Advection-diffusion equations with random coefficients on evolving hypersurfaces

Ana Djurdjevac
Institut für Mathematik
Freie Universität Berlin
Arnimallee 6
14195 Berlin, Germany
adjurdjevac@math.fu-berlin.de

Abstract

We present the analysis of advection-diffusion equations with random coefficients on moving hypersurfaces. We define a weak and a strong material derivative, which account for the spatial movement. Then we define the solution space for these kind of equations, which is the Bochner-type space of random functions defined on a moving domain. We consider both cases, uniform and log-normal distributions of the diffusion coefficient. Under suitable regularity assumptions we prove the existence and uniqueness of weak solutions of the equation under analysis, and also we give some regularity results about the solution.

2010 Mathematics Subject Classification: Primary 35R60, 35K99; Secondary 35R01, 35R37.
Keywords: Advection-diffusion; evolving surfaces; uncertainty quantification; random coefficients; existence.

1 Introduction

There is a growing interest in partial differential equations (PDEs) with random coefficients that model problem parameters which include some uncertainty. The uncertainty can come from intrinsic variability of the physical system or when the input data of the real system are not completely known [26]. This work addresses specifically parabolic PDEs with random coefficients which have so far been studied in several papers (see e.g. [3, 7, 22, 25]). Furthermore, these PDEs occur in many applications, such as hydrogeology, material science, fluid dynamics, biological fluids etc.

All these works have considered equations on some bounded *flat* fixed domain in $\mathbb{R}^d$. On the other hand, it is known and well studied that in a variety of applications these models can be better formulated on both stationary and evolving *curved* domains, cf., e.g. [36]. Over the last years, surface PDEs have garnered increasing interest due to a variety of applications, such as image processing [21], computer graphics [4], biological modelling [27] and engineering [30]. Particularly for this paper, the motivating example is modelling the transport of a surface active agent (surfactant) on the interface between two fluids [23, 37].

The numerical analysis of surface PDEs started with the paper [12] and later it developed in [11, 14, 24] etc. Dziuk and Elliott have introduced the evolving surface finite element method for PDEs on moving hypersurfaces [13, 15]. Recently this topic has been generalized in [1] and [2] to a more abstract level, i.e. to parabolic PDEs on any evolving Hilbert space.

Uncertainty naturally appears in all these applications (for example through randomness of the input data). However, there is no mathematical theory that merges these two frameworks, uncertainty quantification and surface PDEs. This serves to motivate the topic of this paper to consider PDEs with random coefficients on moving surfaces. More precisely, we wish to analyse the following advection-diffusion equation

$$\begin{aligned} \partial^{\bullet} u - \nabla_{\Gamma} \cdot (\alpha \nabla_{\Gamma} u) + u \nabla_{\Gamma} \cdot \mathbf{w} &= f \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

where ∇_{Γ} is a tangential surface gradient, $\partial^{\bullet}$ is the material derivative and $\mathbf{w}$ is a velocity field of the evolution. Note that we assume the surface evolution to be prescribed. In contrast to the deterministic case, the diffusion coefficient α, the source function f and the initial value u_0 are random. Hence the solution u will also be a random function. The equation (1.1) models the transport of a scalar quantity, e.g., a surfactant, along a moving two-dimensional interface [37]. The surfactant is transported by advection via the tangential fluid velocity and by diffusion within the surface.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. In analogy to the elliptic case [28], for the parabolic PDE with random coefficients there exist two weak formulations: path-wise (for fixed sample ω) and "mean" (includes also integration over Ω). The more direct way (as in [3]) of proving the integrability of the solution with respect to $\mathbb{P}$ is when we integrate the equation over the spatial domain and in addition also take expectations, which allows us to apply the Banach–Nečas–Babuška [BNB] theorem directly to the whole solution space. We will call this approach the "mean-weak" formulation. This result guarantees the measurability and the existence of the first and second moments of the solution and bounds of their norms, which motivates us to adopt this approach in the uniform case when the bilinear forms are uniformly bounded. The main task is to define properly the framework for the equation which will take into account the $L^2(\Omega)$ space and to keep track of the constants in estimates that we perform, i.e. to show that the constants are independent of ω. In particular, we will choose an appropriate Gelfand triple, precisely define the material derivative and a solution space. Another difficulty is that our domain changes over time. To deal with this, we will connect the space at arbitrary time t with the fixed initial space and incorporate this pull-back into the definition of the solution space. This construction is adapted from [1] where the abstract setting of the PDE on an evolving Hilbert space has been considered. This setting will enable us to apply [BNB] which gives us the well-posedness of the PDE with uniformly distributed random coefficients on an evolving space. First main result is stated in Theorem 4.3. Furthermore, we prove that for more regular input data, our solution also has more regularity in its material derivative.

In many practical applications in the geosciences but also in biology [9], flow and transfer in porous media are processes that are usually analysed and log-normally distributed random coefficients play an important role. As explained for example in [19], if the diffusion coefficient varies drastically within a layer, it is appropriate to expand its logarithm in an affine series of independent identically distributed normal random variables. The log-normal random parameter has been already analysed for the elliptic equations in many papers, for example in [7, 8, 19, 34] and in parabolic case in [25, 31]. However, in this case the bilinear forms are not uniformly distributed any more, thus we cannot consider the "mean-weak" formulation. Instead, we will consider the path-wise formulation (as in [22] and [25]). In this approach for each realisation we consider the deterministic problem. Therefore, we get the family of deterministic weak formulations over the spatial domain that can be solved $\mathbb{P}$-almost surely by applying the [BNB] theorem in the parabolic case. This implies the existence and uniqueness of the solution $u(\omega)$ for $\mathbb{P}$-a.e. ω. Since we are considering the PDE with random coefficients, we are interested in statistics of the solution, i.e. we want to prove that the solution is in $L^2(\Omega)$. In order to achieve that with the path-wise approach, one needs to prove the measurability of the solution with respect to $\mathbb{P}$ and a uniform bound for the $L^2(\Omega)$-norm (or higher order norm). Usually the latter is reduced to controlling the constants from the existence theory for the solution of the deterministic PDE (for example to bound the inf-sup constant). The final result on regularity and stability of the family of path-wise solutions is stated in Theorem 5.6.

The paper is organized as follows. We start the next section by setting up the notation, description of the hypersurfaces and assumptions on the evolution of the hypersurfaces. Furthermore, since our spaces will have tensor structure, we briefly summarize, without proofs, the relevant material on tensor products. At the end of the section we present notation and results about the log-normal distribution. In the third section we proceed with setting up the function spaces and defining the material derivative. Moreover, we show that the general framework from [1] is applicable. Section 4 concerns the uniformly bounded diffusion coefficient and contains the proofs of the main results about the existence, uniqueness and regularity of solutions. In the fifth section we discuss the case of log-normally distributed random coefficients and we prove the integrability of the solution. In the final section we discuss possible extensions to this paper for further research.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with sample space Ω, a $\sigma-$algebra of events $\mathcal{F}$ and a probability $\mathbb{P} : \mathcal{F} \to [0,1]$. In addition, we assume that $L^2(\Omega)$ is a separable space. For this assumption it suffices to assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is separable [5, Theorem 4.13].

We will only consider a fixed finite time interval $[0,T]$, where $T \in (0,\infty)$. Furthermore, we will denote by $\mathcal{D}([0,T];V)$ the space of infinitely differentiable functions with values in a Hilbert space V and compact support in $(0,T)$.

2.1 Hypersurfaces

Let us first recall some basic theory about hypersurfaces and Sobolev spaces on hypersurfaces. For more details we refer to [10] or [16]. We will assume that Γ is a $\mathcal{C}^2$ compact, connected, orientable, without a boundary, $n-$dimensional hypersurface, embedded in $\mathbb{R}^{n+1}$ for $n = 1, 2,$ or 3. For a function $f : \Gamma \to \mathbb{R}$ which is differentiable in an open neighborhood of Γ we define the *tangential gradient* by

$$\nabla_\Gamma f(x) := \nabla \tilde{f}(x) - \nabla \tilde{f}(x) \cdot \nu(x)\nu(x) \quad x \in \Gamma,$$

where $\nu(x)$ is the unit normal on $T_x\Gamma$ and $\nabla \tilde{f}(x)$ is the usual gradient in $\mathbb{R}^{n+1}$ of an arbitrary smooth extension of f to its neighborhood. Note that $\nabla_\Gamma f(x)$ is the orthogonal projection of $\nabla f(x)$ onto $T_x\Gamma$ (thus it is a tangential vector) and it depends only on the values of f on Γ [16, Lemma 2.4], which makes the previous definition of the tangential gradient independent of the extension $\tilde{f}$. The tangential gradient is a vector-valued quantity and for its components we will use the notation

$$\nabla_\Gamma f(x) = (\underline{D}_1 f(x), \ldots, \underline{D}_{n+1} f(x)).$$

Now we can define the *Laplace-Beltrami* operator by

$$\Delta_\Gamma f(x) = \nabla_\Gamma \cdot \nabla_\Gamma f(x) = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f(x) \quad x \in \Gamma.$$

Let us state the integration by parts formula for function $f \in \mathcal{C}^1(\overline{\Gamma}; \mathbb{R}^{n+1})$ and $\partial\Gamma = \emptyset$

$$\int_\Gamma \nabla_\Gamma \cdot f = \int_\Gamma f \cdot H\nu \tag{2.1}$$

where H is the mean curvature with respect to ν. Furthermore, we state *Green's formula*

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g = -\int_\Gamma f \Delta_\Gamma g. \tag{2.2}$$

From (2.1) and (2.2) we derive the following

$$\int_\Gamma f \nabla_\Gamma g = -\int_\Gamma (\nabla_\Gamma f - f H\nu) g. \tag{2.3}$$

We will consider a weak formulation of PDEs on Γ, which leads to the concept of Sobolev spaces on surfaces. We define $L^2(\Gamma)$ as usual, i.e. as a set of all measurable functions $f : \Gamma \to \mathbb{R}$ such that

$$\|f\|_{L^2(\Gamma)} := \left(\int_\Gamma |f(x)|^2 \right)^{1/2} < \infty.$$

We say that a function $f \in L^2(\Gamma)$ has a weak derivative $g_i = D_i f \in L^2(\Gamma)$, $(i = \{1, \ldots, n+1\})$ if for every function $\phi \in \mathcal{C}^1(\Gamma)$ and every i it holds

$$\int_\Gamma f \underline{D}_i \phi = -\int_\Gamma \phi g_i + \int_\Gamma f \phi H \nu_i.$$

The Sobolev space on Γ is defined by

$$H^1(\Gamma) = \{f \in L^2(\Gamma) \mid D_i f \in L^2(\Gamma), i = 1, \dots, n+1\}$$

with the norm

$$\|f\|_{H^1(\Gamma)} = \sqrt{\|f\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma f\|^2_{L^2(\Gamma)}}.$$

Let us define the family of evolving surfaces $\{\Gamma(t)\}$ for $t \in [0,T]$ that we will consider. For each $t \in [0,T]$ we assume that $\Gamma(t)$ satisfies the same properties as Γ and we set $\Gamma_0 := \Gamma(0)$. Furthermore, we assume the existence of a flow $\overline{\Phi} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$ its restriction $\Phi^0_t := \Phi(t,\cdot) : \Gamma_0 \to \Gamma(t)$, $\Phi \in \mathcal{C}^1([0,T], \mathcal{C}^2(\Gamma_0))$ is a diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt}\Phi^0_t(\cdot) &= \mathbf{w}(t, \Phi^0_t(\cdot)) \\ \Phi^0_0(\cdot) &= \mathrm{Id}(\cdot). \end{aligned}$$

where $\mathbf{w} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a velocity field. We assume that $\mathbf{w}(\cdot,t) \in \mathcal{C}^2(\Gamma(t))$ and that it has uniformly bounded divergence

$$|\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)| \le C_{\mathbf{w}} \quad \text{for all } t \in [0,T]. \tag{2.4}$$

Remark. Besides the normal velocity $\mathbf{w}_\nu = \mathbf{w} \cdot \nu\nu$, which is enough to define the evolution of the surface, we assume that the surface also has an advective tangential velocity $\mathbf{w}_\tau$, that describes the motion of points along the surface. Hence we assume that we are given a global velocity field $\mathbf{w}$ that can be decomposed as $\mathbf{w} = \mathbf{w}_\nu + \mathbf{w}_\tau$. In addition, we assume that the physical velocity agrees with the velocity of the parametrisation. For remark about the different notions of velocities for an evolving hypersurface see for example [2, Remark 2.6].

2.2 Tensor products

Since the function spaces which will be used later have tensor product structure, let us recall some basic results about it (see [20] or [33] for more details). Let H_1 and H_2 be Hilbert spaces and $v_i \in H_i, i = 1, 2$. We define $v_1 \otimes v_2$ as a conjugate bilinear form on $H_1 \times H_2$ by

$$(v_1 \otimes v_2)(w_1, w_2) := (v_1, w_1)_{H_1}(v_2, w_2)_{H_2}.$$

Let S be the set of finite linear combinations of such conjugate bilinear forms. We can define an inner product on S by

$$(v_1 \otimes v_2, w_1 \otimes w_2) := (v_1, w_1)_{H_1}(v_2, w_2)_{H_2} \tag{2.5}$$

and extend it by linearity to S. The *tensor product* $H_1 \otimes H_2$ is the completion of S under the inner product (2.5).

Theorem 2.1. The tensor space $H_1 \otimes H_2$ is a Hilbert space. If $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ are basis of Hilbert spaces H_1 and H_2, then $\{e_j \otimes f_k\}_{j,k\in\mathbb{N}}$ constitute a basis of $H_1 \otimes H_2$.

Proof. The proof can be found for example in [33]. □

Theorem 2.2. Let (X, μ) and (Y, ν) be measure spaces such that $L^2(X, \mu)$ and $L^2(Y, \nu)$ are separable. Then, the following holds:

a) There is a unique isometric isomorphism

$$L^2(X, \mu) \otimes L^2(Y, \nu) \cong L^2(X \times Y, \mu \times \nu)$$

so that $f \otimes g \mapsto fg$.

b) If H is a separable Hilbert space then there is a unique isometric isomorphism

$$L^2(X, \mu) \otimes H \cong L^2(X, \mu; H)$$

so that $f(x) \otimes \varphi \mapsto f(x)\varphi$.

Proof. The proof can be found for example in [33]. □

2.3 Log-normal expansion

In this subsection we will recall some definitions about the log-normal distribution that we will use in Chapter 5. For more details we refer to [19, 31, 34].

Definition 2.3. Let $S \subset \mathbb{R}^n$ and $a : \Omega \times S \to \mathbb{R}$ be a *random field* (RF) i.e. a is a measurable from $(\Omega \times S, \mathcal{F} \otimes \mathcal{B}(S))$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Then

• a is called *Gaussian* if for every $k \in \mathbb{N}$, $x_1, \dots, x_k \in S$ the multivariate random variable $(a(x_1), \dots, a(x_k))$ is multivariate Gaussian distributed, i.e. $\sum_{i=1}^k c_i a(x_i)$ is normally distributed random variable for every $c_i \in \mathbb{R}, i = 1, \dots, k$.

• $\alpha : \Omega \times S \to \mathbb{R}_+$ is *log-normal, RF* if $\log \alpha$ is an Gaussian RF on S.

In our setting, the diffusion coefficient is defined on the space-time domain $\mathcal{G}_T := \bigcup_t \Gamma(t) \times \{t\}$. Hence, let $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}_+$ be a log-normal diffusion coefficient. We will consider a series expansion of its logarithm.

Assumption 2.4. There exists a sequence $(Y_k)_{k\in\mathbb{N}}$ of i.i.d. standard Gaussian random variables on Ω and functions $\alpha_k \in L^\infty(\mathcal{G}_T)$ for $k \in \mathbb{N}$ with $b := (\|\alpha_k\|_{L^\infty(\mathcal{G}_T)})_{k\in\mathbb{N}} \in l^1(\mathbb{N})$ i.e. $\sum_k b_k < \infty$, where $b_k := \|\alpha_k\|_{L^\infty(\mathcal{G}_T)}$, such that the diffusion coefficient has the form

$$\alpha(\omega; x, t) = \exp\left(\sum_{k\geq 1} \alpha_k(x, t) Y_k(\omega)\right). \tag{2.6}$$

Remark 2.5. Without loss of generality we assumed that logarithm of α is a centred Gaussian random field.

Remark 2.6. Necessary conditions to satisfy Assumption 2.4 are discussed e.g. in [29] and the references given therein. It is shown that standard measurability conditions (more precisely: measurability, finite-variance and isotropy) imply mean-square continuity of a random field. This in turn is necessary for representation (2.6) to hold.

Motivated by the analysis in [19] and [34], for the log-normal case, we will reformulate the problem with the parameter domain $\mathbb{R}^{\mathbb{N}}$ instead of Ω (for details how this can be done see [19, 34]). Thus, our probability space is $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}), \gamma)$ with

$$\gamma := \bigotimes_{k\geq 1} N_1 \tag{2.7}$$

where N_1 is the standard Gaussian measure on $\mathbb{R}$. We underline this change by switching from the notation ω to y and from $Y_k(\omega)$ to y_k. Therefore, the diffusion coefficient now has the form

$$\alpha(y; x, t) = \exp\left(\sum_{k\geq 1} \alpha_k(x,t) y_k\right) \tag{2.8}$$

for $y = (y_k)_{k\in\mathbb{N}} \in \mathbb{R}^{\mathbb{N}}$ and we assume that y_k are i.i.d. standard Gaussian random variables on $\mathbb{R}$. In order to have the convergence of the series (2.8) we consider

$$\Theta_b := \left\{ y \in \mathbb{R}^{\mathbb{N}} \mid \sum_{k\geq 1} b_k |y_k| < \infty \right\}. \tag{2.9}$$

With Assumption 2.4, from [19, Lemma 2.2] the series in (2.8) converges in $L^\infty(\mathcal{G}_T)$ in the parameter space Θ_b.

Lemma 2.7. For any $b \in l^1(\mathbb{N})$ it holds $\Theta_b \in \mathcal{B}(\mathbb{R}^{\mathbb{N}})$ and $\gamma(\Theta_b) = 1$.

Proof. We refer to [19, Lemma 2.3]. □

Instead of the whole space $\mathbb{R}^{\mathbb{N}}$, due to Lemma 2.7, we will consider $\Theta = \Theta_b$ as the parameter space with the measure that is restriction of γ on Θ. From Assumption 2.4 it follows that the diffusion coefficient is bounded from above and has positive lower bound for every $y \in \Theta$:

Lemma 2.8. For all $y \in \Theta$, the diffusion coefficient $\alpha(y)$ is well-defined and satisfies

$$0 < \alpha_{\min}(y) := \operatorname*{ess\,inf}_{(x,t)\in\mathcal{G}_T} \alpha(y; x, t) \leq \operatorname*{ess\,sup}_{(x,t)\in\mathcal{G}_T} \alpha(y; x, t) =: \alpha_{\max}(y) < \infty \tag{2.10}$$

with

$$\alpha_{\max}(y) \leq \exp\left(\sum_{k\geq 1} b_k |y_k|\right)$$
$$\alpha_{\min}(y) \geq \exp\left(-\sum_{k\geq 1} b_k |y_k|\right).$$

Proof. The proof can be found in [34, Lemma 2.29], as a direct consequence of Assumption 2.4. □

2.4 Product measures on the probability space

Results of this subsection can be found in [19] or [31], we state those that we will use in the log-normal case, for the convenience of the reader.

For any $\sigma = (\sigma_k)_{k\in\mathbb{N}} \in \exp(l^1(N))$ i.e. $\sigma_k = \exp(s_k)$ with $(s_k)_{k\in\mathbb{N}} \in l^1(\mathbb{N})$, we define the product measure on $(\mathbb{R}^{\mathbb{N}}, \mathcal{B}(\mathbb{R}^{\mathbb{N}}))$ by

$$\gamma_\sigma := \bigotimes_{k\geq 1} N_{\sigma_k^2}$$

where $N_{\sigma_k^2}$ is a centered Gaussian measure on $\mathbb{R}$ with standard deviation σ_k. Note that $\gamma = \gamma_1$ is the standard Gaussian measure on $\mathbb{R}^{\mathbb{N}}$.

Theorem 2.9. For all $\sigma \in \exp(l^1(\mathbb{N}))$, the measure γ_σ is equivalent to γ and the density of γ_σ with respect to γ is given by

$$\zeta_\sigma(y) = \left(\prod_{k\geq 1} \frac{1}{\sigma_k}\right) \exp\left(-\frac{1}{2}\sum_{k\geq 1}(\sigma_k^{-2} - 1)y_k^2\right).$$

Proof. We refer to [19, Proposition 2.11]. □

From the previous theorem we get that $\gamma_\sigma(\Theta) = 1$ for every $\sigma \in \exp(l^1(\mathbb{N}))$, thus restriction of γ_σ on Θ is a probability measure. Let σ be the sequence that depends exponentially on $b = (b_k)_{k\in\mathbb{N}}$, for $b_k := \|\alpha_k\|_{L^\infty(\mathcal{G}_T)}$defined in Assumption 2.4. We will consider the class

$$\sigma_k := \exp(\chi b_k) \qquad \chi \in \mathbb{R}, \quad k \in \mathbb{N}$$

and we will use the following notation $\gamma_\chi := \gamma_{\sigma(\chi)}$ and $\zeta_\chi := \zeta_{\sigma(\chi)}$.

Lemma 2.10. Let $\eta < \chi$ and $m \geq 0$. Then, for every $y \in \Theta$ it holds

$$\frac{\zeta_\eta(y)}{\zeta_\chi(y)} \exp\left(m\sum_{k\geq 1} b_k|y_k|\right) \leq \exp\left(\left(\frac{m^2\exp(2\chi\|b\|_{l^\infty})}{4(\chi-\eta)} + \chi - \eta\right)\|b\|_{l^1}\right).$$

Proof. The proof can be found in [34, Lemma 2.32]. □

We will need the special case from the previous Lemma, when $\eta = 0$, which gives us the bound for the $1/\zeta_\chi(y)\exp\left(m\sum_{k\geq 1} b_k|y_k|\right)$.

3 Function spaces

In this section we will define the function spaces that we will mainly consider in the case when diffusion coefficient has uniform distribution.

3.1 Gelfand triple

In this section, we will define the basic Gelfand triple that will be used in the uniform case to define the solution space for (1.1). We begin by recalling the notion of Gelfand triple. Let V and H be separable Hilbert spaces. A Gelfand triple is a construction

$$V \underset{i}{\hookrightarrow} H \cong H^* \underset{i'}{\hookrightarrow} V^*$$

where both embeddings i and i' are continuous and dense, and H is identified with its dual space H^* via the Ritz representation theorem (see [39, Def 17.1] for more general definition). The duality pairing between V and V^* is compatible with the inner product on H in the sense that

$$\langle u, v \rangle_{V^*,V} = (u, v)_H \quad \text{whenever } u \in H, v \in V.$$

In order to define the Gelfand triple for each $t \in [0, T]$, let us define

$$V(t) := L^2(\Omega, H^1(\Gamma(t))) \quad \text{and} \quad H(t) := L^2(\Omega, L^2(\Gamma(t))).$$

Then the dual space of $V(t)$ is the space $V^*(t) = L^2(\Omega, H^{-1}(\Gamma(t)))$ where $H^{-1}(\Gamma(t))$ is the dual space of $H^1(\Gamma(t))$.

Since all spaces $L^2(\Omega), L^2(\Gamma(t))$ and $H^1(\Gamma(t))$ are separable Hilbert spaces, using Theorem 2.2 we have

$$L^2(\Omega, H^1(\Gamma(t))) \cong L^2(\Omega) \otimes H^1(\Gamma(t)) \tag{3.1}$$

$$L^2(\Omega, L^2(\Gamma(t))) \cong L^2(\Omega) \otimes L^2(\Gamma(t)). \tag{3.2}$$

Remark. For convenience we will often (but not always) write $u(\omega, x)$ instead of $u(\omega)(x)$, which is justified by the aforementioned isomorphisms.

Lemma 3.1. $V(t) \hookrightarrow H(t) \hookrightarrow V^*(t)$ is a Gelfand triple for every $t \in [0, T]$.

Proof. Since $H^1(\Gamma(t))$ is dense in $L^2(\Gamma(t))$, the proof follows from (3.1), (3.2) and [20, Lemma 4.34], using the density argument. □

3.2 Compatibility of spaces

In order to treat the evolving spaces, we need to define special Bochner-type function spaces such that for every $t \in [0, T]$ we have $u(t) \in V(t)$. In general, if we have an evolving family of Hilbert spaces $X = (X(t))_{t \in [0,T]}$, the idea is to connect the space $X(t)$ at any time $t \in [0, T]$ with some fixed space, for example with the initial space $X(0)$. We do that using the family of maps $\phi_t : X(0) \to X(t)$, which we call the pushforward map. We denote the inverse of ϕ_t by $\phi_{-t} : X(t) \to X(0)$ and call it the pullback map. The following definition is adapted from [1].

Remark. This approach is similar to Arbitrary Lagrangian Eulerian [ALE] framework.

Definition 3.2. The pair $\{X, (\phi_t)_{t\in[0,T]}\}$ is *compatible* if the following conditions hold:

- for every $t \in [0, T]$, ϕ_t is linear homeomorphism such that ϕ_0 is the identity map
- there exists a constant C_X which is independent of t such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X(0)} \quad \text{for every } u \in X(0) \\ \|\phi_{-t} u\|_{X(0)} &\leq C_X \|u\|_{X(t)} \quad \text{for every } u \in X(t) \end{aligned}$$

- the map $t \mapsto \|\phi_t u\|_{X(t)}$ is continuous for every $u \in X(0)$.

We will denote the dual operator of ϕ_t by $\phi_t^* : X^*(t) \to X^*(0)$. As a consequence of the previous conditions, we obtain that ϕ_t^* and its inverse are also linear homeomorphisms which satisfy the following conditions

$$\begin{aligned} \|\phi_t^* f\|_{X^*(0)} &\leq C_X \|f\|_{X^*(t)} \quad \text{for every } f \in X^*(t) \\ \|\phi_{-t}^* f\|_{X^*(t)} &\leq C_X \|f\|_{X^*(0)} \quad \text{for every } f \in X^*(0). \end{aligned}$$

For the Gelfand triple $L^2(\Omega, H^1(\Gamma(t))) \subset L^2(\Omega, L^2(\Gamma(t))) \subset L^2(\Omega, H^{-1}(\Gamma(t)))$ we define the pullback operator $\phi_{-t} : L^2(\Omega, L^2(\Gamma(t))) \to L^2(\Omega, L^2(\Gamma_0))$ in the following way

$$(\phi_{-t} u)(\omega)(x) := u(\omega)(\Phi_t^0(x)) \quad \text{for every } x \in \Gamma(0),\ \omega \in \Omega.$$

Remark. Since we are interested only in the dual operator of $\phi_t\big|_V$, we will denote it by $\phi_t^* : V^*(t) \to V_0^*$.

The next step is to prove that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}\big|_{V_0})$ are compatible pairs. The proof is similar to the proof of [38, Lemma 3.2].

Let $J_t^0(\cdot) := \det D_{\Gamma_0} \Phi_t^0(\cdot)$ denote the Jacobian determinant (where $(D_{\Gamma_0} \Phi_t^0)_{ij} := \underline{D}_j (\Phi_t^0)_i$), i.e. it presents the change area of the element when transformed from Γ_0 to $\Gamma(t)$. The assumptions for the flow Φ_t^0 imply $J_t^0 \in \mathcal{C}^1([0, T] \times \Gamma_0)$ and that the field J_t^0 is uniformly bounded

$$\frac{1}{C_J} \leq J_t^0(x) \leq C_J \quad \text{for every } x \in \Gamma_0 \text{ and for all } t \in [0, T], \tag{3.3}$$

where C_J is positive constant.

The substitution formula for integrable functions $\zeta : \Gamma(t) \to \mathbb{R}$ reads

$$\int_{\Gamma(t)} \zeta = \int_{\Gamma_0} (\zeta \circ \Phi_t^0) J_t^0 = \int_{\Gamma_0} \phi_{-t} \zeta J_t^0.$$

Using the Leibniz formula for differentiation of a parameter dependent surface integral [13, Lemma 2.1] it can be shown [38, Lemma 3.2] that

$$\frac{d}{dt} J_t^0 = \phi_{-t} (\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0. \tag{3.4}$$

Lemma 3.3. The pairs $\big(H, (\phi_t)_{t\in[0,T]}\big)$ and $\Big(V, (\phi_t\big|_{V_0})_{t\in[0,T]}\Big)$ are compatible.

Proof. The proof is similar to the proof of [38, Lemma 3.3]. However, we will state the proof in order to show that constants that appear are independent of the sample ω.

First we will prove the statement for the pair $\big(H, (\phi_t)_{t\in[0,T]}\big)$. Let u be from $L^2(\Omega, L^2(\Gamma(t)))$. Then we have

$$\|\phi_{-t}u\|^2_{L^2(\Omega,L^2(\Gamma_0))} = \int_\Omega\int_{\Gamma(t)} |u(\omega)(y)|^2 \frac{1}{J^0_t((\Phi^0_t)^{-1}(y))} \leq C_J \|u\|^2_{L^2(\Omega,L^2(\Gamma(t)))},$$

where we have used the substitution formula and boundedness of J^0_t. It is clear that ϕ_{-t} is linear and that its continuity follows immediately from the previous estimate. Since Φ^0_t is $\mathcal{C}^2-$diffeomorphism, it follows that ϕ_{-t} is bijective and its inverse (the pushforward) is defined by

$$\phi_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma(t))), \quad (\phi_t v)(\omega, x) = v(\omega) \circ (\Phi^0_t)^{-1}(x).$$

Similarly as for ϕ_{-t}, we can prove that ϕ_t is well defined, satisfies the norm boundedness relation and is continuous. Thus, ϕ_t is linear homeomorphism.
Since the probability space does not depend on time, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega,L^2(\Gamma(t)))}$ follows directly from [38, Lemma 3.3.] and the triangle inequality.

In order to prove compatibility of the family $(V, (\phi_t\big|_{V_0})_{t\in[0,T]})$, let $v \in L^2(\Omega, H^1(\Gamma(t)))$ and $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$. Using the substitution formula and integration by parts on $\Gamma(t)$ we get

$$\begin{aligned}
&\int_\Omega\int_{\Gamma_0} \phi_{-t}v(\omega, x)\nabla_\Gamma\varphi(\omega, x) = \int_\Omega\int_{\Gamma(t)} v(\omega, x)(D\bar{\Phi}_t(x))^T\nabla_\Gamma(\phi_t\varphi(\omega, x))J^0_{-t}(x)\\
&= -\int_\Omega\int_{\Gamma(t)} \phi_t\varphi(\omega, x)s(\omega, x)J^0_{-t}(x)\\
&= -\int_\Omega\int_{\Gamma_0} [\phi_{-t}s(\omega, x) - H_0\nu_0\phi_{-t}v(\omega, x)]\varphi(\omega, x) + H_0\nu_0\phi_{-t}v(\omega, x)\varphi(\omega, x),
\end{aligned} \tag{3.5}$$

where s is the function that we get from the partial integration. Note that s depends only on the mean curvature and derivative of $\bar{\Phi}_t$ which can be bounded independently of time and ω. Thus, $\|s(\omega)\|_{L^2(\Gamma(t))^{(n+1)}} \leq C\|v(\omega)\|_{H^1(\Gamma(t))}$, where C does not depend on ω and t. Furthermore, we get

$$\|s\|_{L^2(\Omega,L^2(\Gamma(t))^{n+1})} \leq C\|v\|_{L^2(\Omega,H^1(\Gamma(t)))}.$$

Hence, using the estimate from the first part of the proof we get

$$\phi_{-t}v \in L^2(\Omega, L^2(\Gamma_0)) \quad \text{and} \quad \|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C'\|v\|_{L^2(\Omega,H^1(\Gamma(t)))}. \tag{3.6}$$

On the other hand, from the partial integration on hypersurface we get

$$\int_\Omega\int_{\Gamma_0} \phi_{-t}v(\omega, x)\nabla_\Gamma\varphi(\omega, x) = -\int_\Omega\int_{\Gamma_0} \varphi(\omega, x)(\nabla_\Gamma(\phi_{-t}v)(\omega, x) + \phi_{-t}v(\omega, x)H_0\nu_0).$$

From the last relation and (5.10), since they hold for every $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$, we get

$$\nabla_\Gamma(\phi_{-t}v)(\omega, x) = \phi_{-t}s(\omega, x) - H_0\nu_0(\phi_{-t}v)(\omega, x). \tag{3.7}$$

For $v \in L^2(\Omega, L^2(\Gamma(t)))$, we have already proved that $\|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}$. Therefore, the following estimate follows

$$\|H_0\nu_0(\phi_{-t}v)(\omega, x)\|_{L^2(\Omega,L^2(\Gamma_0))} \leq |H_0|C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Using the last inequality, (3.6) and (3.7), we get

$$\|\phi_{-t}v\|_{L^2(\Omega,H^1(\Gamma_0))} \leq C_V\|v\|_{V(t)},$$

where C_V depends on global bound on $|H_t|$, $\|\partial\bar{\Phi}_t\|$ and $\|\partial_{ij}\bar{\Phi}_t\|$ with $1 \leq i, j \leq n+1$, $t \in [0, T]$ and these bounds are deterministic and independent of time.

Similarly to the previous case, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega,H^1(\Gamma(t)))}$ follows from [38, Lemma 3.3] and the independence of the probability space of time, which completes the proof. □

3.3 Bochner-type spaces

In this section, we want to define Bochner-type spaces of random functions that are defined on evolving spaces. In order to strictly define these spaces we will ask that the pull-back of u belongs to the fixed initial space $V(0)$. These spaces are a special case of general function spaces defined in [1]:

Definition 3.4. For a compatible pair $(X, (\phi_t)_t)$ we define spaces:

$$L^2_X := \left\{ u : [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\},\ t \mapsto (\bar{u}(t), t) \mid \phi_{-(\cdot)}\bar{u}(\cdot) \in L^2(0,T;X_0) \right\}$$

$$L^2_{X^*} := \left\{ f : [0,T] \to \bigcup_{t\in[0,T]} X^*(t) \times \{t\},\ t \mapsto (\bar{f}(t), t) \mid \phi^*_{(\cdot)}\bar{f}(\cdot) \in L^2(0,T;X^*_0) \right\}.$$

Remark. In the following we will identify $u(t) = (\overline{u}(t), t)$ with $\overline{u}(t)$.

The spaces L^2_X and $L^2_{X^*}$ are separable Hilbert spaces ([1], Corollary 2.11) with the inner product defined as

$$(u, v)_{L^2_X} = \int_0^T (u(t), v(t))_{X(t)} dt$$

$$(f, g)_{L^2_{X^*}} = \int_0^T (f(t), g(t))_{X^*(t)} dt.$$

By Lemma 3.3, the spaces L^2_V, $L^2_{V^*}$ and L^2_H are well-defined. Furthermore, from [1, Lemma 2.15] it follows that we can identify $L^2_{V^*}$ and $(L^2_V)^*$. Using Lemma 3.1 and [1, Lemma 2.19] we conclude the following result.

Lemma 3.5.
$$L^2_{L^2(\Omega,H^1(\Gamma(t)))} \hookrightarrow L^2_{L^2(\Omega,L^2(\Gamma(t)))} \hookrightarrow L^2_{L^2(\Omega,H^{-1}(\Gamma(t)))}$$
is a Gelfand triple.

3.4 Material derivative

This subsection is motivated by the abstract framework from the Chapter 2.4 in [1]. We want to define a time derivative that will also take into account the spatial movement, i.e. the material derivative for random functions. First let us consider the spaces of pushed-forward continuously differentiable functions

$$\mathcal{C}^j_V := \{u \in L^2_V \mid \phi_{-(\cdot)} u(\cdot) \in \mathcal{C}^j([0,T], L^2(\Omega, H^1(\Gamma_0)))\} \text{ for } j \in \{0, 1, \dots\}.$$

Definition 3.6. For $u \in \mathcal{C}^1_V$ *the strong material derivative* $\dot{u} \in \mathcal{C}^0_V$ is defined by

$$\dot{u}(t) = \phi_t \left(\frac{d}{dt} \phi_{-t} u(t) \right)$$

for every $t \in [0,T]$.

By smoothness of $\Gamma(t)$ and evolution Φ^0_t, for every $\omega \in \Omega$ each function $u(t,\omega) : \Gamma(t) \to \mathbb{R}$ can be extended to a neighbourhood of $\bigcup_{t\in[0,T]} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+2}$ in which $\nabla u(\omega)$ and $u_t(\omega)$ for the extension are well defined for every ω (for the construction of extension see [16]). Using the chain rule, for $u \in \mathcal{C}^1_V$ and $y \in \Gamma_0$, we get

$$\begin{aligned}
\frac{d}{dt}\phi_{-t}u(t) &= \frac{d}{dt}(u(t,\omega,\Phi^0_t(y)) \\
&= u_t(t,\omega,\Phi^0_t(y)) + \nabla u \mid_{(t,\omega,\Phi^0_t(y))} \cdot \mathbf{w}(t,\Phi^0_t(y)) \\
&= \phi_{-t}u_t(t,\omega,y) + \phi_{-t}\nabla u(t,\omega,y) \cdot \phi_{-t}(\mathbf{w}(t,y)).
\end{aligned}$$

Thus, we get the following explicit formula for the *strong material derivative*

$$\dot{u}(t,\omega,x) = u_t(t,\omega,x) + \nabla u(t,\omega,x) \cdot \mathbf{w}(t,x), \tag{3.8}$$

for every $x \in \Gamma(t)$ and $\omega \in \Omega$.

Remark. Note that the right hand side of (3.8) does not depend on extension, so it is irrelevant that every extension (i.e. neighbourhood) will depend on ω.

Just as in the deterministic case, it might happen that the equation does not have a solution if we ask that $u \in \mathcal{C}^1_V$. Hence, we want to define a weak material derivative that needs less regularity. In addition to the case when we consider a fixed domain, we will have an extra term that will take into account the movement of the domain. As usual in this setting (see for example[1]), the idea is to pull-back the inner product on $L^2(\Omega, L^2(\Gamma(t)))$ onto the fixed space $L^2(\Omega, L^2(\Gamma_0))$, which will be the bilinear form $\hat{b}$. Furthermore, we define $\hat{c}$ as a regular time derivative of this bilinear form. Thus, the extra term c in the weak material derivative will be the push-forward of $\hat{c}$ onto $H(t) \times H(t)$.

Let us define the bounded bilinear form $\hat{b}(t, \cdot, \cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ for every $t \in [0, T]$

$$\begin{aligned}\hat{b}(t, u_0, v_0) &:= (\phi_t u_0, \phi_t v_0)_{L^2(\Omega, L^2(\Gamma(t)))} \\ &= \int_\Omega \int_{\Gamma(0)} u_0(\omega, x) v_0(\omega, x) J_t^0(x).\end{aligned}$$

Moreover, we define the map $\theta : [0, T] \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ that is the classical time derivative of the norm on $L^2(\Omega, L^2(\Gamma(t)))$

$$\theta(t, u_0) := \frac{d}{dt} \|\phi_t u_0\|^2_{L^2(\Omega, L^2(\Gamma(t)))} \quad \forall u_0 \in L^2(\Omega, L^2(\Gamma_0)).$$

Lemma 3.7. a) The map θ is well defined and for each $t \in [0, T]$ the map

$$u_0 \mapsto \theta(t, u_0) \quad u_0 \in L^2(\Omega, L^2(\Gamma_0)) \tag{3.9}$$

is continuous.

b) For every $t \in [0, T]$ there exists deterministic constant C that is independent of time such that

$$|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| \leq C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}.$$

Proof. a) Using the substitution formula, the formula (3.4) and the assumption (2.4) we get:

$$\theta(t, u_0) = \int_\Omega \int_{\Gamma(0)} (u_0(\omega, x))^2 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t, x)) J_t^0(x) \leq C \|u_0\|^2_{L^2(\Omega, L^2(\Gamma_0))}.$$

Hence, θ is well-defined. In order to prove continuity of (3.9) note that $u \in L^2(\Omega, L^2(\Gamma_0))$ implies $u^2 \in L^1(\Omega, L^1(\Gamma_0))$. This implies that if $u_n \to u$ in $L^2(\Omega, L^2(\Gamma_0))$, then $u_n^2 \to u^2$ in $L^1(\Omega, L^1(\Gamma_0))$. Now continuity follows from:

$$\begin{aligned}|\theta(t, u_n) - \theta(t, u)| &\leq \int_\Omega \int_{\Gamma_0} |u_n^2(\omega, x) - u^2(\omega, x)| |\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t, x)) J_t^0(x)| \\ &\leq C \|u_n^2 - u^2\|_{L^1(\Omega, L^1(\Gamma_0))} \to 0.\end{aligned}$$

b) Using the Cauchy-Schwarz inequality, (3.3) and (3.4) we get the estimate:

$$\begin{aligned}|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| &= \|4 \frac{d}{dt} \hat{b}(t; u_0, v_0)\| \\ &= 4 |\int_\Omega \int_{\Gamma_0} u_0(\omega, x) v_0(\omega, x) \frac{d}{dt} J_t^0(x)| \\ &\leq C |\,(u_0, v_0)\,|_{L^2(\Omega, L^2(\Gamma_0))} \\ &\leq C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}. \quad \square\end{aligned}$$

Now we can define the bilinear form $\hat{c}(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ as a partial time derivative of $\hat{b}$

$$\hat{c}(t;u_0,v_0) := \frac{\partial}{\partial t}\hat{b}(t;u_0,v_0) = \frac{1}{4}(\theta(t,u_0+v_0) - \theta(t,u_0-v_0))$$
$$= \int_\Omega \int_{\Gamma_0} u_0(\omega,x)v_0(\omega,x)\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x))J_t^0(x).$$

From [1, Lemma 2.27] it follows that for every $u, v \in \mathcal{C}^1([0,T]; L^2(\Omega, L^2(\Gamma_0)))$ the map

$$t \mapsto \hat{b}(t; u(t), v(t))$$

is differentiable in the classical sense and the formula for differentiation of the scalar product on $L^2(\Omega, L^2(\Gamma(t)))$ is

$$\frac{d}{dt}\hat{b}(t;u(t),v(t)) = \hat{b}(t;u'(t),v(t)) + \hat{b}(t;u(t),v'(t)) + \hat{c}(t;u(t),v(t)).$$

We will generalise this result in Section 3.5, to less regular functions u and v.

Now we can define the extra term that appears in the definition of the weak material derivative. As we have already announced, we pull the functions back to $\Gamma(0)$ and apply bilinear form $\hat{c}$ to them. More precisely, we define the bilinear form $c(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma(t))) \times L^2(\Omega, L^2(\Gamma(t))) \to \mathbb{R}$ by

$$c(t;u,v) := \hat{c}(t;\phi_{-t}u, \phi_{-t}v) = \int_\Omega \int_{\Gamma(t)} u(\omega,z)v(\omega,z)(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x)).$$

Lemma 3.8. For every $u, v \in L^2_V$, the map

$$t \mapsto c(t; u(t), v(t))$$

is measurable. Furthermore, c is bounded independently of t by deterministic constant:

$$|c(t;u,v)| \leq C\|u\|_{L^2(\Omega,L^2(\Gamma(t)))}\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Proof. From Lemma 3.7 it follows that we can apply a corollary of [1, Lemma 2.26], which proves the Lemma. □

Now we can define the weak material derivative.

Definition 3.9. We say that $\partial^\bullet u \in L^2_{V^*}$ is *a weak material derivative* of $u \in L^2_V$ if and only if

$$\int_0^T \langle \partial^\bullet u(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t), \dot{\eta}(t))_{H(t)} - \int_0^T c(t; u(t), \eta(t))$$
$$= \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\dot{\eta}(t,\omega,x) - \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\eta(t,\omega,x)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x),$$

holds for all $\eta \in \mathcal{D}_V(0,T) = \{\eta \in L^2_V \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0,T); L^2(\Omega, H^1(\Gamma_0)))\}$.

Note that it can be directly shown that if it exists, the weak material derivative is unique and every strong material derivative is also a weak material derivative.

3.5 Solution space

We will ask for the solution of the equation (1.1) to be in the space L^2_V and also to have a weak material derivative. Hence, we define the solution space as:

$$W(V,V^*) := \{u \in L^2_V \mid \partial^\bullet u \in L^2_{V^*}\}.$$

In order to prove that the solution space is Hilbert space and also that it has some additional properties, we will connect it with the standard Sobolev-Bochner space for which these properties are known. Thus, let us define the following space:

$$\mathcal{W}(V_0,V_0^*) = \{u \in L^2(0,T;L^2(\Omega,H^1(\Gamma_0))) \mid u' \in L^2(0,T;L^2(\Omega,H^{-1}(\Gamma_0)))\}.$$

The space $\mathcal{W}(V_0,V_0^*)$ is Hilbert space with the inner product defined via:

$$(u,v)_{\mathcal{W}(V_0,V_0^*)} := \int_0^T\int_\Omega (u(t,\omega),v(t,\omega))_{H^1(\Gamma_0)} + \int_0^T\int_\Omega (u'(t,\omega),v'(t,\omega))_{H^{-1}(\Gamma_0)}.$$

We will use that the embedding

$$\mathcal{D}([0,T];V_0) \subset \mathcal{W}(V_0,V_0^*) \tag{3.10}$$

is dense. More properties of this space can be found for example in [38, Lemma 2.2].

We want to show that the previous two types of spaces are connected in a natural way, i.e. that the pull-back of the functions from the solution space belong to Sobolev-Bochner space and vice versa. In addition, we also have the equivalence of the norms. First we will prove the technical result which is similar to [38, Lemma 3.6.].

Lemma 3.10. Let $w \in \mathcal{W}(V_0,V_0^*)$ and $f \in \mathcal{C}^1([0,T]\times\Gamma_0)$. Then $fw \in \mathcal{W}(V_0,V_0^*)$ and

$$(fw)' = \partial_t f w + fw', \tag{3.11}$$

where $\langle fw',\varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma_0)),L^2(\Omega,H^1(\Gamma_0))} = \langle w',f\varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma_0)),L^2(\Omega,H^1(\Gamma_0))}$.

Proof. We will first prove the Lemma for $\varphi \in \mathcal{D}([0,T],L^2(\Omega,H^1(\Gamma_0)))$. From the proof of [38, Lemma 3.6] it follows that $f \in \mathcal{C}^1([0,T]\times\Gamma_0)$ implies

$$f \in \mathcal{C}([0,T],\mathcal{C}^1(\Gamma_0)) \text{ and } f \in \mathcal{C}^1([0,T],\mathcal{C}(\Gamma_0)). \tag{3.12}$$

In order to prove that $f\varphi \in L^2([0,T];L^2(\Omega,H^1(\Gamma_0)))$ we can treat deterministic function f as a random function that is constant in ω. More precisely, if we define the function $\tilde f(t,\omega,x) := f(t,x)$, from (3.12) it follows $\tilde f \in \mathcal{C}([0,T],L^2(\Omega,\mathcal{C}^1(\Gamma_0))$. This can be strictly shown by defining the function $g : \mathcal{C}(\Gamma_0) \to L^2(\Omega,\mathcal{C}(\Gamma_0))$, $g(f)(\omega,x) := f(x)$. Note that g is linear, thus a $\mathcal{C}^\infty$-function and for every t we have $g(f(t)) = \tilde f(t)$.

It is then clear that we have

$$\tilde{f}\varphi \in \mathcal{C}([0,T], L^2(\Omega, H^1(\Gamma_0))) \cap \mathcal{C}^1([0,T], L^2(\Omega, L^2(\Gamma_0)))$$

which implies $\tilde{f}\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$ and hence, $f\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$.

It is left to prove that formula (3.11) is valid. We will prove this using the characterisation of the weak derivative [1, Theorem 2.2] and partial integration [1, Lemma 2.1(3)]:

$$\int_0^T \langle fw', \varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma(t))),L^2(\Omega,H^1(\Gamma(t)))} = -\int_0^T \langle w, (f\varphi)'\rangle_{L^2(\Omega,H^1(\Gamma(t))),L^2(\Omega,H^{-1}(\Gamma(t)))}$$
$$= -\int_0^T \langle \partial_t f w, \varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma(t))),L^2(\Omega,H^1(\Gamma(t)))} - \int_0^T \langle fw, \varphi'\rangle_{L^2(\Omega,H^{-1}(\Gamma(t))),L^2(\Omega,H^1(\Gamma(t)))}.$$

It follows

$$\int_0^T \langle fw, \varphi'\rangle_{L^2(\Omega,L^2(\Gamma_0))} = \int_0^T \langle \partial_t f w + fw', \varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma(t))),L^2(\Omega,H^1(\Gamma(t)))},$$

i.e. $(fw)' = \partial_t f w + fw'$. Using the density result (3.10) we can approximate every function fw by continuous $L^2(\Omega, H^1(\Gamma_0))-$valued functions and conclude that $fw \in L^2(\Omega, H^1(\Gamma_0))$. The similar argument implies that $(fw)' \in L^2(\Omega, H^{-1}(\Gamma_0))$. □

Corollary 3.11. If $T_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ is defined as $T_t u_0(\omega, x) := u_0(\omega, x) J_t^0(x)$, then it holds:

$$u \in \mathcal{W}(V_0, V_0^*) \text{ if and only if } T_{(\cdot)}u(\cdot) \in \mathcal{W}(V_0, V_0^*). \tag{3.13}$$

Proof. Apply Lemma 3.10 to the functions $f = J^0_{(\cdot)}$ and $f = \frac{1}{J^0_{(\cdot)}}$, which are both from the space $\mathcal{C}^1([0,T] \times \Gamma_0)$. □

Theorem 3.12. The following equivalence holds

$$v \in W(V, V^*) \text{ if and only if } \phi_{-(\cdot)}v(\cdot) \in \mathcal{W}(V_0, V_0^*), \tag{3.14}$$

and the norms are equivalent

$$C_1 \|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0,V_0^*)} \le \|v\|_{W(V,V^*)} \le C_2 \|\phi_{-(\cdot)}v(\cdot)\|_{\mathcal{W}(V_0,V_0^*)}. \tag{3.15}$$

Remark. Following the notation from [1], we say that there exists an *evolving space equivalence* between the spaces $W(V, V^*)$ and $\mathcal{W}(V_0, V_0^*)$ if and only if they satisfy (3.14) and (3.15).

Proof. Let $u \in \mathcal{W}(V_0, V_0^*)$. For every $t \in [0,T]$ we define a map $\hat{S}(t) : V_0^* \to V_0^*$ by

$$\hat{S}(t)u'(t) := J_t^0 u'(t).$$

Note that since J_t^0 is bounded independently of t and has an inverse, this implies that $\hat{S}(t)$ has an inverse, and both $\hat{S}(t)$ and $\hat{S}^{-1}(t)$ are bounded independently of t. Furthermore, from the uniform bound on J_t^0 we have that $\hat{S}(\cdot)u'(\cdot) \in L^2(0,T;V_0^*)$. In the end, using the product rule (3.11), we get

$$(T_t u(t))' = (J_t^0 u(t))' = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0 u(t) + J_t^0 u'(t) = \hat{S}(t)u'(t) + \hat{C}(t)u(t),$$

where T_t is defined in the previous corollary and $\hat{C}(t) : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ is defined as $\hat{C}(t,\omega,x) = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0(x)$, i.e. $\left\langle \hat{C}(t)u_0, v_0 \right\rangle := \hat{c}(t;u_0,v_0)$. Thus, using in addition Corollary 3.11, we can apply [1, Theorem 2.32.], which yields that there exists the evolving space equivalence between $W(V,V^*)$ and $\mathcal{W}(V_0,V_0^*)$. □

Corollary 3.13. The solution space $W(V,V^*)$ is a Hilbert space with the inner product defined via

$$(u,v)_{W(V,V^*)} = \int_0^T \int_\Omega (u(t),v(t))_{H^1(\Gamma(t))} + \int_0^T \int_\Omega (\partial^\bullet u(t), \partial^\bullet v(t))_{H^{-1}(\Gamma(t))}.$$

More properties of the space $W(V,V^*)$ can be found in [1].
We have shown how to differentiate the inner product of functions from $\mathcal{C}_H^1$ on $H(t) = L^2(\Omega, L^2(\Gamma(t)))$. We can generalize this result to functions from the solution space.

Theorem 3.14. (Transport theorem.) For all $u,v \in W(V,V^*)$, the map

$$t \mapsto (u(t),v(t))_{L^2(\Omega,L^2(\Gamma(t)))}$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(u(t),v(t))_{H(t)} = \langle \partial^\bullet u(t), v(t) \rangle_{V^*(t),V(t)} + \langle \partial^\bullet v(t), u(t) \rangle_{V^*(t),V(t)} + c(t;u(t),v(t)), \tag{3.16}$$

for almost all $t \in [0,T]$.

Proof. The proof is based on the density of the space $D_V[0,T]$ in the space $W(V,V^*)$ and the Transport formula for the functions from $\mathcal{C}_H^1$. For a detailed proof, we refer the reader to [1, Theorem 2.38.]. □

4 Uniform random diffusion coefficient

In this section we will consider the case when the diffusion coefficient is uniformly bounded away from zero and from above, which allows us to consider the ”mean-weak” formulation and directly apply the [BNB] theorem about the existence and uniqueness of the solution. The formulation of the [BNB] theorem can be found for example in [17].

4.1 Formulation of the problem

We want to consider the following equation

$$\begin{aligned} \partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + u \nabla_\Gamma \cdot \mathbf{w} = f \quad \text{in } L^2_{V^*} \\ u(0) = u_0. \end{aligned} \tag{4.1}$$

Remark. The initial condition is meaningful thanks to the embedding $W(V, V^*) \subset \mathcal{C}^0_V$ [1, Lemma 2.35].

Let us state assumptions for the initial data that we need in order to prove the existence and uniqueness of the solution.

Assumption 4.1. The initial value u_0 belongs to $L^2(\Omega, L^2(\Gamma_0))$. For the source term we assume $f \in L^2_{V^*}$. Moreover, $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}$ is assumed to be a random $\mathcal{F} \otimes \mathcal{B}(\mathcal{G}_T)-$measurable function, where $\mathcal{G}_T$ is the space-time surface $\mathcal{G}_T := \bigcup_t \Gamma(t) \times \{t\}$. Furthermore, we assume that the diffusion coefficient α is bounded and uniformly coercive in the sense that there are constants $\alpha_{\min}$, $\alpha_{\max}$ such that

$$0 < \alpha_{\min} \leq \alpha(\omega, x, t) \leq \alpha_{\max} < \infty \quad \forall (x,t) \in \mathcal{G}_T \tag{4.2}$$

holds for $\mathbb{P}-$a.e. $\omega \in \Omega$.

Definition 4.2. We say that u is *a "mean-weak" solution* of (4.1) if it satisfies the initial condition $u(0) = u_0$ and $u \in W(V, V^*)$ and a.e. in $[0, T]$:

$$\begin{aligned} \langle \partial^\bullet u(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))} + \int_\Omega \int_{\Gamma(t)} \alpha(t) \nabla_\Gamma u(t) \cdot \nabla_\Gamma v \\ + \int_\Omega \int_{\Gamma(t)} u(t) v \nabla_\Gamma \cdot \mathbf{w} = \langle f(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))}, \end{aligned} \tag{4.3}$$

for every $v \in L^2(\Omega, H^1(\Gamma(t)))$.

In order to simplify the notation we define the bilinear form $a(t; \cdot, \cdot) : V(t) \times V(t) \to \mathbb{R}$ by

$$a(t; u, v) := \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u(\omega, x) \cdot \nabla_\Gamma v(\omega, x). \tag{4.4}$$

Let us state some of the properties of the bilinear form a.

Lemma 4.3. The map

$$t \mapsto a(t; u(t), v(t)) \tag{4.5}$$

is measurable for all $u, v \in L^2_V$. Furthermore, there exist positive deterministic constants C_1, C_2 and C_3 that are independent of t such that the following holds for almost every $t \in [0, T]$

$$a(t; v, v) \geq C_1 \|v\|^2_{L^2(\Omega, H^1(\Gamma(t)))} - C_2 \|v\|^2_{L^2(\Omega, L^2(\Gamma(t)))} \quad \forall v \in V(t) \tag{4.6}$$

$$|a(t; u, v)| \leq C_3 \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))} \quad \forall u, v \in V(t). \tag{4.7}$$

Proof. The measurability of (4.5) follows directly from the Fubini-Tonelli theorem. Moreover, the assumption (4.2) directly implies that

$$a(t; v, v) \geq \alpha_{min} \|\nabla_\Gamma v\|^2_{L^2(\Omega, L^2(\Gamma))},$$

thus we can take $C_1 = C_2 = \alpha_{\min}$. Using again (4.2) and the Cauchy-Schwarz inequality we get that $C_3 = \alpha_{\max}$

$$\begin{aligned} \left| \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u \cdot \nabla_\Gamma v \right| &\leq \alpha_{\max} \left| \langle \nabla_\Gamma u, \nabla_\Gamma v \rangle_{L^2(\Omega, L^2(\Gamma(t)))} \right| \\ &\leq \alpha_{\max} \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))}. \quad \square \end{aligned}$$

4.2 Existence and uniqueness

After developing all the necessary results, we can now formulate the theorem about the existence and uniqueness of a "mean-weak" solution of the equation (4.3).

Theorem 4.4. Under the Assumption 4.1 for given $f \in L^2_{V^*}$ and $u_0 \in H_0$, there exists a unique "mean-weak" solution $u \in W(V, V^*)$ satisfying (4.3) such that

$$\|u\|_{W(V,V^*)} \leq C(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}})$$

where $V = (V(t))_{t \in [0,T]}$ is the family of spaces $V(t) = L^2(\Omega, H^1(\Gamma(t)))$, V^* is the family of corresponding dual spaces and $H_0 = L^2(\Omega, L^2(\Gamma_0))$.

Proof. Lemma 3.3, Theorem 3.12 and Lemma 4.3 imply that we can apply [1, Theorem 3.6] about the existence and uniqueness of the solution of the parabolic PDE on an abstract evolving space. The main idea of the proof of [1, Theorem 3.6] is to use the Banach-Nečas-Babuška theorem. This proves the theorem. □

4.3 Regularity

Let us now assume more regularity of the input data. More precisely, let $f \in L^2_H$ and $u_0 \in V_0$. We will prove that in this case we also have more regularity for the solution, i.e. its material derivative. Before we state this result, we will prove some technical results.

First we define the solution space for the case when the solution has more regularity.

Definition 4.5. We define

$$W(V, H) := \{u \in L^2_V \,|\, \partial^\bullet u \in L^2_H\}.$$

Lemma 4.6. There is an evolving space equivalence between $W(V, H)$ and $\mathcal{W}(V_0, H_0) \equiv \{v \in L^2(0, T; L^2(\Omega, H^1(\Gamma_0))) \,|\, v' \in L^2(0, T; L^2(\Omega, L^2(\Gamma_0)))\}$.

Proof. Since The Jacobian J_t^0 is uniformly bounded, both in time and space (see 3.3), applying [1, Theorem 2.33] to the restriction $\hat{S}(t): H_0 \to H_0$ of the map defined in the proof of Theorem 3.12, completes the proof. $\square$

Corollary 4.7. $W(V, H)$ is a Hilbert space.

If $u_0 \in V_0$ and $f \in L^2_H$, the Definition 4.2 of the "mean-weak" solution transforms to:
find $u \in W(V, H)$ such that $u(0) = u_0$ and a.e. in $[0, T]$ holds

$$\int_\Omega (\partial^\bullet u(t), v)_{H^1(\Gamma(t))} + \int_\Omega \int_{\Gamma(t)} \alpha(t) \nabla_\Gamma u(t) \cdot \nabla_\Gamma v + \int_\Omega \int_{\Gamma(t)} u(t) v \nabla_\Gamma \cdot \mathbf{w}(t) = \int_\Omega \int_{\Gamma(t)} f v, \tag{4.8}$$

for every $v \in L^2(\Omega, H^1(\Gamma(t)))$.

Lemma 4.8. There exists a basis $\{\chi_j^0\}_{j \in \mathbb{N}}$ of $V_0 \equiv L^2(\Omega, H^1(\Gamma_0))$ and for every $u_0 \in V_0$ there exists a sequence $\{u_{0k}\}_{k \in \mathbb{N}}$ such that $u_{0k} \in \operatorname{span}\{\chi_1^0, \dots, \chi_k^0\}$ for every k, such that

$$u_{0k} \to u_0 \text{ in } V_0$$
$$\|u_{0k}\|_{H_0} \leq \|u_0\|_{H_0}$$
$$\|u_{0k}\|_{V_0} \leq \|u_0\|_{V_0}.$$

Proof. Since $H^1(\Gamma_0)$ is compactly embedded in $L^2(\Gamma_0)$, there exists an orthonormal basis $\{w_m\}$ in $L^2(\Gamma_0)$ such that

$$(u, w_m)_{L^2(\Gamma_0)} = \lambda_m^{-1} (u, w_m)_{H^1(\Gamma_0)} \quad \forall u \in H^1(\Gamma_0) \tag{4.9}$$

and in addition, $\{\lambda_m^{-1/2} w_m\}_{m \in \mathbb{N}}$ is an orthonormal basis of $H^1(\Gamma_0)$ (see for instance [32, Theorem 6.2-1]). On the other hand, since $L^2(\Omega)$ is separable, it has an orthonormal basis $\{e_n\}_{n \in \mathbb{N}}$. It follows according to Theorem 2.1 that $\{w_m e_n\}_{m,n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega, L^2(\Gamma_0))$ and $\{\lambda^{-1/2} w_m e_n\}_{m,n \in \mathbb{N}}$ is an orthonormal basis of $L^2(\Omega, H^1(\Gamma_0))$. Let $u_0 \in L^2(\Omega, H^1(\Gamma_0))$ be arbitrary. Then, (4.9) implies

$$(u_0, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} = \lambda_m^{-1} (u_0, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))}. \tag{4.10}$$

Thus we have

$$u_0 = \sum_{m,n} (u_0, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} e_n w_m = \sum_{m,n} (u_0, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))} \lambda_m^{-1} e_n w_m.$$

Now we can define

$$u_{0k} := \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} e_n w_m = \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))} \lambda_m^{-1} e_n w_m,$$

where the last equality follows from (4.10). We choose M_k and N_k such that they both converge to ∞, as $k \to \infty$. Defined like this, u_{0k} satisfies the conditions from the Lemma. $\square$

If we write $\chi_j^t := \phi_t(\chi_j^0)$, where $\{\chi_j^0\}_{j\in\mathbb{N}}$ is a basis of V_0, then by [1, Lemma 5.1] it follows that $\{\chi_j^t\}_{j\in\mathbb{N}}$ is a countable basis of $V(t)$. Now we define the space

$$\tilde{C}_V^1 := \{u \,|\, u(t) = \sum_{j=1}^{m} \alpha_j(t)\chi_j^t, m \in \mathbb{N}, \alpha_j \in AC([0,T]) \text{ and } \alpha_j' \in L^2(0,T)\},$$

where $AC([0,T])$ is the space of absolutely continuous functions from $[0,T]$.

For improved regularity of the solution, we will also need the following assumption on the material derivative of the random coefficient α. More precisely, we assume that there exists a deterministic constant C that does not depend on time such that

$$|\dot{\alpha}(\omega, x, t)| \leq C, \quad \mathbb{P} - \text{almost everywhere} \tag{4.11}$$

where $\dot{\alpha}$ is a strong material derivative. For this assumption to be fulfilled, it suffices to assume that $\alpha(\omega,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{G}_T)$ holds $\mathbb{P}-$almost everywhere, which implies the boundedness of $|\dot{\alpha}(\omega)|$ on $\mathcal{G}_T$ and in addition we assume that this bound is uniform in ω.

Lemma 4.9. a) The map

$$t \mapsto a(t; y(t), y(t))$$

is an absolutely continuous function on $[0,T]$ for all $y \in \tilde{C}_V^1$.

b) $a(t; v, v) \geq 0$ for all $v \in V(t)$.

c)

$$\frac{d}{dt}a(t; y(t), y(t)) = 2a(t; y(t), \partial^\bullet y(t)) + r(t; y(t)) \quad \forall y \in \tilde{C}_V^1,$$

where the derivative is taken in the classical sense and $r(t;\cdot) : V(t) \to \mathbb{R}$ satisfies

$$|r(t; v)| \leq C_3 \|v\|^2_{V(t)} \quad \forall v \in V(t).$$

Proof. Part b) follows immediately from the assumption (4.2). In order to prove parts a) and c), let us first take $\eta \in C_V^\infty$. Since the probability space Ω does not depend on time, it does not have any influence in taking time derivative, thus the analogue Transport formulae from the deterministic case (that can be found in [15, Lemma 2.1]) still hold in our setting. By applying this formula to the bilinear form $a(t;\cdot,\cdot)$ we get

$$\frac{d}{dt}a(t; \eta(t), \eta(t)) = 2a(t; \eta(t), \partial^\bullet \eta(t)) + r(t; \eta(t)), \tag{4.12}$$

where the function $r(t; \eta(t))$ is defined by

$$r(t; \eta(t)) := \int_\Omega \int_{\Gamma(t)} \dot{\alpha}|\nabla_\Gamma \eta|^2 + \alpha|\nabla_\Gamma \eta|^2 \nabla_\Gamma \cdot \mathbf{w} - 2\nabla_\Gamma \eta (D_\Gamma(\mathbf{w}))\nabla_\Gamma \eta$$

with the deformation tensor $(D_\Gamma \mathbf{w}(t))_{ij} := \underline{D}_j \mathbf{w}^i(t)$.

By the similar arguments as in [2, Ch. 5.1], which are based on the density result of space C_V^∞ in $\tilde{C}_V^1$, we can conclude that the previous formula is also true for every function $\eta \in \tilde{C}_V^1$. Furthermore, the boundedness of $r(t;\cdot)$ follows directly from the assumptions about the velocity (2.4) and assumption (4.11). This proves c). It remains to prove part a). This claim follows directly from the previous calculation, which implies that both the function $a(t;\eta(t),\eta(t))$ and its time derivative (i.e. the right hand side of (4.12)) belong to $L^1(0,T)$, from which it follows that $t \mapsto a(t;\eta(t),\eta(t))$ has an absolutely continuous representative. □

Theorem 4.10. Let Assumption 4.1 hold and additionally assume (4.11). Then for given $f \in L^2_H$ and $u_0 \in V_0$, there exists a unique "mean-weak" solution $u \in W(V,H)$ satisfying (4.8) and the following a priori estimate holds

$$\|u\|_{W(V,H)} \leq C(\|u_0\|_{V_0} + \|f\|_{L^2_H}).$$

Proof. From Lemma 4.3, Lemma 4.8 and Lemma 4.9, it follows that we can apply the general theorem [1, Theorem 3.13] about the regularity of the solution of parabolic PDEs on evolving space, which implies the theorem. □

5 Log-normal random diffusion coefficient

In this section we will consider the case when the diffusion coefficient has a log-normal distribution introduced by Definition 2.3 and satisfies Assumption 2.4. We will use results and definitions from subsections 2.3 and 2.4, especially our sample space Θ will be defined by (2.9) with measure γ defined by (2.7). Since in this case the random coefficient is not uniformly bounded in the parameter $y \in \Theta$, integration of the path-wise formulation over Θ with respect to γ does not lead to a well-posed "mean-weak" formulation. Thus we can not apply the [BNB] theorem as we did in the uniform case in the section 4. Instead, we will consider for each realization y a path-wise formulation for which we know from the deterministic case that it has a unique solution $u(y)$. Since we are interested in the statistics of the solution, especially expectation and variance, we want to prove p-integrability of the solution with respect to γ. This consists of two steps, first, proving the measurability of the map $y \mapsto u(y)$ and second, proving the bound for the norm.

5.1 Path-wise formulation of the problem

For the path-wise formulation we will consider the Gelfand triple $H^1(\Gamma(t)) \subset L^2(\Gamma(t)) \subset H^{-1}(\Gamma(t))$. Let us define

$$\mathcal{V}(t) := H^1(\Gamma(t)) \quad \text{and} \quad \mathcal{H}(t) := L^2(\Gamma(t)).$$

For simplicity we will assume that the source term $f \in L^2_{\mathcal{V}^*}$ and the initial data $u_0 \in L^2_{\mathcal{H}}$ are deterministic. Furthermore, let us remark that we can transform the problem (1.1) into a PDE with zero initial condition, the reader can find a more detailed argument in [1]. Thus, from now we will assume that $u_0 = 0$.

The solution space for the path-wise formulation will be

$$W_0(\mathcal{V}, \mathcal{V}^*) = \{u \in L^2_{\mathcal{V}} \,|\, \partial^\bullet u \in L^2_{\mathcal{V}^*}, u(0) = 0\}$$

which is a Hilbert space, as a closed linear subspace of $W(\mathcal{V}, \mathcal{V}^*)$.
Let us now state the path-wise weak formulation of (1.1):

For every $y \in \Theta$ find $u(y) \in W_0(\mathcal{V}, \mathcal{V}^*)$ such that almost everywhere in $[0, T]$ it holds

$$\langle \partial^\bullet u(y), v \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \int_{\Gamma(t)} \alpha(y) \nabla_\Gamma u(y) \cdot \nabla_\Gamma v + \int_{\Gamma(t)} u(y) v \nabla_\Gamma \cdot \mathbf{w} \quad = \langle f, v \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)}\,, \tag{5.1}$$

for every $v \in \mathcal{V}(t)$.

In order to get a coercive bilinear form, we write (1.1) as

$$\partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + (\lambda + \nabla_\Gamma \cdot \mathbf{w}) u - \lambda u = f \tag{5.2}$$

for any $\lambda \in \mathbb{R}$. Introducing

$$\hat{u}(y) := e^{-\lambda t} u(y) \quad \text{and} \quad \hat{f}(y) := e^{-\lambda t} f(y)$$

and using the product rule, we can rewrite (5.2) as

$$\partial^\bullet \hat{u} - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma \hat{u}) + (\lambda + \nabla_\Gamma \cdot \mathbf{w}) \hat{u} = \hat{f}. \tag{5.3}$$

Furthermore, the path-wise weak form of (5.3) is given by:

for every $y \in \Theta$ find $\hat{u}(y) \in W_0(\mathcal{V}, \mathcal{V}^*)$ such that almost everywhere in $[0, T]$ it holds

$$\langle \partial^\bullet \hat{u}(y), \hat{v} \rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \hat{a}(y, t; \hat{u}, \hat{v}) = \left\langle \hat{f}, \hat{v} \right\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} \qquad \forall \hat{v} \in \mathcal{V}(t), \tag{5.4}$$

where the parametric bilinear form $\hat{a}(y, t; \cdot, \cdot) : \mathcal{V}(t) \times \mathcal{V}(t) \to \mathbb{R}$ is defined by

$$\hat{a}(y, t; \xi, \eta) := \int_{\Gamma(t)} \alpha(y) \nabla_\Gamma \xi \cdot \nabla_\Gamma \eta + (\lambda + \nabla_\Gamma \cdot \mathbf{w}) \xi \eta.$$

The advantage of writing the equation in this form is that now the induced bilinear form $\hat{a}(y, t; \cdot, \cdot)$ is coercive and bounded, for sufficiently large λ. Namely for $\lambda > C_{\mathbf{w}}$ and $C_\lambda := \lambda - C_{\mathbf{w}}$ we have

$$\hat{a}(y, t; \eta, \eta) \geq m(y) \|\eta\|^2_{\mathcal{V}(t)} \tag{5.5}$$

$$|\hat{a}(y, t; \eta, \xi)| \leq M(y) \|\eta\|_{\mathcal{V}(t)} \|\xi\|_{\mathcal{V}(t)} \tag{5.6}$$

where $m(y) := \min(\alpha_{\min}(y), C_\lambda)$ and $M(y) := \max(\alpha_{\max}(y), \lambda + C_{\mathbf{w}})$.

Furthermore, we will also use the following estimate

$$\hat{a}(y, t; \eta, \eta) \geq \min\left(\alpha_{\min}(y), \frac{C_\lambda}{2}\right) \|\eta\|^2_{\mathcal{V}(t)} + \frac{C_\lambda}{2} \|\eta\|^2_{\mathcal{H}(t)}. \tag{5.7}$$

We define the bilinear form $d(y) : W_0(\mathcal{V},\mathcal{V}^*) \times L^2_{\mathcal{V}} \to \mathbb{R}$ by

$$d(y;\xi,\eta) := \int_0^T \langle \partial^\bullet \xi, \eta\rangle_{\mathcal{V}^*,\mathcal{V}} + \hat{a}(y,t;\xi,\eta).$$

Then the inf-sup constant is given by

$$\beta(y) := \inf_{\eta\in W_0(\mathcal{V},\mathcal{V}^*)} \sup_{\xi\in L^2_{\mathcal{V}}} \frac{|d(y;\eta,\xi)|}{\|\eta\|_{W_0(\mathcal{V},\mathcal{V}^*)}\|\xi\|_{L^2_{\mathcal{V}}}}.$$

Lemma 5.1. Let Assumption 2.4 hold and additionally assume $\lambda \geq 3C_{\mathbf{w}}$ and (2.4). Then for every $y \in \Theta$, there exists a unique solution $\hat{u}(y) \in W_0(\mathcal{V},\mathcal{V}^*)$ to the problem (5.4). Moreover, the following estimate holds

$$\|\hat{u}(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)} \leq \frac{1}{\beta(y)}\|\hat{f}\|_{L^2_{\mathcal{V}^*}} \tag{5.8}$$

where the inf-sup constant is bounded from below by

$$\beta(y) \geq \frac{\min\left(\frac{m(y)}{M(y)^2}, \alpha_{\min}(y), \frac{C_\lambda}{2}\right)}{\sqrt{2\max(m(y)^{-2},1)}}. \tag{5.9}$$

Proof. Under Assumption 2.4, the existence and uniqueness of the solution, as well as the estimate (5.8) follow from the deterministic result for $\lambda \geq 3C_{\mathbf{w}}$, which can be found in [2] and [13]. In order to prove the bound (5.9) we will use the idea from [35]. The main difference in the proof is that our domain is curved and changing in time, therefore we can not use the standard partial integration formula, but instead we will use partial integration that follows from the Transport theorem and has an additional term that reflects the spatial change in time.

Let $y \in \Theta$ be arbitrary. We start with defining the linear operator $A(y,t) : \mathcal{V}(t) \to \mathcal{V}^*(t)$ induced by

$$\langle A(y,t)\eta, \xi\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} := \hat{a}(y,t;\eta,\xi).$$

Given an arbitrary $0 \neq w(y) \in W_0(\mathcal{V},\mathcal{V}^*)$, we define

$$z_w(y,t) := A^{-1}(y,t)\partial^\bullet w(y,t) \in \mathcal{V}(t)$$

and select the test function

$$v_w(y,t) := z_w(y,t) + w(y,t) \in \mathcal{V}(t).$$

Using (5.5) and (5.6) we obtain

$$\langle \partial^\bullet w, z_w\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} \geq \frac{m(y)}{M(y)^2}\|\partial^\bullet w\|^2_{\mathcal{V}^*(t)}. \tag{5.10}$$

The definition of z_w directly implies

$$\hat{a}(y,t;w,z_w) = \left\langle Aw, A^{-1}\partial^\bullet w\right\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} = \langle w, \partial^\bullet w\rangle_{\mathcal{V}(t),\mathcal{V}^*(t)}. \tag{5.11}$$

Analogous to Theorem 3.14, the Transport formula for the scalar product in $\mathcal{H}(t)$ holds with

$$c(t;u,v) := \int_{\Gamma(t)} uv\nabla_\Gamma \cdot \mathbf{w}.$$

As a consequence, we obtain the following integration by parts formula (see [1, Corollary 2.41])

$$(u(T),v(T)))_{\mathcal{H}(t)} - (u(0),v(0))_{\mathcal{H}(t)} = \int_0^T \langle \partial^\bullet u, v\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \langle \partial^\bullet v, u\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + c(t;u,v). \quad (5.12)$$

Using (5.10) and (5.11) we arrive at

$$\begin{aligned} d(y;w,v_w) &\geq \int_0^T \frac{m(y)}{M(y)^2}\|\partial^\bullet w\|^2_{\mathcal{V}^*(t)} + \langle \partial^\bullet w, w\rangle_{\mathcal{V}^*(t),\mathcal{V}(t)} + \langle w, \partial^\bullet w\rangle_{\mathcal{V}(t),\mathcal{V}^*(t)} + \hat{a}(y,t;w,w) \\ &\geq \int_0^T \frac{m(y)}{M(y)^2}\|\partial^\bullet w\|^2_{\mathcal{V}^*(t)} - C_{\mathbf{w}}\|w\|^2_{\mathcal{H}(t)} + \frac{C_\lambda}{2}\|w\|^2_{\mathcal{H}(t)} + \min\left(\alpha_{\min}(y), \frac{C_\lambda}{2}\right)\|w\|^2_{\mathcal{V}(t)} \end{aligned}$$

where for the last inequality we used (5.7), (5.12) and (2.4). Taking $\lambda \geq 3C_{\mathbf{w}}$ gives $C_\lambda \geq 2C_{\mathbf{w}}$ and we get

$$d(y;w,v_w) \geq \min\left(\frac{m(y)}{M(y)^2}, \alpha_{\min}(y), \frac{C_\lambda}{2}\right)\|w\|^2_{W_0(\mathcal{V},\mathcal{V}^*)}. \quad (5.13)$$

It is left to estimate the norm $\|v_w\|_{L^2_\mathcal{V}}$, which follows directly from (5.5)

$$\begin{aligned} \|v_w\|^2_{L^2_\mathcal{V}} &\leq 2\left(\|A^{-1}\partial^\bullet w\|^2_{L^2_\mathcal{V}} + \|w\|^2_{L^2_\mathcal{V}}\right) \\ &\leq 2\max(m(y)^{-2},1)\|w\|^2_{W_0(\mathcal{V},\mathcal{V}^*)}. \end{aligned}$$

Since w is arbitrary, the last estimate together with (5.13) implies the bound (5.9). □

Using Lemma 5.1 we can prove the bound for the path-wise solution.

Theorem 5.2. Let Assumptions 2.4 hold and additionally assume (2.4). Then problem (5.1) has a unique solution $u(y) \in W_0(\mathcal{V},\mathcal{V}^*)$ for every $y \in \Theta$ and it satisfies

$$\|u(y)\|_{W(\mathcal{V},\mathcal{V}^*)} \leq \frac{\hat{C}}{\beta(y)}\|f\|_{L^2_{\mathcal{V}^*}}$$

where $\hat{C}$ is independent of y and the inf-sup constant $\beta(y)$ is bounded from below by (5.9).

Proof. Similarly as in the previous Lemma, the existence and uniqueness of the path-wise solution follow from the deterministic results (see [2, 13]). In order to get the estimate of the solution norm, we compare the norms $\|u(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)}$ and $\|\hat{u}(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)}$. Since

$$\|\partial^\bullet u(y)\|^2_{L^2_{\mathcal{V}^*}} \leq 2e^{2\lambda T}\left(C\lambda\|\hat{u}(y)\|^2_{L^2_\mathcal{V}} + \|\partial^\bullet \hat{u}(y)\|^2_{L^2_{\mathcal{V}^*}}\right)$$

where C is the embedding constant of $L^2_\mathcal{V}$ into $L^2_{\mathcal{V}^*}$, using Lemma 5.1 we obtain

$$\begin{aligned} \|u(y)\|^2_{W_0(\mathcal{V},\mathcal{V}^*)} &\leq e^{2\lambda T}\left(\|\hat{u}(y)\|^2_{L^2_\mathcal{V}} + 2C\lambda\|\hat{u}(y)\|^2_{L^2_\mathcal{V}} + 2\|\partial^\bullet \hat{u}(y)\|^2_{L^2_{\mathcal{V}^*}}\right) \\ &\leq e^{2\lambda T}\max(2,1+2C\lambda)\frac{1}{\beta(y)^2}\|\hat{f}\|^2_{L^2_{\mathcal{V}^*}} \leq \hat{C}^2\frac{1}{\beta(y)^2}\|f\|^2_{L^2_{\mathcal{V}^*}} \end{aligned}$$

where $\hat{C}^2 = e^{\lambda T}\max(2,1+2C\lambda)$ is independent of y, which completes the proof. □

Remark 5.3. Without loss of generality we can assume

$$\alpha_{\min}(y) \leq C_{\mathbf{w}} \leq \frac{\alpha_{\max}(y)}{4}$$

for almost every y. Furthermore, without loss of generality we can assume that $\alpha_{\min}(y) \leq 1$ and $\alpha_{\max}(y) \geq 1$ for almost every y. Previous assumptions are without loss of generality and just makes the calculations less technical, since it simplifies the bound of the inf-sup constant.

Under Assumption 5.3, by taking $\lambda = 3C_{\mathbf{w}}$, the bound (5.9) becomes

$$\beta(y) \geq \frac{1}{\sqrt{2}} \frac{\alpha_{\min}(y)^2}{\alpha_{\max}(y)^2} \qquad \text{for a.e. } y.$$

Previous together with Lemma 2.8 imply

$$\|u(y)\|_{W_0(\mathcal{V},\mathcal{V}^*)} \leq \frac{\sqrt{2}}{\hat{C}} \frac{\alpha_{\min}(y)^2}{\alpha_{\max}(y)^2} \leq \frac{\sqrt{2}}{\hat{C}} \left(4 \sum_{k \geq 1} b_k |y_k| \right) \tag{5.14}$$

for almost every y.

5.2 Integrability of the solution

In this section we will prove the p-integrability of the solution u with respect to γ. The first step is to show the measurability of the map $y \mapsto u(y)$, $\Theta \to W_0(\mathcal{V}, \mathcal{V}^*)$. The main idea of the proof is adopted from [19, Lemma 3.4]. It consists of proving that the solution u is almost surely the limit of measurable functions u_n that are the "mean-weak" solutions of (1.1) in the uniform case.

Remark 5.4. Let us note that since the sample space Θ is independent of time, it holds

$$L^2(\Theta, L^2_{\mathcal{V}}) \cong L^2(\Theta) \otimes L^2_{\mathcal{V}} \cong L^2_{L^2(\Theta,\mathcal{V})}.$$

From this we deduce

$$W(V, V^*) \cong L^2(\Theta) \otimes W(\mathcal{V}, \mathcal{V}^*) \cong L^2(\Theta, W(\mathcal{V}, \mathcal{V}^*)).$$

We will exploit this isomorphism in the proof of the p-integrability of the solution u with respect to γ, where we consider the problem in a path-wise sense.

Theorem 5.5. The solution $u : \Theta \to W(\mathcal{V}, \mathcal{V}^*)$, $y \mapsto u(y)$ of (5.1) is $\mathcal{B}(\mathbb{R}^{\mathbb{N}})-$measurable.

Proof. Since we have proved the well-posedness of the "mean-weak" formulation in the uniform case, the proof of the measurability can be adopted from [19, Lemma 3.4]. Here we just sketch its main idea. We start with defining a subspace Θ_n of Θ, for every $n \in \mathbb{N}$, where the diffusion coefficient is uniformly bounded

$$\Theta_n := \{y \in \Theta \,|\, \alpha_{\max}(y) < n, \alpha_{\min}(y) > \frac{1}{n}\} \subset \Theta.$$

Note that Θ_n is increasing and $\Theta = \cup_n \Theta_n$. Then we consider the "mean-weak" formulation on the parameter space Θ_n. In the uniform case, from the Theorem 4.4 it follows that there exists a unique solution $u_n \in L^2(\Theta_n, \gamma; W_0(\mathcal{V}, \mathcal{V}^*))$. In particular, u_n is a measurable function on Θ_n. The last step is proving that u is a.s. limit of u_n, thus it is measurable. This follows because u_n also solves the path-wise equation (5.1) for a.e. $y \in \Theta_n$. □

Now we can state the result about the $p-$integrability of the solution.

Theorem 5.6. Let $0 < p < \infty$, $\chi > 0$ and $f \in L^2_{\mathcal{V}^*}$. If Assumption 2.4 holds and additionally we assume (2.4), then the solution u of (5.1) belongs to $L^p(\Theta, \gamma; W_0(\mathcal{V}, \mathcal{V}^*))$ and satisfies

$$\|u\|_{L^p(\Theta,\gamma;W_0(\mathcal{V},\mathcal{V}^*))} \leq \bar{c}_{p,\chi} \|f\|_{L^2_{\mathcal{V}^*}}$$

with

$$\bar{c}_{p,\chi} = \frac{\sqrt{2}}{\hat{C}} \exp\left(\frac{4p\exp(2\chi\|b\|_{l^\infty})}{\chi} + \frac{\chi}{p}\right) \|b\|_{l^1}.$$

Proof. With previous results in mind, the proof is similar to the proof stated in [31, Prop. 3.3.2]. However, since the bound for the inf-sup constant β is a bit different in our case, we give the main ideas of the proof. From Theorem 5.2 and Theorem 5.5 we obtain

$$\begin{aligned}\int_\Theta \|u(y)\|^p_{W_0(\mathcal{V},\mathcal{V}^*)} d\gamma &\leq \int_\Theta \frac{1}{\beta(y)^p} \|f\|^p_{L^2_{\mathcal{V}^*}} d\gamma \\ &= \int_\Theta \zeta_\chi(y)^{-1} \frac{1}{\beta(y)^p} \|f\|^p_{L^2_{\mathcal{V}^*}} d\gamma_\chi \leq \operatorname*{ess\,sup}_y \left(\frac{1}{\zeta_\chi(y)\beta(y)^p}\right) \|f\|^p_{L^2_{\mathcal{V}^*}},\end{aligned}$$

where ξ_χ and γ_χ are defined in Section 2.4. In order to bound $\frac{1}{\zeta_\chi(y)\beta(y)^p}$ we use Lemma 2.10 and bound (5.14), which completes the proof. □

6 Outlook

Although we have stated and solved the problem of finding the unique solution of advection-diffusion PDEs with random coefficients on a moving hypersurface, only the continuous case has been discussed. The next step is to consider the numerical approximation of the solution of the equation. More strictly, since the solution is a random variable, we are interested in a numerical approximation of the expected value of the solution. One approach for discretization in space would be to use the evolving surface finite element method from [13], for which we approximate the hypersurface by an evolving interpolated polyhedral surface. In order to deal with uncertainty, one could use the Monte Carlo method which approximates the expected value. The goal would be to find the error estimate for this approximation. These results are the subject of ongoing research and a paper is in preparation.

Acknowledgments

I would like to thank Prof. Ralf Kornhuber for helpful discussions and for providing a stimulating research environment in the associated project AP01 of CRC 1114. Furthermore, I would like to express my gratitude to Prof. Charles M. Elliott for many helpful suggestions and insightful conversations during the preparation of the paper. I also acknowledge the support of the Berlin Mathematical School (BMS) and the Hilda Geiringer scholarship.

I would like to thank the two reviewers for their careful reading and their insightful comments which improved this paper.

References

[1] ALPHONSE, A., ELLIOTT, C. M., & STINNER, B. An abstract framework for parabolic PDEs on evolving spaces. *Port. Math.* **72** (2015), 1–46.

[2] ALPHONSE, A., ELLIOTT, C. M., & STINNER, B. On some linear parabolic PDEs on moving hypersurfaces. *Interfaces Free Bound* **17** (2015), 157-187.

[3] BABUŠKA, I., NOBILE, F., & TEMPONE, R. A stochastic collocation method for elliptic partial differential equations with random input data. *SIAM J. Numer. Anal.* **45**, no. 3 (2007), 1005–1034.

[4] BERTALMIO M., SAPIRO G., CHENG L.T., & OSHER S. A framework for solving surface partial differential equations for computer graphics applications. *CAM Report, UCLA* (2000).

[5] BREZIS, H. *Functional analysis, Sobolev spaces and partial differential equations. Springer Science & Business Media*, New York (2010).

[6] CHARRIER, J. Strong and weak error estimates for elliptic partial differential equations with random coefficients. *SIAM J. Numer. Anal.* **50**, no. 1 (2012), 216–246.

[7] CHARRIER, J. Numerical analysis of the advection diffusion of a solute in porous media with uncertainty. *SIAM/ASA J. on Uncertainty Quantification* **3** (2015), 650–685.

[8] CHARRIER J., SCHEICHL R., & TECKENTRUP A.L. Finite Element Error Analysis of Elliptic PDEs with Random Coefficients and its Application to Multilevel Monte Carlo Methods. *SIAM J. on Numer. Anal.* **51**, no.1 (2013), 322–352.

[9] CLASS H., NIESSNER, J., & HELMIG R. Transfer of model concepts for flow and transport processes from porous media to biological tissues. *ASME International Mechanical Engineering Congress and Exhibition. Proceedings.* (2008), 29–32.

[10] DECKELNICK, K., DZIUK, G., & ELLIOT, C.M. Computation of geometric partial differential equations and mean curvature flow. *Acta Numer.* **14** (2005), 139–232.

[11] DEMLOW, A. Higher-order finite element methods and pointwise error estimates for elliptic problems on surfaces. *SIAM J. Numer. Anal.* **47**, no. 2

[12] DZIUK, G. Finite elements for the Beltrami operator on arbitrary surfaces. In: *Partial differential equations and calculus of variations*, S. Hildebrandt and R. Leis, eds., Lecture Notes in Math. **1357**, Springer, Berlin (1988), 142–155.

[13] DZIUK, G., & ELLIOTT, C. M. Finite elements on evolving surfaces. *IMA J. Numer. Anal.*, **27**, no. 2 (2007), 262–292.

[14] DZIUK, G., & ELLIOTT, C. M. Surface finite elements for parabolic equations. *J. Comp. Math.* **25** (2007), 385–407.

[15] DZIUK, G., & ELLIOTT, C. M. L^2 estimates for the evolving surface finite element method. *Math. Comp.* **82**, no. 1 (2013), 1–24.

[16] DZIUK, G., & ELLIOTT, C. M. Finite element methods for surface PDEs. *Acta Numer.* **22** (2013), 289-396.

[17] ERN A., & GUERMOND J. L. *Theory and practice of finite elements.* Applied Mathematical Sciences. 159. Springer-Verlag, New York (2004).

[18] FORSTER, R. On the Stochastic Richards Equation. PhD thesis, Freie Univ. Berlin (2010).

[19] GITTELSON C. J. Stochastic Galerkin discretization of the log-normal isotropic diffusion problem. *Math. Models Methods Appl. Sci.* **20** (2010), 237–263.

[20] HACKBUSCH, W. Tensor Spaces and Numerical Tensor Calculus. Springer, Heidelberg (2011).

[21] HAILIN JIN, YEZZI A. J., & SOATTO S. Region based segmentation on evolving surfaces with application to 3D reconstruction of shape and piecewise constant radiance. *UCLA preprint* (2004).

[22] HOANG, V. H., & SCHWAB, CH. Sparse Tensor Galerkin Discretizations for Parametric and Random Parabolic PDEs - Analytic Regularity and GPC Approximation. *SIAM J. Math. Anal.* **45**, no. 5 (2013), 3050–3083.

[23] JAMES, A. J., & LOWENGRUB, J. A surfactant-conserving volume-of-fluid method for interfacial flows with insoluble surfactant. *J. Comput. Physics* **201** (2004), 685–722.

[24] JIAN-JU X., & HONG-KAI Z. An Eulerian formulation for solving partial differential equations along a moving interface. *J. Sci. Comput.* **19** (2003), 573–594.

[25] LARSSON S., C. MOLLET C., & MOLTENI M. Quasi-optimality of Petrov-Galerkin discretizations of parabolic problems with random coefficients. arXiv:1604.06611 (2016).

[26] LE MAÎTRE, O. P., & KNIO, O. M. *Spectral Methods for Uncertainty Quantification.* Springer, New York (2010).

[27] LEUNG C. H.,& BERZINS M. A computational model for organism growth based on surface mesh generation. *J. Comput. Physics* **188** (2003), 75–99.

[28] LORD G., POWELL C. E., & SHARDLOW T. *An Introduction to Computational Stochastic PDEs* . Cambridge University Press, Cambridge (2014).

[29] MARINUCCI D., & PECCATI G. Mean-square continuity on homogeneous spaces of compact groups. *Electron. Commun. Probab.* **18**, no. 37 (2013), 10.

[30] MORROW N. R., & MASON G. Recovery of oil by spontaneous imbibition. *Current Opinion in Colloid and Interface Science* **6**, no. 4 (2001), 321–337.

[31] PRESTIPINO G. Numerical methods for parabolic PDEs with time-dependent random-field-coefficients. Master thesis, Universitá degli Studi di Milano (2015).

[32] RAVIART, P. A., & THOMAS, J. M. *Introduction à l'analyse numérique des équations aux dérivées partielles.* Masson, Paris etc (1992).

[33] REED, M., & SIMON, B. *Methods of modern mathematical physics.* **I**: Functional analysis, *Academic Press*, New York, San Francisco, London (1972).

[34] SCHWAB C., & GITTELSON C. J. Sparse tensor discretizations of high-dimensional parametric and stochastic PDEs. *Acta Numerica* **20** (2011), 291–467.

[35] SCHWAB C., & STEVENSON R. Space-time adaptive wavelet methods for parabolic evolution problems. *Math. Comp.* **78**, no. 267 (2009), 1293–1318.

[36] SCRIVEN L.E. Dynamics of a fluid interface. *Chem. Eng. Sci.* **12** (1960), 98–108.

[37] STONE, H. A. A simple derivation of the time-dependent convective-diffusion equation for surfactant transport along a deforming interface. *Phys. Fluids* **A2** (1990), 111–112.

[38] VIERLING, M. Parabolic optimal control problems on evolving surfaces subject to point- wise box constraints on the control – theory and numerical realization. *Interfaces Free Bound.* **16** no. 2 (2014), 137–173.

[39] WLOKA, J. *Partial Differential Equations.* Cambridge University Press, Cambridge (1987).