathcal{N}_1} \varepsilon_{\mathcal{S}}(P) a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \tilde{\phi}_P).$$

Now Lemma 3.8 yields

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \tilde{\phi}_P) = \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle$$

for all $P \in \mathcal{N}_1 \subset \mathcal{N}$ satisfying $\varepsilon_{\mathcal{Q}}(P) = \varepsilon_{\mathcal{S}}(P) > 0 \geq \psi(P) - u_{\mathcal{S}}(P)$. By definition of $\mathcal{N}_b \subset \mathcal{N}_1$, we have

$$I_1 \leq \sum_{P\in\mathcal{N}_1} \varepsilon_{\mathcal{S}}(P) \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \leq \sum_{P\in\mathcal{N}_b} \varepsilon_{\mathcal{S}}(P) \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle. \tag{3.19}$$

Now let $P \in \mathcal{N}_b$. As $\mathcal{N}_b \subset \mathcal{N}_1$, it is clear that $\gamma_P^1 \neq \emptyset$ so that there is an $E_P \in \gamma_P^1$ with the property

$$\varepsilon_{\mathcal{Q}}(x_{E_P}) = \min\{\varepsilon_{\mathcal{Q}}(x_E) \mid E \in \gamma_P^1\}.$$

By Lemma 3.9, we have

$$|\varepsilon_{\mathcal{Q}}(P) - \varepsilon_{\mathcal{Q}}(x_{E_P})| \lesssim \|\varepsilon_{\mathcal{Q}}\|_{\omega_P} \|\phi_P\|^{-1}$$

Either $E_P \in \mathcal{E}_1^-$ leads to

$$\begin{aligned} \varepsilon_{\mathcal{S}}(P) \quad &= \quad \varepsilon_{\mathcal{Q}}(P) \leq \varepsilon_{\mathcal{Q}}(P) - \varepsilon_Q(x_{E_P}) \\ &\lesssim \quad \|\varepsilon_{\mathcal{Q}}\|_{\omega_P} \|\phi_P\|^{-1}, \end{aligned}$$

or $E_P \in \mathcal{E}_1^+$ provides

$$\begin{aligned} \varepsilon_{\mathcal{S}}(P) \quad &= \quad \varepsilon_{\mathcal{Q}}(P) = \varepsilon_{\mathcal{Q}}(P) - \varepsilon_Q(x_{E_P}) + \varepsilon_Q(x_{E_P}) \\ &\lesssim \quad \|\varepsilon_{\mathcal{Q}}\|_{\omega_P} \|\phi_P\|^{-1} + \varepsilon_Q(x_{E_P}). \end{aligned}$$

In any case, we get

$$\varepsilon_{\mathcal{S}}(P) \leq \|\varepsilon_{\mathcal{Q}}\|_{\omega_P} \|\phi_P\|^{-1} + \max\{0, \varepsilon_Q(x_{E_P})\}.$$

We insert this estimate into (3.19) and apply the Cauchy-Schwarz inequality, to obtain

$$\begin{aligned} I_1 \quad &\leq \quad \sum_{P\in\mathcal{N}_b} \|\varepsilon_{\mathcal{Q}}\|_{\omega_P} \rho_P + \sum_{P\in\mathcal{N}_b} \max\{0, \varepsilon_Q(x_{E_P})\} \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \\ &\lesssim \quad \|\varepsilon_{\mathcal{Q}}\| \Big(\sum_{P\in\mathcal{N}_b} \rho_P^2 \Big)^{\frac{1}{2}} + \sum_{P\in\mathcal{N}_b, E_P\in\mathcal{E}_1^+} \varepsilon_Q(x_{E_P}) \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle. \end{aligned} \tag{3.20}$$

We concentrate on the second term in (3.20). Let $E_P \in \mathcal{E}_1^+$. Then $\gamma_P^1 \subset \mathcal{E}_1^+$. As $\varepsilon_{\mathcal{Q}}(x_{E_P}) > 0 \geq \psi(x_E) - u_{\mathcal{S}}(x_E)$ holds for all $E \in \mathcal{E}_1^+ \subset \mathcal{E}_1$ and $\langle \sigma_{\mathcal{S}}, \phi_E \rangle \leq 0$ is valid for all $E \in \mathcal{E}_1$, Lemma 3.4 provides

$$a_{\mathcal{Q}}(\varepsilon_Q, \phi_E) = \langle \sigma_{\mathcal{S}}, \phi_E \rangle \leq 0 \qquad \forall E \in \mathcal{E}_1^+. \tag{3.21}$$

Hence, utilizing $\langle \sigma_{\mathcal{S}}, \phi_P \rangle \leq 0$, we obtain

$$\begin{aligned} \varepsilon_{\mathcal{Q}}(x_{E_P}) \langle \sigma_{\mathcal{S}}, \tilde{\phi}_P \rangle \quad &\leq \quad \varepsilon_Q(x_{E_P}) \langle \sigma_{\mathcal{S}}, -\sum_{E\in\gamma_P^1} \phi_P(x_E) \phi_E \rangle \\ &\leq \quad \sum_{E\in\gamma_P^1} \varepsilon_{\mathcal{Q}}(x_E) \langle \sigma_{\mathcal{S}}, -\phi_P(x_E) \phi_E \rangle \\ &= \quad \sum_{E\in\gamma_P^1} \varepsilon_{\mathcal{Q}}(x_E) a_{\mathcal{Q}}(\varepsilon_Q, -\phi_P(x_E) \phi_E). \end{aligned}$$

Exploiting again (3.21), it is easily checked that

$$\begin{aligned}\sum_{P\in\mathcal{N}_b, E_P\in\mathcal{E}_1^+} \varepsilon_{\mathcal{Q}}(x_{E_P})\langle\sigma_{\mathcal{S}}, \tilde{\phi}_P\rangle &\leq \sum_{P\in\mathcal{N}_b}\sum_{E\in\mathcal{E}_1^+\cap\gamma_P^1} \phi_P(x_E)\varepsilon_{\mathcal{Q}}(x_E)a_{\mathcal{Q}}(\varepsilon_Q, -\phi_E)\\ &\leq \sum_{E\in\mathcal{E}_1^+}(\sum_{P\in\mathcal{N}_b\cap\mathcal{N}_E} \phi_P(x_E))\varepsilon_{\mathcal{Q}}(x_E)a_{\mathcal{Q}}(\varepsilon_Q, -\phi_E)\\ &\leq -\sum_{E\in\mathcal{E}_1^+} \varepsilon_{\mathcal{Q}}(x_E)a_{\mathcal{Q}}(\varepsilon_Q, \phi_E) = -I_2.\end{aligned}$$

In the light of (3.20) and the norm equivalence (2.11), we have shown

$$I_1 + I_2 \lesssim \|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}\Big(\sum_{P\in\mathcal{N}_b}\rho_P^2\Big)^{\frac{1}{2}}. \tag{3.22}$$

In the final step, we insert the inequalities (3.18) and (3.22) into (3.17), to obtain

$$a_{\mathcal{Q}}(\varepsilon_{\mathcal{Q}}, \varepsilon_{\mathcal{Q}}) \leq \Big(\sum_{P\in\mathcal{N}_b}\rho_P^2\Big)^{\frac{1}{2}}\|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}} + \Big(\sum_{E\in\mathcal{E}_1} d_E^2\Big)^{\frac{1}{2}}\|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}} + \sum_{E\in\mathcal{E}_2}\rho_E^2. \tag{3.23}$$

Utilizing Lemma 3.6 and the right equivalence in (2.12), we conclude

$$\left(\sum_{E\in\mathcal{E}_2}\rho_E^2\right)^{\frac{1}{2}} \leq \left(\sum_{E\in\mathcal{E}}\varepsilon_{\mathcal{V}}(x_E)^2 a(\phi_E, \phi_E)\right)^{\frac{1}{2}} \lesssim \|\varepsilon_{\mathcal{V}}\|_{\mathcal{Q}} \leq \|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}}$$

so that (3.23) takes the form

$$\|\varepsilon_{\mathcal{Q}}\|_{\mathcal{Q}} \lesssim \Big(\sum_{P\in\mathcal{N}_b}\rho_P^2\Big)^{\frac{1}{2}} + \Big(\sum_{E\in\mathcal{E}_1}\tilde{\varepsilon}_{\mathcal{V}}(x_E)^2\Big)^{\frac{1}{2}} + \Big(\sum_{E\in\mathcal{E}_2}\tilde{\varepsilon}_{\mathcal{V}}(x_E)^2\Big)^{\frac{1}{2}}$$

and the assertion follows from the Cauchy-Schwarz inequality. □

4. Numerical Results

In our numerical experiments, we will consider sequences of triangulations $\mathcal{T}_j$, $j = 0, 1, \ldots, J$, as resulting from j local refinement steps of an initial triangulation $\mathcal{T}_0$. The subscript j will always refer to the corresponding triangulation $\mathcal{T}_j$ as, for example, in $\mathcal{N}_j$, $\mathcal{E}_j$, $\mathcal{S}_j$, $u_{\mathcal{S}_j}$, and so on. We either apply uniform refinement, i.e., we connect the midpoints of all edges $E \in \mathcal{E}_j$, or we apply local adaptive refinement based on the local contributions η_E^2, ρ_P^2 to the hierarchical error estimator

$$\eta_j = \sum_{E\in\mathcal{E}_j}\eta_E^2 + \rho_j, \qquad \rho_j = \sum_{P\in\mathcal{N}_{j,b}}\rho_P^2,$$

as introduced in Theorem 3.2. Here, we use a variant of the refinement strategy suggested by Dörfler [7] to be described as follows. First, the local contributions η_E^2, ρ_P^2 are ordered according to their size. Then, proceeding from the largest to smaller contributions, we collect all entries from this list until they sum up to $(1-\theta)^2\eta_j$. Finally, if η_E^2 or ρ_P^2 are contained in this collection, then all triangles in the support of ϕ_E or ϕ_P are marked for refinement. Like Dörfler [7], we select $\theta = 0.2$ in our computations. Note that in general this strategy does not preserve symmetry, because only the first of more than one entry with equal size might be collected for refinement.

4.1. **Constant Obstacle.** Following Nochetto et al. [15], we consider the constant obstacle $\psi \equiv 0$, the domain $\Omega = (0,1)^2$, and the radially symmetric right-hand side

$$\ell(v) = \int_\Omega fv\, dx, \qquad f(x) = \begin{cases} -4(2|x|^2 + 2(|x|^2 - r^2)), & |x| > r \\ -8r^2(1 - (|x|^2 - r^2)), & |x| \le r \end{cases},$$

providing the radially symmetric exact solution

$$u(x) = (\max\{|x|^2 - r^2, 0\})^2.$$

with corresponding boundary conditions. Like Nochetto et al. [15], we select $r = 0.7$ in our numerical computations. In our first experiment, the triangulations $\mathcal{T}_j$, $j = 1, \dots, 9$, are obtained by uniform refinement of an initial triangulation $\mathcal{T}_0$ consisting of four congruent triangles.

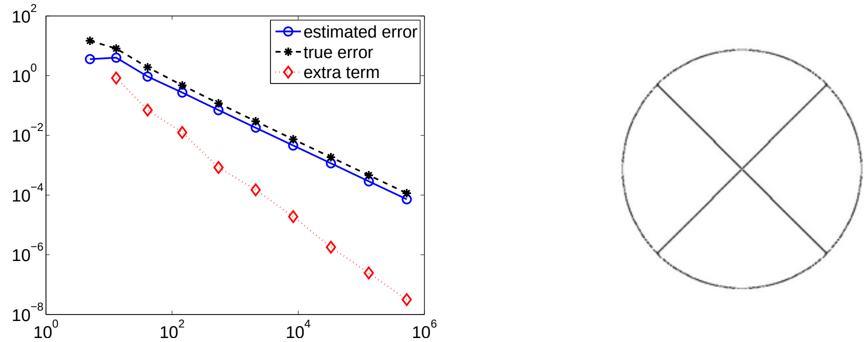

FIGURE 1. Comparison of the squared error $\|u - u_{\mathcal{S}_j}\|^2$ with the hierarchical estimator η_j, and distribution of the exceptional nodes $\mathcal{N}_{9,b}$.

The left picture in Figure 1 shows the squared discretization error $\|u - u_{\mathcal{S}_j}\|^2$, the hierarchical estimator η_j, and the extra term ρ_j over the number of unknowns n_j. Obviously, the true error is approximated quite well. More precisely, for $j = 1, \dots, 9$ the effectivity indices $\|u - u_{\mathcal{S}_j}\|^2/\eta_j$ are ranging from 0.63 to 0.79 and seem to saturate at 0.79. This behavior is in perfect agreement with saturation (2.8) and preconditioning (2.11). Like the squared error, the estimator η_j is proportional to $n_j^{-1} = \mathcal{O}(h_j^2)$ with h_j denoting the mesh size of $\mathcal{T}_j$. Moreover, we observe $\rho_j = \mathcal{O}(n_j^{-3/2}) = \mathcal{O}(h_j^3)$, i.e., the extra term ρ_j is of higher order, as predicted in Remark 3.3. On the other hand, the distribution of exceptional nodes $P \in \mathcal{N}_{9,b}$, as illustrated in the right picture of Figure 1, is partly surprising at first sight. A subset of the exceptional nodes is concentrated at the circular free boundary of $u_{\mathcal{S}_j}$ which is supporting the heuristic reasoning in Remark 3.1. However, there is another subset of exceptional nodes located along the diagonals which seems to contradict our expectation that there are no exceptional nodes inside of the coincidence set. The simple reason is the *inherent instability of quadratic finite elements*: In this example, piecewise quadratic approximation $u_{\mathcal{Q}_9}$ creates a spurious free boundary along the diagonals! As the exceptional nodes are intended to account for the deviation of $\tilde{\varepsilon}_{\mathcal{V}_j}$ from the piecewise quadratic approximation $e_{\mathcal{Q}_j} = u_{\mathcal{Q}_j} - u_{\mathcal{S}_j}$ of the error and *not* from the true error $u - u_{\mathcal{S}_j}$, it is natural that the exceptional nodes $\mathcal{N}_{j,b}$ cluster along the spurious free boundary as well. This is exactly what we observe. Note that the spurious contributions ρ_P^2 at the diagonals are by several magnitudes smaller than the others. We emphasize that such instability effects can be easily avoided by selecting a stable hierarchical extension $\mathcal{V}$ as obtained, e.g., from piecewise linear finite elements associated with triangulation $\mathcal{T}_j'$ as obtained from $\mathcal{T}_j$ by uniform refinement. Recall that efficiency, reliability and all our other theoretical considerations carry over to this case (cf. Remark 2.1).

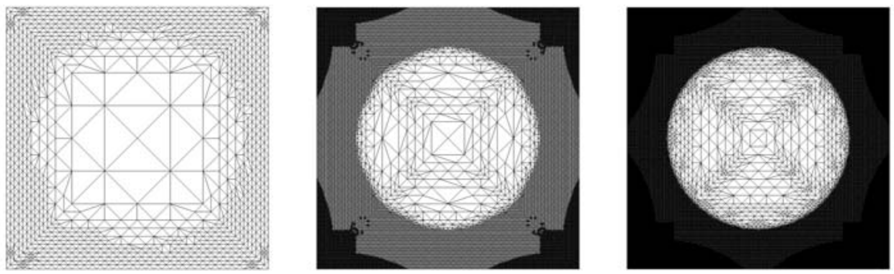

FIGURE 2. Adaptively refined triangulations $\mathcal{T}_6$, $\mathcal{T}_{10}$, $\mathcal{T}_{12}$.

In order to illustrate the locality of the hierarchical error estimator η_j, Figure 2 shows the triangulations $\mathcal{T}_j$, $j = 6, 10, 12$, as resulting from the adaptive refinement strategy described above. Note that the quadratic instability hardly influences the refinement process, because the corresponding local contributions are very small. However, effects of quadratic instability become slightly visible with increasing refinement. Though the adaptively refined triangulations no longer fulfill the regularity condition (3.4), we observe that the effectivity indices $\|u - u_{\mathcal{S}_j}\|^2/\eta_j$ are still quite satisfying, ranging from 0.63 to 0.82, and that the extra term ρ_j is still of higher order.

4.2. **Lipschitz Obstacle.** Following Nochetto et al. [15] again, we consider the domain $\Omega = \{x \in \mathbb{R}^2 \mid |x_1| + |x_2| < 1\}$, the right hand side $\ell(v) = -5\int_\Omega v(x)\,dx$, the Lipschitz obstacle

$$\psi(x) = \operatorname{dist}(x, \partial\Omega) - \tfrac{1}{5},$$

and homogeneous Dirichlet boundary conditions. The triangulations $\mathcal{T}_j$, $j = 1, 2, \ldots, 12$, are resulting from local adaptive refinement of the initial triangulation $\mathcal{T}_0$ consisting of four congruent triangles. The final approximate solution u_{12} is depicted in the left picture of Figure 3 while the right picture shows the corresponding approximate free boundary. Observe the cusps approximated by "antennas" of sole edges. Note that this effect can be regarded as a lack of regularity of the discrete coincidence set [4]. As no exact solution is available, we cannot compare our estimator with the true error. However, as in the first example, we still observe $\eta_j = \mathcal{O}(n_j^{-1})$ and extra terms ρ_j of higher order. In contrast to the first example the exceptional nodes $\mathcal{N}_{j,b}$ are now concentrated along the approximate free boundary. In order to illustrate the strong locality of our hierarchical error estimator, Figure 4 shows $\mathcal{T}_6$, $\mathcal{T}_9$ and a zoom into the upper corner of $\mathcal{T}_{12}$. Observe that there is no refinement within the coincidence set, where the obstacle ψ is exactly resolved by the underlying mesh. The triangulation is locally refined in the neighborhood of

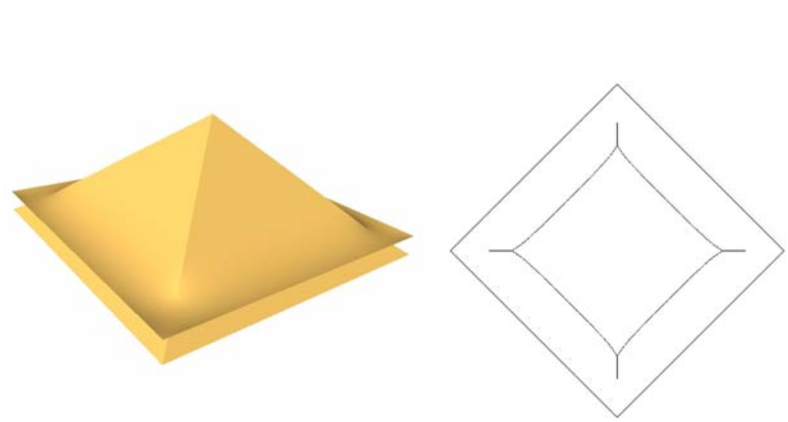

FIGURE 3. Approximate solution u_{12} with obstacle function ψ and associated free boundary

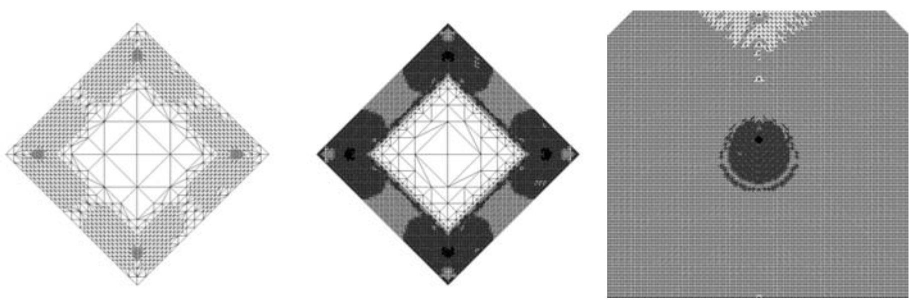

FIGURE 4. Adaptively refined triangulations $\mathcal{T}_6$, $\mathcal{T}_9$ and a zoom into the upper corner of $\mathcal{T}_{12}$.

the free boundary which is in agreement with the corresponding lack of regularity. Finally, strong local refinement takes place at the cusps which perfectly reflects the corresponding singularity of the solution.

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