Advection-diffusion equations with random coefficients on evolving hypersurfaces

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Abstract

We present the analysis of advection-diffusion equations with random coefficients on moving hypersurfaces. We define weak and strong material derivative, that take into account also the spacial movement. Then we define the solution space for these kind of equations, which is the Bochner-type space of random functions defined on moving domain. Under suitable regularity assumptions we prove the existence and uniqueness of solutions of the concerned equation, and also we give some regularity results about the solution.

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1 Introduction

There is a growing interest in partial differential equations (PDEs) with random coefficients that are used as model equations. These equations contain random variables that model parameters which include some uncertainty. The uncertainty can come from intrinsic variability of the physical system or when the input data of the real system are not completely known [15]. This work addresses specifically parabolic PDEs with random coefficients which have been so far studied in several papers ([18], [19], [22]). Furthermore, these PDEs occur in many applications, such as hydrogeology, material science, fluid dynamics, biological fluids etc.

All these papers have considered equations on some bounded fixed domain in $\mathbb{R}^d$. Some of these models can be better formulated on the moving domain, especially in biological applications ([16],[17]). However, there is no mathematical theory for PDEs with random coefficients on moving surfaces and for this reason in this paper we consider this type of problem. The deterministic counterpart for these equations, surface PDEs, have been introduced in [10] and then later developed in

[6], [13], [14], etc. Dziuk and Elliott have introduced the evolving surface finite element method for PDEs on moving hypersurfaces ([5], [7]). Recently this topic has been generalized ([1], [2]) to a more abstract level, i.e. to parabolic PDEs on any evolving Hilbert space.

The aim of this paper is to combine surface PDEs on moving hypersurfaces with PDEs with random coefficients. More precisely, we wish to analyse the following advection-diffusion equation

$$\begin{aligned} \partial^{\bullet} u - \nabla_{\Gamma} \cdot (\alpha \nabla_{\Gamma} u) + u \nabla_{\Gamma} \cdot \mathbf{w} &= f \\ u(0) &= u_0 \end{aligned} \tag{1.1}$$

where ∇_{Γ} is a tangential surface gradient, $\partial^{\bullet}$ is a material derivative and $\mathbf{w} = \mathbf{w}_{\nu} + \mathbf{w}_{\tau}$ is a velocity field. Thus, besides the normal velocity, which is enough to define the evolution of the surface, the surface also has an advective tangential velocity. Note that we assume the surface evolution to be prescribed. In contrast to the deterministic case, a source function f and initial function u_0 are random. The diffusion coefficient α is also random and uniformly bounded from above and away from zero. Hence the solution u will also be a random function. The equation (1.1) models the transport of a scalar quantity, e.g., a surfactant, along a moving two-dimensional interface [12]. The surfactant is transported by advection via the tangential fluid velocity and by diffusion within the surface. A natural next step would be to consider the case when α is a lognormal random field. This model is widely used for the flow equation in porous media and has been considered on the fixed domain, for example in [22]. Note that in this case the random field no longer satisfies uniform coercivity w.r.t. ω. The analysis of this case is left to future work.

Our goal is to prove the existence and uniqueness of a solution of the equation (1.1). Furthermore, we will prove that for more regular input data, our solution will also have more regularity in its material derivative. The main task is first to define properly the framework for the equation (choose an appropriate Gelfand triple, precisely define the material derivative and a solution space), and then prove the properties of this setting. This will enable us to apply the theorem of well-posedness of the PDE on an evolving space in the abstract case given in [1].

The paper is organized as follows. We start the second section by setting up the notation, description of the hypersurfaces and assumptions on the evolution of the hypersurfaces. Furthermore, since our spaces will have tensor structure, we briefly summarize without proofs the relevant material on tensor products. In the third section we proceed with setting up the function spaces and defining the material derivative. Moreover, we show that the framework from [1] is applicable. The Section 4 contains the precise formulation of the problem and assumptions about random coefficients and random source. Also, the proofs of the main results about existence, uniqueness and regularity of solutions are given. In the final section we discuss possible extensions to this paper for further research.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with sample space Ω, a $\sigma-$algebra of events $\mathcal{F}$ and a probability $\mathbb{P} : \mathcal{F} \to [0,1]$. In addition, we assume that $L^2(\Omega)$ is a separable space. We will only consider a fixed finite time interval $[0,T]$, where $T \in (0,\infty)$. Furthermore, we will denote by $\mathcal{D}([0,T];V)$ the space of infinitely differentiable functions with values in V and compact support in $[0,T]$.

2.1 Hypersurfaces

Let us first recall some basic theory about hypersurfaces and Sobolev spaces on hypersurfaces. For more details we refer to [9] or [8]. We will assume that Γ is a $\mathcal{C}^2$ compact, connected, orientable, $n-$dimensional hypersurface, embedded in $\mathbb{R}^{n+1}$ for $n = 1, 2$, or 3. For a function $f : \Gamma \to \mathbb{R}$ which is differentiable in an open neighbourhood of Γ we define the *tangential gradient* by

$$\nabla_\Gamma f(x) := \nabla f(x) - \nabla f(x) \cdot \nu(x)\nu(x) \quad x \in \Gamma,$$

where $\nu(x)$ is normal on $T_x\Gamma$. Note that $\nabla_\Gamma f(x)$ is the orthogonal projection of $\nabla f(x)$ onto $T_x\Gamma$ and it depends only on the values of f on Γ [8, Lemma 2.4]. Now we can define the *Laplace-Beltrami* operator by

$$\Delta_\Gamma f(x) = \nabla_\Gamma \cdot \nabla_\Gamma f(x) = \sum_{i=1}^{n+1} \underline{D}_i \underline{D}_i f(x) \quad x \in \Gamma,$$

where $\nabla_\Gamma = (\underline{D}_1, \ldots, \underline{D}_{n+1})$. Let us state the integration by parts formula for functions $f \in \mathcal{C}^1(\overline{\Gamma}; \mathbb{R}^{n+1})$:

$$\int_\Gamma \nabla_\Gamma \cdot f = \int_\Gamma f \cdot H\nu + \int_{\partial\Gamma} f \cdot \mu, \tag{2.1}$$

where μ is the unit conormal vector and H is the mean curvature. Note that since we have assumed that Γ is compact, it has no boundary. Furthermore, we state *Green's formula*

$$\int_\Gamma \nabla_\Gamma f \cdot \nabla_\Gamma g = \int_{\partial\Gamma} f \nabla_\Gamma g \cdot \mu - \int_\Gamma f \Delta_\Gamma g. \tag{2.2}$$

From (2.1) and (2.2), in the case when $\partial\Gamma = \emptyset$, we can derive the following

$$\int_\Gamma f \nabla_\Gamma g = -\int_\Gamma (\nabla_\Gamma f - f H\nu) g. \tag{2.3}$$

We will consider the Sobolev space on a hypersurface. We define $L^2(\Gamma)$ as usual, i.e. as a set of all measurable functions $f : \Gamma \to \mathbb{R}$ such that

$$\|f\|_{L^2(\Gamma)} := \left(\int_\Gamma |f(x)|^2 \right)^{1/2} < \infty.$$

We say that a function $f \in L^2(\Gamma)$ has a weak derivative $g_i = D_i f \in L^2(\Gamma)$, $i = 1, \ldots, n+1$ if for every function $\phi \in \mathcal{C}_c^1(\Gamma)$ and every i it holds

$$\int_\Gamma f \underline{D}_i \phi = -\int_\Gamma \phi g_i + \int_\Gamma f \phi H \nu_i.$$

Now, we can define the Sobolev space

$$H^1(\Gamma) = \{f \in L^2(\Gamma) \mid D_i f \in L^2(\Gamma), i = 1, \ldots, n+1\}$$

with the norm

$$\|f\|_{H^1(\Gamma)} = \sqrt{\|f\|^2_{L^2(\Gamma)} + \|\nabla_\Gamma f\|^2_{L^2(\Gamma)}}.$$

We will consider the family of evolving surfaces $\{\Gamma(t)\}$ for $t \in [0,T]$ which evolves according to a given velocity field **w**. For each $t \in [0,T]$ we assume that $\Gamma(t)$ satisfies the same properties as Γ and we set $\Gamma_0 := \Gamma(0)$. We also assume the existence of a flow $\Phi : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ such that for all $t \in [0,T]$ its restriction $\Phi_t^0 := \Phi(t, \cdot) : \Gamma_0 \to \Gamma(t)$ is $\mathcal{C}^2-$diffeomorphism that satisfies

$$\begin{aligned} \frac{d}{dt}\Phi_t^0(\cdot) &= \mathbf{w}(t, \Phi_t^0(\cdot)) \\ \Phi_0^0(\cdot) &= \mathrm{Id}(\cdot), \end{aligned}$$

where $\mathbf{w} : [0,T] \times \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is the $\mathcal{C}^2-$velocity field that is uniformly bounded

$$|\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)| \leq C \quad \text{for all } t \in [0,T]. \tag{2.4}$$

2.2 Tensor products

Since the function spaces which will be used later has tensor product structure, let us recall some basic results about it. Let H_1 and H_2 be Hilbert spaces. The *tensor product* $v_1 \otimes v_2$ is defined as a conjugate bilinear form:

$$(v_1 \otimes v_2)(w_1, w_2) := (v_1, w_1)_{H_1}(v_2, w_2)_{H_2}$$

on $H_1 \times H_2$. Let S be the set of finite linear combinations of such tensor products. We can define an inner product on S by

$$(v_1 \otimes v_2, w_1 \otimes w_2) := (v_1, w_1)_{H_1}(v_2, w_2)_{H_2} \tag{2.5}$$

and extend it by linearity to S. The *tensor product* $H_1 \otimes H_2$ is the completion of S under the inner product (2.5).

Theorem 2.1. The tensor space $H_1 \otimes H_2$ is a Hilbert space. If $\{e_j\}_{j\in\mathbb{N}}$ and $\{f_k\}_{k\in\mathbb{N}}$ are basis of Hilbert spaces H_1 and H_2, then $\{e_j \otimes f_k\}_{j,k\in\mathbb{N}}$ constitute a basis of $H_1 \otimes H_2$.

Proof. The proof can be found for example in [21]. □

Theorem 2.2. Let (X, μ) and (Y, ν) be measure spaces such that $L^2(X, \mu)$ and $L^2(Y, \nu)$ are separable. Then, the following holds:

a) There is a unique isometric isomorphism

$$L^2(X, \mu) \otimes L^2(Y, \nu) \cong L^2(X \times Y, \mu \times \nu)$$

so that $f \otimes g \mapsto fg$.

b) If H is a separable Hilbert space then there is a unique isometric isomorphism

$$L^2(X, \mu) \otimes H \cong L^2(X, \mu; H)$$

so that $f(x) \otimes \varphi \mapsto f(x)\varphi$.

Proof. The proof can be found for example in [21]. □

3 Function spaces

3.1 Gelfand triple

In this section, we will define the basic Gelfand triple that will be used to define the solution space for (1.1). For each $t \in [0, T]$, let us define

$$V(t) := L^2(\Omega, H^1(\Gamma(t))) \quad \text{and} \quad H(t) := L^2(\Omega, L^2(\Gamma(t))).$$

Then the dual space of $V(t)$ is the space $V^*(t) = L^2(\Omega, H^{-1}(\Gamma(t)))$ where $H^{-1}(\Gamma(t))$ is the dual space of $H^1(\Gamma(t))$.

Since all spaces $L^2(\Omega), L^2(\Gamma(t))$ and $H^1(\Gamma(t))$ are separable Hilbert spaces, using Theorem 2.2 we have:

$$L^2(\Omega, H^1(\Gamma(t))) \cong L^2(\Omega) \otimes H^1(\Gamma(t)) \tag{3.1}$$

$$L^2(\Omega, L^2(\Gamma(t))) \cong L^2(\Omega) \otimes L^2(\Gamma(t)) \tag{3.2}$$

Remark. For convenience we will often (but not always) write $u(\omega, x)$ instead of $u(\omega)(x)$, which is justified by the aforementioned isomorphisms.

Lemma 3.1. $V(t) \subset H(t) \subset V^*(t)$ is a Gelfand triple for every $t \in [0, T]$.

Proof. Since $H^1(\Gamma(t))$ is dense in $L^2(\Gamma(t))$, the proof follows from (3.1), (3.2) and [11, Lemma 4.34], using the density argument. □

3.2 Compatibility of spaces

In order to treat the evolving spaces, we need to define special Bochner-type function spaces such that for every $t \in [0, T]$ we have $u(t) \in V(t)$. In general, if we have an evolving family of Hilbert spaces $X = (X(t))_{t\in[0,T]}$, the idea is to connect the space $X(t)$ at any time $t \in [0, T]$ with some fixed space, for example with the initial space $X(0)$. We do that using the family of maps $\phi_t : X(0) \to X(t)$, which we call the pushforward map. We denote the inverse of ϕ_t by $\phi_{-t} : X(t) \to X(0)$ and call it the pullback map. The following definition is adapted from [1].

Definition 3.2. The pair $\{X, (\phi_t)_{t\in[0,T]}\}$ is *compatible* if the following conditions hold:

- for every $t \in [0, T],\ \phi_t$ is linear homeomorphism such that ϕ_0 is the identity map
- there exists a constant C_X which is independent of t such that

$$\begin{aligned} \|\phi_t u\|_{X(t)} &\leq C_X \|u\|_{X(0)} \quad \text{for every } u \in X(0) \\ \|\phi_{-t} u\|_{X(0)} &\leq C_X \|u\|_{X(t)} \quad \text{for every } u \in X(t) \end{aligned}$$

- the map $t \mapsto \|\phi_t u\|_{X(t)}$ is continuous for every $u \in X(0)$.

We will denote the dual operator of ϕ_t by $\phi_t^* : X^*(t) \mapsto X^*(0)$. As a consequence of the previous conditions, we obtain that ϕ_t^* and its inverse are also linear homeomorphisms which satisfy the following conditions

$$\begin{aligned} \|\phi_t^* f\|_{X^*(0)} &\leq C_X \|f\|_{X^*(t)} \quad \text{for every } f \in X^*(t) \\ \|\phi_{-t}^* f\|_{X^*(t)} &\leq C_X \|f\|_{X^*(0)} \quad \text{for every } f \in X^*(0). \end{aligned}$$

For the Gelfand triple $L^2(\Omega, H^1(\Gamma(t))) \subset L^2(\Omega, L^2(\Gamma(t))) \subset L^2(\Omega, H^{-1}(\Gamma(t)))$ we define the pullback operator $\phi_{-t} : L^2(\Omega, L^2(\Gamma(t))) \to L^2(\Omega, L^2(\Gamma_0))$ in the following way:

$$(\phi_{-t} u)(\omega)(x) := u(\omega)(\Phi_t^0(x)) \quad \text{for every } x \in \Gamma(0),\ \omega \in \Omega.$$

Remark. Since we are interested only in the dual operator of $\phi_t\big|_V$, we will denote it by $\phi_t^* : V^*(t) \to V_0^*$.

The next step is to prove that $(H, \phi_{(\cdot)})$ and $(V, \phi_{(\cdot)}\big|_{V_0})$ are compatible pairs. The proof is similar to the proof of [3, Lemma 3.2].

Let $J_t^0(\cdot) := \det D_{\Gamma_0} \Phi_t^0(\cdot)$ denote the Jacobian determinant, i.e. the change area of the element when transformed from Γ_0 to $\Gamma(t)$. The assumptions for the flow Φ_t^0 imply $J_t^0 \in \mathcal{C}^1([0, T] \times \Gamma_0)$ and that the field J_t^0 is uniformly bounded

$$\frac{1}{C_J} \leq J_t^0(x) \leq C_J \quad \text{for every } x \in \Gamma_0 \text{ and for all } t \in [0, T], \tag{3.3}$$

where C_J is positive constant.

The substitution formula for integrable functions $\zeta : \Gamma(t) \to \mathbb{R}$ reads

$$\int_{\Gamma(t)} \zeta = \int_{\Gamma_0} (\zeta \circ \Phi_t^0) J_t^0 = \int_{\Gamma_0} \phi_{-t}\zeta J_t^0.$$

Using the Leibniz formula for differentiation of a parameter dependent surface integral [5, Lemma 2.1] it can be shown [3, Lemma 3.2.] that

$$\frac{d}{dt} J_t^0 = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0. \tag{3.4}$$

Lemma 3.3. The pairs $(H, (\phi_t))$ and $\left(V, (\phi_t\big|_{V_0})\right)$ are compatible.

Proof. First we will prove the statement for the pair $(H, (\phi_t))$. Let u be from $L^2(\Omega, L^2(\Gamma(t)))$. Then we have

$$\|\phi_{-t}u\|^2_{L^2(\Omega, L^2(\Gamma_0))} = \int_\Omega \int_{\Gamma(t)} |u(\omega)(y)|^2 \frac{1}{J_t^0((\Phi_t^0)^{-1}(y))} \leq C_J \|u\|^2_{L^2(\Omega, L^2(\Gamma(t)))},$$

where we have used the substitution formula and boundeness of J_t^0. It is clear that ϕ_{-t} is linear and that its continuity follows immediately from the previous estimate. Since Φ_t^0 is $\mathcal{C}^2-$diffeomorphism, it follows that ϕ_{-t} is bijective and its inverse (the pushforward) is defined by

$$\phi_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma(t))), \quad (\phi_t v)(\omega, x) = v(\omega) \circ (\Phi_t^0)^{-1}(x).$$

Similarly as for ϕ_{-t}, we can prove that ϕ_t is well defined, satisfies the norm boundeness relation and is continuous. Thus, ϕ_t is linear homeomorphism.
Since the probability space does not depend on time, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega, L^2(\Gamma(t)))}$ follows directly from [3, Lemma 3.3.] and the triangle inequality.

In order to prove compatibility of the family $(V, \phi_t\big|_{V_0})$, let $v \in L^2(\Omega, H^1(\Gamma(t)))$ and $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$. Using the substitution formula and integration by parts on $\Gamma(t)$ we get

$$\begin{aligned}
&\int_\Omega \int_{\Gamma_0} \phi_{-t} v(\omega, x) \nabla_\Gamma \varphi(\omega, x) = \int_\Omega \int_{\Gamma(t)} v(\omega, x) (D\bar{\Phi}_t(x))^T \nabla_\Gamma (\phi_t \varphi(\omega, x)) J_{-t}^0(x) \\
&= -\int_\Omega \int_{\Gamma(t)} \phi_t \varphi(\omega, x) s(\omega, x) J_{-t}^0(x) \\
&= -\int_\Omega \int_{\Gamma_0} [\phi_{-t} s(\omega, x) - H_0 \nu_0 \phi_{-t} v(\omega, x)] \varphi(\omega, x) + H_0 \nu_0 \phi_{-t} v(\omega, x) \varphi(\omega, x),
\end{aligned} \tag{3.5}$$

where s is the function that we get from the partial integration. Note that s depends only on the mean curvature and derivative of $\bar{\Phi}_t$ which can be bounded independently of time and ω. Thus, $\|s(\omega)\|_{L^2(\Gamma(t))^(n+1)} \leq C \|v(\omega)\|_{H^1(\Gamma(t))}$, where C does not depend on ω and t. Furthermore, we get

$$\|s\|_{L^2(\Omega, L^2(\Gamma(t))^{n+1})} \leq C \|v\|_{L^2(\Omega, H^1(\Gamma(t)))}.$$

Hence, using the estimate from the first part of the proof we get

$$\phi_{-t}v \in L^2(\Omega, L^2(\Gamma_0)) \quad \text{and} \quad \|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C'\|v\|_{L^2(\Omega,H^1(\Gamma(t)))}. \tag{3.6}$$

On the other hand, from the partial integration on hypersurface we get

$$\int_\Omega \int_{\Gamma_0} \phi_{-t}v(\omega,x)\nabla_\Gamma \varphi(\omega,x) = -\int_\Omega \int_{\Gamma_0} \varphi(\omega,x)(\nabla_\Gamma(\phi_{-t}v)(\omega,x) + \phi_{-t}v(\omega,x)H_0\nu_0).$$

From the last relation and (3.5), since they hold for every $\varphi \in L^2(\Omega, \mathcal{C}^1(\Gamma_0))$, we get

$$\nabla_\Gamma(\phi_{-t}v)(\omega,x) = \phi_{-t}s(\omega,x) - H_0\nu_0(\phi_{-t}v)(\omega,x). \tag{3.7}$$

For $v \in L^2(\Omega, L^2(\Gamma(t)))$, we have already proved that $\|\phi_{-t}v\|_{L^2(\Omega,L^2(\Gamma_0))} \leq C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}$. Therefore, the following estimate follows

$$\|H_0\nu_0(\phi_{-t}v)(\omega,x)\|_{L^2(\Omega,L^2(\Gamma_0))} \leq |H_0|C_H\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Using the last inequality, (3.6) and (3.7), we get

$$\|\phi_{-t}v\|_{L^2(\Omega,H^1(\Gamma_0))} \leq C_V\|v\|_{V(t)},$$

where C_V depends on global bound on $|H_t|$, $\|\partial\bar{\Phi}_t\|$ and $\|\partial_{ij}\bar{\Phi}_t\|$ with $1 \leq i,j \leq n+1$, $t \in [0,T]$ and these bounds are deterministic and independent of time.

Similarly to the previous case, the continuity of the map $t \mapsto \|\phi_t u\|_{L^2(\Omega,H^1(\Gamma(t)))}$ follows from [3, Lemma 3.3] and the independence of the probability space of time, which completes the proof. □

3.3 Bochner-type spaces

In this section, we want to define Bochner-type spaces of random functions that are defined on evolving spaces. In order to strictly define these spaces we will ask that the pull-back of u belongs to the fixed initial space $V(0)$. These spaces are a special case of general function spaces defined in [1]:

Definition 3.4. For a compatible pair $(X, (\phi_t)_t)$ we define spaces:

$$L^2_X := \left\{ u : [0,T] \to \bigcup_{t\in[0,T]} X(t) \times \{t\},\, t \mapsto (\bar{u}(t),t) \mid \phi_{-(\cdot)}\bar{u}(\cdot) \in L^2(0,T;X_0) \right\}$$

$$L^2_{X^*} := \left\{ f : [0,T] \to \bigcup_{t\in[0,T]} X^*(t) \times \{t\},\, t \mapsto (\bar{f}(t),t) \mid \phi^*_{(\cdot)}\bar{f}(\cdot) \in L^2(0,T;X^*_0) \right\}.$$

The spaces L^2_X and $L^2_{X^*}$ are separable Hilbert spaces ([1], Corollary 2.11) with the inner product defined as

$$(u,v)_{L^2_X} = \int_0^T (u(t),v(t))_{X(t)}dt$$

$$(f,g)_{L^2_{X^*}} = \int_0^T (f(t),g(t))_{X^*(t)}dt.$$

By Lemma 3.3, the spaces L^2_V, $L^2_{V^*}$ and L^2_H are well-defined. Since $V \subset H \subset V^*$ is Gelfand triple, using identification between $L^2_{V^*}$ and $(L^2_V)^*$ [1, Lemma 2.15] and by [1, Lemma 2.19] we get:

Lemma 3.5.

$$L^2_{L^2(\Omega,H^1(\Gamma(t)))} \subset L^2_{L^2(\Omega,L^2(\Gamma(t)))} \subset L^2_{L^2(\Omega,H^{-1}(\Gamma(t)))}$$

is a Gelfand triple.

3.4 Material derivative

We want to define a time derivative that will also take into account the spatial movement, i.e. the material derivative for random functions. First let us consider the push-forward of continuously differentiable functions:

$$\mathcal{C}^1_V := \{u \in L^2_V \mid \phi_{-(\cdot)}u(\cdot) \in \mathcal{C}^1([0,T], L^2(\Omega, H^1(\Gamma_0))\}.$$

Definition 3.6. For $u \in \mathcal{C}^1_V$ *the strong material derivative* is defined by

$$\dot{u} = \phi_t \left(\frac{d}{dt} \phi_{-t} u \right).$$

By smoothness of $\Gamma(t)$, for every $\omega \in \Omega$ each function $u(t,\omega) : \Gamma(t) \to \mathbb{R}$ can be extended to a neighbourhood of $\bigcup_{t\in[0,T]} \Gamma(t) \times \{t\} \subset \mathbb{R}^{n+2}$. Thus $\nabla u(\omega)$ and $u_t(\omega)$ are well defined for every ω. Using the chain rule, for $u \in \mathcal{C}^1_V$ and $y \in \Gamma_0$, we get

$$\begin{aligned} \frac{d}{dt}\phi_{-t}u(t) &= \frac{d}{dt}(u(t,\omega,\Phi^0_t(y)) \\ &= u_t(t,\omega,\Phi^0_t(y)) + \nabla u \mid_{(t,\omega,\Phi^0_t(y))} \cdot \mathbf{w}(t,\Phi^0_t(y)) \\ &= \phi_{-t}u_t(t,\omega,y) + \phi_{-t}\nabla u(t,\omega,y) \cdot \phi_{-t}(\mathbf{w}(t,y)). \end{aligned}$$

Thus, we get the following explicit formula for the *strong material derivative*

$$\dot{u}(t,\omega,x) = u_t(t,\omega,x) + \nabla u(t,\omega,x) \cdot \mathbf{w}(t,x), \tag{3.8}$$

for every $x \in \Gamma(t)$.

Remark. Note that the right hand side of (3.8) does not depend on extension, so it is irrelevant that every extension (i.e. neighbourhood) will depend on ω.

Just as in the deterministic case, it might happen that the equation does not have a solution if we ask that $u \in \mathcal{C}^1_V$. Hence, we want to define a weak material derivative that needs less regularity. In addition to the case when we consider a fixed domain, we will have an extra term that will take into account the movement of the domain. As usual in this setting, the idea is to pull-back the inner product on $L^2(\Omega, L^2(\Gamma(t)))$ onto the fixed space $L^2(\Omega, L^2(\Gamma_0))$. Which will be the bilinear form $\hat{b}$.

Furthermore, we define $\hat{c}$ as a regular time derivative of this bilinear form. Thus, the extra term c in the weak material derivative will be the push-forward of $\hat{c}$ onto $H(t) \times H(t)$.

Let us define the bounded bilinear form $\hat{b}(t, \cdot, \cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ for every $t \in [0, T]$:

$$\begin{aligned}\hat{b}(t, u_0, v_0) &:= (\phi_t u_0, \phi_t v_0)_{L^2(\Omega, L^2(\Gamma(t)))} \\ &= \int_\Omega \int_{\Gamma(0)} u_0(\omega, x) v_0(\omega, x) J_t^0(x).\end{aligned}$$

Moreover, we define the map $\theta : [0.T] \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ that is the classical time derivative of the norm on $L^2(\Omega, L^2(\Gamma(t)))$:

$$\theta(t, u_0) := \frac{d}{dt} \|\phi_t u_0\|^2_{L^2(\Omega, L^2(\Gamma(t)))} \quad \forall u_0 \in L^2(\Omega, L^2(\Gamma_0)).$$

Lemma 3.7. a) The map θ is well defined and for each $t \in [0, T]$ the map

$$u_0 \mapsto \theta(t, u_0) \quad u_0 \in L^2(\Omega, L^2(\Gamma_0)) \tag{3.9}$$

is continuous.

b) For every $t \in [0, T]$ there exists deterministic constant C that is independent of time such that

$$|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| \le C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}.$$

Proof. a) Using the substitution formula, the formula (3.4) and the assumption (2.4) we get:

$$\begin{aligned}\theta(t, u_0) &= \int_\Omega \int_{\Gamma(0)} (u_0(\omega, x))^2 \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t, x)) J_t^0(x) \\ &= \int_\Omega \int_{\Gamma(t)} (\phi_t u_0(\omega, x))^2 \nabla_{\Gamma(t)} \cdot \mathbf{w}(t, x) \\ &\le C \|\phi_t u_0\|^2_{L^2(\Omega, L^2(\Gamma(t)))}.\end{aligned}$$

Hence, θ is well-defined. In order to prove continuity of (3.9) note that $u \in L^2(\Omega, L^2(\Gamma_0))$ implies $u^2 \in L^1(\Omega, L^1(\Gamma_0))$. This implies that if $u_n \to u$ in $L^2(\Omega, L^2(\Gamma_0))$, then $u_n^2 \to u^2$ in $L^1(\Omega, L^1(\Gamma_0))$. Now continuity follows from:

$$\begin{aligned}|\theta(t, u_n) - \theta(t, u)| &\le \int_\Omega \int_{\Gamma_0} |u_n^2(\omega, x) - u^2(\omega, x)||\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t, x)) J_t^0(x)| \\ &\le C \|u_n^2 - u^2\|_{L^1(\Omega, L^1(\Gamma_0))} \to 0.\end{aligned}$$

b) Using the Cauchy-Schwarz inequality, (3.3) and (3.4) we get the estimate:

$$\begin{aligned}|\theta(t, u_0 + v_0) - \theta(t, u_0 - v_0)| &= \|4 \frac{d}{dt} \hat{b}(t; u_0, v_0)\| \\ &= 4 |\int_\Omega \int_{\Gamma_0} u_0(\omega, x) v_0(\omega, x) \frac{d}{dt} J_t^0(x)| \\ &\le C |\, (u_0, v_0)\, |_{L^2(\Omega, L^2(\Gamma_0))} \\ &\le C \|u_0\|_{L^2(\Omega, L^2(\Gamma_0))} \|v_0\|_{L^2(\Omega, L^2(\Gamma_0))}. \quad \square\end{aligned}$$

Now we can define the bilinear form $\hat{c}(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma_0)) \times L^2(\Omega, L^2(\Gamma_0)) \to \mathbb{R}$ as a partial time derivative of $\hat{b}$:

$$\hat{c}(t;u_0,v_0) := \frac{\partial}{\partial t}\hat{b}(t;u_0,v_0) = \frac{1}{4}(\theta(t,u_0+v_0) - \theta(t,u_0-v_0))$$
$$= \int_\Omega \int_{\Gamma_0} u_0(\omega,x)v_0(\omega,x)\phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x))J_t^0(x).$$

From [1, Lemma 2.27] it follows that for every $u, v \in \mathcal{C}^1([0,T]; L^2(\Omega, L^2(\Gamma_0)))$ the map

$$t \mapsto \hat{b}(t;u(t),v(t))$$

is differentiable in the classical sense and the formula for differentiation of the scalar product on $L^2(\Omega, L^2(\Gamma(t)))$ is

$$\frac{d}{dt}\hat{b}(t;u(t),v(t)) = \hat{b}(t;u'(t),v(t)) + \hat{b}(t;u(t),v'(t)) + \hat{c}(t;u(t),v(t)).$$

We will generalise this result in Section 3.5, to less regular functions u and v.

Now we can define the extra term that appears in the definition of the weak material derivative. As we have already announced, we pull-back the functions on $\Gamma(0)$ and apply bilinear form $\hat{c}$ on them. More precisely, we define the bilinear form $c(t;\cdot,\cdot) : L^2(\Omega, L^2(\Gamma(t))) \times L^2(\Omega, L^2(\Gamma(t))) \to \mathbb{R}$ by

$$c(t;u,v) := \hat{c}(t;\phi_{-t}u, \phi_{-t}v) = \int_\Omega \int_{\Gamma(t)} u(\omega,z)v(\omega,z)(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x)).$$

Lemma 3.8. For every $u, v \in L^2_V$, the map

$$t \mapsto c(t;u(t),v(t))$$

is measurable. Furthermore, c is bounded independently of t by deterministic constant:

$$|c(t;u,v)| \leq C\|u\|_{L^2(\Omega,L^2(\Gamma(t)))}\|v\|_{L^2(\Omega,L^2(\Gamma(t)))}.$$

Proof. From Lemma 3.7 it follows that we can apply a corollary of [1, Lemma 2.26], which proves the Lemma. □

Now we can define the weak material derivative.

Definition 3.9. We say that $\partial^\bullet u \in L^2_{V^*}$ is *a weak material derivative* of $u \in L^2_V$ iff

$$\int_0^T \langle \partial^\bullet u(t), \eta(t)\rangle_{V^*(t),V(t)} = -\int_0^T (u(t),\dot{\eta}(t))_{H(t)} - \int_0^T c(t;u(t),\eta(t))$$
$$= \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\dot{\eta}(t,\omega,x) - \int_0^T \int_\Omega \int_{\Gamma(t)} u(t,\omega,x)\eta(t,\omega,x)\nabla_{\Gamma(t)} \cdot \mathbf{w}(t,x),$$

holds for all $\eta \in \mathcal{D}_V(0,T) = \{\eta \in L^2_V \mid \phi_{-(\cdot)}\eta(\cdot) \in \mathcal{D}((0,T); L^2(\Omega, H^1(\Gamma_0)))\}$.

Note that it can be directly shown that if it exists, the weak material derivative is unique and every strong material derivative is also a weak material derivative.

3.5 Solution space

We will ask for the solution of the equation (1.1) to be in the space L^2_V and also to have a weak material derivative. Hence, we define the solution space as:

$$W(V,V^*) := \{u \in L^2_V \mid \partial^\bullet u \in L^2_{V^*}\}.$$

In order to prove that the solution space is Hilbert space and also that it has some additional properties, we will connect it with the standard Sobolev-Bochner space for which these properties are known. Thus, let us define the following space:

$$\mathcal{W}(V_0,V_0^*) = \{u \in L^2(0,T;L^2(\Omega,H^1(\Gamma_0))) \mid u' \in L^2(0,T;L^2(\Omega,H^{-1}(\Gamma_0)))\}.$$

The space $\mathcal{W}(V_0,V_0^*)$ is Hilbert space with the inner product defined via:

$$(u,v)_{\mathcal{W}(V_0,V_0^*)} := \int_0^T \int_\Omega (u(t,\omega),v(t,\omega))_{H^1(\Gamma_0)} + \int_0^T \int_\Omega (u'(t,\omega),v'(t,\omega))_{H^{-1}(\Gamma_0)}.$$

We will use that the embedding

$$\mathcal{D}([0,T];V_0) \subset \mathcal{W}(V_0,V_0^*) \tag{3.10}$$

is dense. More properties of this space can be found for example in [3, Lemma 2.2].

We want to show that the previous two types of spaces are connected in a natural way, i.e. that the pull-back of the functions from the solution space belong to Sobolev-Bochner space and vice versa. In addition, we also have the equivalence of the norms. First we will prove the technical result which is similar to [3, Lemma 3.6.].

Lemma 3.10. Let $w \in \mathcal{W}(V_0,V_0^*)$ and $f \in \mathcal{C}^1([0,T]\times\Gamma_0)$. Then $fw \in \mathcal{W}(V_0,V_0^*)$ and

$$(fw)' = \partial_t f w + f w', \tag{3.11}$$

where $\langle fw',\varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma)),L^2(\Omega,H^1(\Gamma))} = \langle w', f\varphi\rangle_{L^2(\Omega,H^{-1}(\Gamma)),L^2(\Omega,H^1(\Gamma))}$.

Proof. We will first prove the Lemma for $\varphi \in \mathcal{D}([0,T],L^2(\Omega,H^1(\Gamma_0)))$. From the proof of [3, Lemma 3.6] it follows that $f \in \mathcal{C}^1([0,T]\times\Gamma_0)$ implies

$$f \in \mathcal{C}([0,T],\mathcal{C}^1(\Gamma_0)) \text{ and } f \in \mathcal{C}^1([0,T],\mathcal{C}(\Gamma_0)). \tag{3.12}$$

In order to prove that $f\varphi \in L^2([0,T];L^2(\Omega,H^1(\Gamma_0)))$ we can treat deterministic function f as a random function that is constant in ω. More precisely, if we define the function $\tilde{f}(t,\omega,x) := f(t,x)$, from (3.12) it follows $\tilde{f} \in \mathcal{C}([0,T],L^2(\Omega,\mathcal{C}^1(\Gamma_0))$. This can be strictly shown by defining the function $g : \mathcal{C}(\Gamma_0) \to L^2(\Omega,\mathcal{C}(\Gamma_0))$, $g(f)(\omega,x) := f(x)$. Note that g is linear and thus a $\mathcal{C}^\infty$ function and also that for every t we have $g(f(t)) = \tilde{f}(t)$.

It is then clear that we have

$$\tilde{f}\varphi \in \mathcal{C}([0,T], L^2(\Omega, H^1(\Gamma_0))) \cap \mathcal{C}^1([0,T], L^2(\Omega, L^2(\Gamma_0)))$$

which implies $\tilde{f}\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$ and hence, $f\varphi \in L^2([0,T]; L^2(\Omega, H^1(\Gamma_0)))$.

It is left to prove that formula (3.11) is valid. We will prove this using the characterisation of the weak derivative ([1, Theorem 2.2]) and partial integration ([1, Lemma 2.1(3)],) we get

$$\int_0^T \langle fw', \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma)), L^2(\Omega, H^1(\Gamma))} = -\int_0^T \langle w, (f\varphi)' \rangle_{L^2(\Omega, H^1(\Gamma)), L^2(\Omega, H^{-1}(\Gamma))}$$
$$= -\int_0^T \langle \partial_t f w, \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma)), L^2(\Omega, H^1(\Gamma))} - \int_0^T \langle fw, \varphi' \rangle_{L^2(\Omega, H^{-1}(\Gamma)), L^2(\Omega, H^1(\Gamma))}.$$

It follows

$$\int_0^T \langle fw, \varphi' \rangle_{L^2(\Omega, L^2(\Gamma_0))} = \int_0^T \langle \partial_t f w + fw', \varphi \rangle_{L^2(\Omega, H^{-1}(\Gamma)), L^2(\Omega, H^1(\Gamma))},$$

i.e. $(fw)' = \partial_t f w + fw'$. Using the density result (3.10) we can approximate every function fw by continuous $L^2(\Omega, H^1(\Gamma_0))-$valued functions and conclude $fw \in L^2(\Omega, H^1(\Gamma_0))$. The similar argument implies that $(fw)' \in L^2(\Omega, H^{-1}(\Gamma_0))$. □

Corollary 3.11. If $T_t : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ is defined as $T_t u_0(\omega, x) := u_0(\omega, x) J_t^0(x)$, then it holds:

$$u \in \mathcal{W}(V_0, V_0^*) \text{ if and only if } T_{(\cdot)} u(\cdot) \in \mathcal{W}(V_0, V_0^*). \tag{3.13}$$

Proof. Apply Lemma 3.10 to the functions $f = J^0_{(\cdot)}$ and $f = \frac{1}{J^0_{(\cdot)}}$, which are both from the space $\mathcal{C}^1([0,T] \times \Gamma_0)$. □

Theorem 3.12. The following equivalence holds:

$$v \in W(V, V^*) \text{ if and only if } \phi_{-(\cdot)} v(\cdot) \in \mathcal{W}(V_0, V_0^*), \tag{3.14}$$

and the norms are equivalent

$$C_1 \|\phi_{-(\cdot)} v(\cdot)\|_{\mathcal{W}(V_0, V_0^*)} \le \|v\|_{W(V,V^*)} \le C_2 \|\phi_{-(\cdot)} v(\cdot)\|_{\mathcal{W}(V_0, V_0^*)}. \tag{3.15}$$

Remark. Following the notation from [1], we say that there exists an *evolving space equivalence* between the spaces that satisfy (3.14) and (3.15).

Proof. Let us define for every $t \in [0,T]$ a map $\hat{S}(t) : V_0^* \to V_0^*$ by

$$\hat{S}(t) u(\omega, x) := u(\omega, x) J_t^0(x).$$

Note that since J_t^0 is bounded independently of t and has an inverse, this implies that $\hat{S}(t)$ has an inverse, and both $\hat{S}(t)$ and $\hat{S}^{-1}(t)$ are bounded independently of t. Furthermore, from the uniform bound on J_t^0 we have that $\hat{S}(\cdot)u'(\cdot) \in L^2(0,T;V_0^*)$ for $u \in \mathcal{W}(V_0, V_0^*)$. In the end, using the product rule (3.11), we get

$$(T_t u(t))' = (J_t^0 u(t))' = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t)) J_t^0 u(t) + J_t^0 u'(t) = \hat{S}(t)u'(t) + \hat{C}(t)u(t),$$

where T_t is defined in the previous corollary and $\hat{C}(t) : L^2(\Omega, L^2(\Gamma_0)) \to L^2(\Omega, L^2(\Gamma_0))$ is defined as $\hat{C}(t,\omega,x) = \phi_{-t}(\nabla_{\Gamma(t)} \cdot \mathbf{w}(t))J_t^0(x)$, i.e. $< \hat{C}(t)u_0, v_0 >:= \hat{c}(t; u_0, v_0)$. Thus, using in addition Corollary 3.11, we can apply [1, Theorem 2.32.], which yields that there exists the evolving space equivalence between $W(V,V^*)$ and $\mathcal{W}(V_0, V_0^*)$. □

Corollary 3.13. The solution space $W(V,V^*)$ is a Hilbert space with the inner product defined via

$$(u,v)_{V,V^*} = \int_0^T \int_\Omega (u(t), v(t))_{H^1(\Gamma(t))} + \int_0^T \int_\Omega (\partial^\bullet u(t), \partial^\bullet v(t))_{H^{-1}(\Gamma(t))}.$$

More properties of the space $W(V,V^*)$ can be found in [1].
We have shown how to differentiate the inner product of functions from C_H^1 on $H(t) = L^2(\Omega, L^2(\Gamma(t)))$. We can generalize this result to functions from the solution space.

Theorem 3.14. (Transport theorem.) For all $u, v \in W(V,V^*)$, the map

$$t \mapsto (u(t), v(t))_{L^2(\Omega, L^2(\Gamma(t)))}$$

is absolutely continuous on $[0,T]$ and

$$\frac{d}{dt}(u(t), v(t))_{H(t)} = \langle \partial^\bullet u(t), v(t) \rangle_{V^*(t),V(t)} + \langle \partial^\bullet v(t), u(t) \rangle_{V^*(t),V(t)} + c(t; u(t), v(t)), \quad (3.16)$$

for almost all $t \in [0,T]$.

Proof. The proof is based on the density of the space $D_V[0,T]$ in the space $W(V,V^*)$ and the transport formula for the functions from $\mathcal{C}_H^1$. For a detailed proof, we refer the reader to [1, Theorem 2.38.]. □

4 Existence, uniqueness and regularity of solutions

4.1 Formulation of the problem

We want to consider the following equation

$$\begin{aligned} \partial^\bullet u - \nabla_\Gamma \cdot (\alpha \nabla_\Gamma u) + u \nabla_\Gamma \cdot \mathbf{w} &= f \quad \text{in } L^2_{V^*} \\ u(0) &= u_0. \end{aligned} \quad (4.1)$$

Let us state assumptions for the initial data that we need in order to prove the existence and uniqueness of the solution. The initial value u_0 belongs to $L^2(\Omega, L^2(\Gamma_0))$. For the source term we assume

$f \in L^2_{V^*}$. Moreover, $\alpha : \Omega \times \mathcal{G}_T \to \mathbb{R}$ is assumed to be a random $\mathcal{F} \times \mathcal{B}(\mathcal{G}_T)-$measurable function, where $\mathcal{G}_T$ is the space-time surface $\mathcal{G}_T := \bigcup_t \Gamma(t) \times \{t\}$. Furthermore, we assume that the diffusion coefficient α is bounded and uniformly coercive in the sense that there are constants $\alpha_{min}, \alpha_{max}$ such that

$$0 < \alpha_{min} < \alpha(\omega, x, t) < \alpha_{max} < \infty \quad \text{a.e. } (x,t) \in \mathcal{G}_T \tag{4.2}$$

holds for $\mathbb{P}-$a.e. $\omega \in \Omega$.

Remark. As we have mentioned in the introduction, one could consider the case with weaker assumptions for the coefficient α. A frequently studied case is when α is a homogeneous lognormal random field. More precisely, let $\alpha(\omega, x) = e^{g(\omega,x)}$ be a lognormal homogeneous random field, where g is a mean-free Gaussian field. We can define $\alpha_{\text{min}}(\omega) = \min_x \alpha(\omega, x)$ and $\alpha_{\text{max}}(\omega) = \max_x \alpha(\omega, x)$ and these random variables satisfy $\alpha_{\text{min}}, \alpha_{\text{max}} \in L^p(\Omega),\ p > 1$. The proof of this result can be found for example in [20]. Thus, it follows from [5] that for all $\omega \in \Omega$ there exists a unique solution. What is still lacking is the regularity of the solution w.r.t. ω, but this result requires a more thorough investigation.

Definition 4.1. We say that u is *a weak solution* of (4.1) if it satisfies the initial condition $u(0) = u_0$ and $u \in W(V, V^*)$ and a.e. in $[0, T]$:

$$\begin{aligned} \langle \partial^\bullet u(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))} &+ \int_\Omega \int_{\Gamma(t)} \alpha(t) \nabla_\Gamma u(t) \cdot \nabla_\Gamma v \\ + \int_\Omega \int_{\Gamma(t)} u(t) v \nabla_\Gamma \cdot \mathbf{w} &= \langle f(t), v \rangle_{L^2(\Omega, H^{-1}(\Gamma(t))), L^2(\Omega, H^1(\Gamma(t)))}, \end{aligned} \tag{4.3}$$

for every $v \in L^2(\Omega, H^1(\Gamma(t)))$.

In order to simplify the notation we define the bilinear form $a(t; \cdot, \cdot) : V(t) \times V(t) \to \mathbb{R}$ by

$$a(t; u, v) := \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u(\omega, x) \cdot \nabla_\Gamma v(\omega, x).$$

Let us state some of the properties of the bilinear form a.

Lemma 4.2. The map

$$t \mapsto a(t; u, v) \tag{4.4}$$

is measurable. Furthermore, there exist positive deterministic constants C_1, C_2 and C_3 that are independent of t such that

$$a(t; v, v) \geq C_1 \|v\|^2_{L^2(\Omega, H^1(\Gamma(t)))} - C_2 \|v\|^2_{L^2(\Omega, L^2(\Gamma(t)))}, \tag{4.5}$$

$$|a(t; u, v)| \leq C_3 \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))}. \tag{4.6}$$

Proof. The measurability of (4.4) follows directly from Fubini-Tonelli theorem. Moreover, the assumption (4.2) directly implies

$$a(t; v, v) \geq \alpha_{min} \|\nabla_\Gamma v\|_{L^2(\Omega, L^2(\Gamma))},$$

thus we can take $C_1 = C_2 = \alpha_{min}$. Using again (4.2) and the Cauchy-Schwarz inequality we get that $C_3 = \alpha_{max}$:

$$\begin{aligned} \left| \int_\Omega \int_{\Gamma(t)} \alpha(\omega, x, t) \nabla_\Gamma u \cdot \nabla_\Gamma v \right| &\leq \alpha_{max} \left| \langle \nabla_\Gamma u, \nabla_\Gamma v \rangle_{L^2(\Omega, L^2(\Gamma(t)))} \right| \\ &\leq \alpha_{max} \|\nabla_\Gamma u\|_{L^2(\Omega, L^2(\Gamma(t)))} \|\nabla_\Gamma v\|_{L^2(\Omega, L^2(\Gamma(t)))} \\ &\leq \alpha_{max} \|u\|_{L^2(\Omega, H^1(\Gamma(t)))} \|v\|_{L^2(\Omega, H^1(\Gamma(t)))}. \quad \square \end{aligned}$$

4.2 Existence and uniqueness

Developing all the necessary results, we can now formulate the theorem about the existence and uniqueness of a solution of the equation (4.3).

Theorem 4.3. Under the aforementioned assumptions on $f \in L^2_{V^*}$, the diffusion coefficient α and $u_0 \in L^2(\Omega, L^2(\Gamma_0))$, there exists a unique solution $u \in W(V, V^*)$ satisfying (4.3) such that

$$\|u\|_{W(V,V^*)} \leq C(\|u_0\|_{H_0} + \|f\|_{L^2_{V^*}})$$

where $V = (V(t))_{t \in [0,T]}$ is the family of spaces $V(t) = L^2(\Omega, H^1(\Gamma(t)))$, V^* is the family of corresponding dual spaces and $H_0 = L^2(\Omega, L^2(\Gamma_0))$.

Proof. Lemma 3.3, Theorem 3.12 and Lemma 4.2 imply that we can apply [1, Theorem 3.6] about the existence and uniqueness of the solution of the parabolic PDE on an abstract evolving space. The main idea of the proof of [1, Theorem 3.6] is to use the Banach-Nečas-Babuška theorem. This proves the theorem. □

4.3 Regularity

Let us now assume more regularity of the input data. More precisely, let $f \in L^2_H$ and $u_0 \in V_0$. We will prove that in this case we also have more regularity for the solution, i.e. its material derivative. Before we state this result, we will prove some technical results.

First we define the solution space for the case when the solution has more regularity:

Definition 4.4. We define

$$W(V, H) := \{u \in L^2_V \,|\, \partial^\bullet u \in L^2_H\}.$$

Lemma 4.5. There is an evolving space equivalence between $W(V, H)$ and $\mathcal{W}(V_0, H_0) \equiv \{v \in L^2(0, T; L^2(\Omega, H^1(\Gamma_0))) \,|\, v' \in L^2(0, T; L^2(\Omega, L^2(\Gamma_0)))\}$.

Proof. Since The Jacobian J_t^0 is uniformly bounded, both in time and space (see 3.3), applying [1, Theorem 2.33] on the restriction $\hat{S}(t): H_0 \to H_0$ of the map defined in the proof of Theorem 3.12, we prove the lemma. $\square$

Corollary 4.6. $W(V, H)$ is a Hilbert space.

If $u_0 \in V_0$ and $f \in L^2_H$, the Definition 4.1 of the weak solution transforms to:

Find $u \in W(V, H)$ such that $u(0) = u_0$ and a.e. in $[0, T]$:

$$\int_\Omega\int_{\Gamma(t)} (\partial^\bullet u(t), v)_{H^1(\Gamma(t))} + \int_\Omega\int_{\Gamma(t)} \alpha(t)\nabla_\Gamma u(t)\cdot\nabla_\Gamma v + \int_\Omega\int_{\Gamma(t)} u(t)v\nabla_\Gamma\cdot\mathbf{w}(t) = \int_\Omega\int_{\Gamma(t)} fv, \quad (4.7)$$

for every $v \in L^2(\Omega, H^1(\Gamma(t)))$.

Lemma 4.7. There exists a basis $\{\chi_j^0\}_{j\in\mathbb{N}}$ of $V_0 \equiv L^2(\Omega, H^1(\Gamma_0))$ and for every $u_0 \in V_0$ there exists a sequence $\{u_{0K}\}_{K\in\mathbb{N}}$ such that $u_{0K} \in \operatorname{span}\{\chi_1^0, \dots, \chi_K^0\}$ for every K, such that

$$u_{0K} \to u_0 \text{ in } V_0$$
$$\|u_{0K}\|_{H_0} \le \|u_0\|_{H_0}$$
$$\|u_{0K}\|_{V_0} \le \|u_0\|_{V_0}.$$

Proof. Since $H^1(\Gamma_0)$ is compactly embedded in $L^2(\Gamma_0)$, there exists an orthonormal basis $\{w_m\}$ in $H^1(\Gamma_0)$ such that $\{\lambda_m^{-1/2} w_m\}$ is an orthonormal basis of $L^2(\Gamma_0)$ where

$$(u, w_m)_{L^2(\Gamma)} = \lambda_m^{-1}(u, w_m)_{H^1(\Gamma)} \quad \forall u \in H^1(\Gamma_0). \quad (4.8)$$

On the other hand, since $L^2(\Omega)$ is separable, it has an orthonormal basis $\{e_n\}$. It follows by Theorem 2.1 that $\{w_m e_n\}$ is the orthonormal basis of $L^2(\Omega, L^2(\Gamma_0))$ and $\{\lambda^{-1/2} w_m e_n\}$ is the orthonormal basis of $L^2(\Omega, H^1(\Gamma_0))$. Let $u_0 \in L^2(\Omega, H^1(\Gamma_0))$ be arbitrary. Then, (4.8) implies

$$(u_0, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} = \lambda_m^{-1}(u_0, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))}. \quad (4.9)$$

Thus we have

$$\begin{aligned} u_0 &= \sum_{m,n} (u_0, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} e_n w_m \\ &= \sum_{m,n} (u_0, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))} \lambda_m^{-1} e_n w_m. \end{aligned}$$

Now we can define

$$u_{0K} := \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega, L^2(\Gamma_0))} e_n w_m = \sum_{\substack{n=1,\dots,N_k \\ m=1,\dots,M_k}} (u, e_n w_m)_{L^2(\Omega, H^1(\Gamma_0))} \lambda_m^{-1} e_n w_m,$$

where the last equality follows from (4.9). We choose M_k and N_k such that they both converge to ∞, as $K \to \infty$. Defined like this, u_{0K} satisfies the conditions from the Lemma. $\square$

If we write $\chi_j^t := \phi_t(\chi_j^0)$, where $\{\chi_j^0\}_{j\in\mathbb{N}}$ is a basis of V_0, then by [1, Lemma 5.1] it follows that $\{\chi_j^t\}_{j\in\mathbb{N}}$ is a countable basis of $V(t)$. Now we define the space

$$\tilde{C}_V^1 := \{u \,|\, u(t) = \sum_{j=1}^{m} \alpha_j(t)\chi_j^t, m \in \mathbb{N}, \alpha_j \in AC([0,T]) \text{ and } \alpha_j' \in L^2(0,T)\},$$

where $AC([0,T])$ is the space of absolutely continuous functions from $[0,T]$.

For improved regularity of the solution, we will also need the following assumption on the material derivative of the random coefficient α. More precisely, we assume that there exists a deterministic constant C that does not depend on time such that

$$|\dot{\alpha}| \leq C, \tag{4.10}$$

where $\dot{\alpha}$ is a strong material derivative. Thus, in order for this assumption to make sense and be satisfied, it is sufficient to assume $\alpha(\omega,\cdot,\cdot) \in \mathcal{C}^1(\mathcal{G}_T)$, $\mathbb{P}-$a.e..

Lemma 4.8. a) The map

$$t \mapsto a(t; y(t), y(t))$$

is an absolutely continuous function on $[0,T]$ for all $y \in \tilde{C}_V^1$.

b) $a(t; v, v) \geq 0$ for all $v \in V(t)$.

c)

$$\frac{d}{dt} a(t; y(t), y(t)) = 2a(t; y(t), \partial^\bullet y(t)) + r(t; y(t)) \quad \forall y \in \tilde{C}_V^1,$$

where the derivative is taken in the classical sense and $r(t;\cdot) : V(t) \to \mathbb{R}$ satisfies

$$|r(t; v)| \leq C_3 \|v\|_{V(t)}^2 \quad \forall v \in V(t).$$

Proof. The part b) follows immediately from the assumption (4.2). In order to prove parts a) and c), let us first take $\eta \in C_V^\infty$. Since the probability space Ω does not depend on time, it does not have any influence in taking time derivative, thus the analogue transport formulae from the deterministic case (that can be found in [7]) still hold in our setting. By applying formula to the bilinear form $a(t;\cdot,\cdot)$ we get:

$$\begin{aligned}
\frac{d}{dt} a(t; \eta(t), \eta(t)) &= 2\int_\Omega \int_{\Gamma(t)} \alpha \nabla_\Gamma \eta \cdot \nabla_\Gamma(\partial^\bullet \eta) + \int_\Omega \int_{\Gamma(t)} \dot{\alpha} |\nabla_\Gamma \eta|^2 \\
&+ \int_\Omega \int_{\Gamma(t)} \alpha |\nabla_\Gamma \eta|^2 \nabla_\Gamma \cdot \mathbf{w} - \int_\Omega \int_{\Gamma(t)} 2\alpha D_\Gamma(\mathbf{w}) |\nabla_\Gamma \eta|^2 \\
&= 2a(t; \eta, \partial^\bullet \eta) + r(t; \eta),
\end{aligned}$$

where $(D_\Gamma \mathbf{w}(t))_{ij} := \underline{D}_j \mathbf{w}^i(t)$ and

$$r(t; \eta(t)) := \int_\Omega \int_{\Gamma(t)} \dot{\alpha} |\nabla_\Gamma \eta|^2 + \alpha |\nabla_\Gamma \eta|^2 \nabla_\Gamma \cdot \mathbf{w} - 2\alpha D_\Gamma(\mathbf{w}) |\nabla_\Gamma \eta|^2.$$

By the similar arguments as in [2, Ch. 5.2], which are based on the density result of space C_V^∞ in $\tilde{C}_V^1$, we can conclude that the previous formula is true also for every function $\eta \in \tilde{C}_V^1$. Furthermore, the boundedness of $r(t;\cdot)$ follows directly from the assumptions about the velocity (2.4) and assumption (4.10). This proves c). It remains to prove the part a). This claim follows directly from the previous calculation, which implies that both the function $\int_\Omega \int_{\Gamma(t)} \alpha(t)|\nabla_\Gamma \eta(t)|^2$ (w.r.t. time) and its time derivative, are in $L^1(0,T)$. □

Theorem 4.9. Under general assumptions on the diffusion coefficient α, assumption (4.10) and in addition $f \in L_H^2$ and $u_0 \in V_0$, the unique solution u of (4.7) satisfies $u \in W(V,H)$ and the following estimate holds

$$\|u\|_{W(V,H)} \leq C(\|u_0\|_{V_0} + \|f\|_{L_H^2}).$$

Proof. From Lemma 4.2, Lemma 4.7 and Lemma 4.8, it follows that we can apply the general [1, Theorem 3.13] about regularity of the solution of parabolic PDEs on evolving space, which implies the theorem. □

5 Outlook

Although we have stated and solved the problem of finding the unique solution of advection-diffusion PDE with random coefficients on a moving hypersurface, only the continuous case has been discussed. The next step is to consider the numerical approximation of the solution of the equation. More strictly, since the solution is a random variable, we are interested in a numerical approximation of the expected value of the solution. One approach for discretization in space would be to use the evolving surface finite element method from [5], for which we approximate the hypersurface by an evolving interpolated polyhedral surface. In order to deal with uncertainty, one could use the Monte Carlo method which approximates the expected value. The goal would be to find the error estimate for this approximation. These results are subject of ongoing research and a paper is in preparation.

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