
On Nonlinear Dirichlet–Neumann Algorithms for Jumping Nonlinearities

Heiko Berninger, Ralf Kornhuber, and Oliver Sander

FU Berlin, FB Mathematik und Informatik
(<http://www.math.fu-berlin.de/rd/we-02/numerik/>)

Summary. We consider a quasilinear elliptic transmission problem where the nonlinearity changes discontinuously across two subdomains. By a reformulation of the problem via Kirchhoff transformation we first obtain linear problems on the subdomains together with nonlinear transmission conditions and then a nonlinear Steklov–Poincaré interface equation. We introduce a Dirichlet–Neumann iteration for this problem and prove convergence to a unique solution in one space dimension. Finally we present numerical results in two space dimensions suggesting that the algorithm can be applied successfully in more general cases.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain divided into two non-overlapping subdomains Ω_1, Ω_2 with the interface $\Gamma = \overline{\Omega_1} \cap \overline{\Omega_2}$.

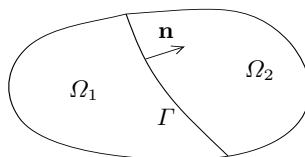


Fig. 1. Non-overlapping partition of the domain Ω .

Given $f \in L^2(\Omega)$, $k_1, k_2 \in L^\infty(\mathbb{R})$ with $k_i \geq \alpha > 0$ for $i = 1, 2$ we consider the following quasilinear elliptic transmission problem:

Find a function p on Ω , $p|_{\Omega_i} = p_i \in H^1(\Omega_i)$, $i = 1, 2$, $p|_{\partial\Omega} = 0$, such that

$$-\operatorname{div}(k_i(p_i)\nabla p_i) = f \quad \text{on } \Omega_i, \quad i = 1, 2 \quad (1)$$

$$p_1 = p_2 \quad \text{on } \Gamma \quad (2)$$

$$k_1(p_1)\nabla p_1 \cdot \mathbf{n} = k_2(p_2)\nabla p_2 \cdot \mathbf{n} \quad \text{on } \Gamma. \quad (3)$$

Observe that the nonlinearities k_i need not be differentiable. Hence, Newton-type linearization suffers from a lack of smoothness. However, by standard *Kirchhoff transformation* [1], we can reformulate the two nonlinear pdes (1) as linear *Poisson equations* in each of the subdomains. Based on this observation, we introduce a nonlinear Dirichlet–Neumann algorithm for (1)–(3) which requires the solution of two linear problems in each iteration step but does *not* involve any further linearization. We present a convergence analysis including sufficient conditions for convergence in one space dimension. As a by-product, we obtain an existence and uniqueness result for (1)–(3). Numerical computations suggest that the algorithm can also be applied successfully to higher-dimensional problems. Related Robin methods will be treated in a forthcoming paper [3].

This paper is organized as follows. In Section 2 we apply the Kirchhoff transformation to (1)–(3) in order to derive an interface problem with linear problems on the subdomains and nonlinear transmission conditions. The transformed problem can be rewritten as a nonlinear Steklov–Poincaré interface equation. In analogy to the linear case, the nonlinear Dirichlet–Neumann algorithm can be regarded as a preconditioned Richardson iteration applied to the nonlinear Steklov–Poincaré equation. In Section 3, we present a convergence theorem in 1D generalizing related results in the linear case [5]. Finally, in Section 4, we illustrate the numerical properties of the nonlinear Dirichlet–Neumann method in a nontrivial two-dimensional setting.

2 An elliptic problem with jumping nonlinearities

We introduce (for $i = 1, 2$) the spaces

$$V_i := \{v_i \in H^1(\Omega_i) \mid v_i|_{\partial\Omega \cap \partial\Omega_i} = 0\}, \quad V_i^0 := H_0^1(\Omega_i), \quad \Lambda := H_{00}^{1/2}(\Gamma)$$

and for $w_i, v_i \in V_i$ the forms

$$a_i(w_i, v_i) := (\nabla w_i, \nabla v_i)_{\Omega_i}, \quad b_i(w_i, v_i) := (k(w_i) \nabla w_i, \nabla v_i)_{\Omega_i},$$

where $(\cdot, \cdot)_{\Omega_i}$ stands for the L^2 inner product on Ω_i . The norm in $H^1(\Omega_i)$ will be denoted by $\|\cdot\|_{1, \Omega_i}$, the norm in Λ with $\|\cdot\|_{\Lambda}$. We point out that much of what we present in the first two sections are generalizations of the linear theory given in [5]. The notation here is used accordingly.

Let R_i , $i = 1, 2$, be any continuous extension operator from Λ to V_i . Then the variational formulation of problem (1)–(3) reads as follows:

Find $p_i \in V_i$, $i = 1, 2$, such that

$$b_i(p_i, v_i) = (f, v_i)_{\Omega_i} \quad \forall v_i \in V_i^0, \quad i = 1, 2 \quad (4)$$

$$p_1|_{\Gamma} = p_2|_{\Gamma} \quad \text{in } \Lambda \quad (5)$$

$$b_1(p_1, R_1\mu) - (f, R_1\mu)_{\Omega_1} = -b_2(p_2, R_2\mu) + (f, R_2\mu)_{\Omega_2} \quad \forall \mu \in \Lambda. \quad (6)$$

We now introduce new variables u_i , $i = 1, 2$, by Kirchhoff transformation κ_i (cf. [1]):

$$u_i(x) := \kappa_i(p_i(x)) = \int_0^{p_i(x)} k_i(q) dq \quad \text{a.e. in } \Omega_i,$$

which yields $k_i(p_i)\nabla p_i = \nabla u_i$. Further properties of κ_i are listed in the following

Proposition 1. $\kappa_i : \mathbb{R} \rightarrow \mathbb{R}$ is a.e. differentiable with $\kappa_i' = k_i$, strictly monotonically increasing and Lipschitz continuous with Lipschitz constant $\|k_i\|_\infty$. κ_i^{-1} is Lipschitz continuous with Lipschitz constant $\|k_i^{-1}\|_\infty^{-1}$. Furthermore there exist positive constants c, C with

$$c \|p_i\|_{1,\Omega_i} \leq \|\kappa_i(p_i)\|_{1,\Omega_i} \leq C \|p_i\|_{1,\Omega_i} \quad \text{and}$$

$$c \|p_i|_\Gamma\|_\Lambda \leq \|\kappa_i(p_i)|_\Gamma\|_\Lambda \leq C \|p_i|_\Gamma\|_\Lambda.$$

Thus with this transformation problem (4)–(6) becomes:

Find $u_i \in V_i$, $i = 1, 2$, such that

$$a_i(u_i, v_i) = (f, v_i)_{\Omega_i} \quad \forall v_i \in V_i^0, \quad i = 1, 2 \quad (7)$$

$$\kappa_1^{-1}(u_1|_\Gamma) = \kappa_2^{-1}(u_2|_\Gamma) \quad \text{in } \Lambda \quad (8)$$

$$a_1(u_1, R_1\mu) - (f, R_1\mu)_{\Omega_1} = -a_2(u_2, R_2\mu) + (f, R_2\mu)_{\Omega_2} \quad \forall \mu \in \Lambda. \quad (9)$$

For given $\lambda \in \Lambda$ we now consider for $i = 1, 2$ the harmonic extensions $u_i^0 = H_i(\kappa_i(\lambda)) \in V_i$ of the Dirichlet boundary value $\kappa_i(\lambda)$ on Γ . (From now on, the brackets are mostly skipped to simplify the notation.) Furthermore let $u_i^* = \mathcal{G}_i f$ be the solutions of the subproblems (7) with homogeneous Dirichlet data $u_i^0|_{\partial\Omega} = 0$. Due to the linearity of the local problems (7) the functions $u_i = H_i\kappa_i\lambda + \mathcal{G}_i f$ satisfy (7)–(9) if and only if

$$\begin{aligned} & a_1(H_1\kappa_1\lambda, R_1\mu) + a_2(H_2\kappa_2\lambda, R_2\mu) = \\ & (f, R_1\mu)_{\Omega_1} - a_1(\mathcal{G}_1 f, R_1\mu) + (f, R_2\mu)_{\Omega_2} - a_2(\mathcal{G}_2 f, R_2\mu) \quad \forall \mu \in \Lambda. \end{aligned} \quad (10)$$

Since the extension operators R_i , $i = 1, 2$, can be chosen arbitrarily, we set $R_i = H_i$. Denoting by $\langle \cdot, \cdot \rangle$ the duality pairing between Λ' and Λ we recall the definition of the Steklov–Poincaré operators $S_i : \Lambda \rightarrow \Lambda'$:

$$\langle S_i \eta, \mu \rangle = a_i(H_i \eta, H_i \mu) \quad \forall \eta, \mu \in \Lambda, \quad i = 1, 2$$

and furthermore the functional $\chi = \chi_1 + \chi_2 \in \Lambda'$:

$$\langle \chi_i, \mu \rangle = (f, H_i \mu)_{\Omega_i} - a_i(\mathcal{G}_i f, H_i \mu) \quad \forall \mu \in \Lambda, \quad i = 1, 2.$$

Now (10) can be written as the nonlinear Steklov–Poincaré interface equation

$$\text{find } \lambda \in \Lambda : \quad (S_1\kappa_1 + S_2\kappa_2)\lambda = \chi \quad (11)$$

or equivalently

$$\text{find } \lambda_2 \in \Lambda : \quad (S_1 \kappa_1 \kappa_2^{-1} + S_2) \lambda_2 = \chi \quad (12)$$

if we set $\lambda_2 = \kappa_2 \lambda$. Note that if $\kappa_2^{-1} : \Lambda \rightarrow \Lambda$ is Lipschitz continuous the convergence of a sequence of iterates λ_2^k to λ_2 implies the convergence of $\lambda^k = \kappa_2^{-1} \lambda_2^k$ to λ .

We state the main result of this section:

Proposition 2. *Solving problem (4)–(6) is equivalent to solving the nonlinear Steklov–Poincaré equations (11) or (12).*

We point out at this stage that the reformulation of the problem (4)–(6) by Kirchhoff transformation is not only a powerful tool for the analysis of the problem that will be subject of the following two sections but also for its numerical treatment due to the linearity of transformed subproblems. In more complicated cases like the time-discretized Richards equation Kirchhoff transformation allows a reformulation of the quasilinear subproblems as convex minimization problems which can be solved efficiently and robustly using monotone multigrid methods [4].

3 Nonlinear Dirichlet–Neumann iteration

Now we consider the Dirichlet–Neumann algorithm applied to our problem (4)–(6). However, since it turns out that for a rigorous analysis the damping has to be carried out in the transformed variables, we state it for the transformed version (7)–(9):

Given $\lambda_2^0 \in \Lambda$, successively find $u_1^{k+1} \in V_1$ and $u_2^{k+1} \in V_2$ for each $k \geq 0$ such that

$$a_1(u_1^{k+1}, v_1) = (f, v_1)_{\Omega_1} \quad \forall v_1 \in V_1^0 \quad (13)$$

$$u_{1|\Gamma}^{k+1} = \kappa_1 \kappa_2^{-1}(\lambda_2^k) \quad \text{in } \Lambda \quad (14)$$

and

$$a_2(u_2^{k+1}, v_2) = (f, v_2)_{\Omega_2} \quad \forall v_2 \in V_2^0 \quad (15)$$

$$a_2(u_2^{k+1}, H_2 \mu) - (f, H_2 \mu)_{\Omega_2} = -a_1(u_1^{k+1}, H_1 \mu) + (f, H_1 \mu)_{\Omega_1} \quad \forall \mu \in \Lambda. \quad (16)$$

Then, with some damping parameter $\theta \in (0, 1)$, the new iterate is

$$\lambda_2^{k+1} := \theta u_{2|\Gamma}^{k+1} + (1 - \theta) \lambda_2^k. \quad (17)$$

Considering the harmonic extensions $H_i u_{i|\Gamma}^{k+1}$ and the solutions $\mathcal{G}_i f$ of the problems (7) with homogeneous boundary data for $i = 1, 2$ the intermediate iterates are obtained by

$$u_1^{k+1} = H_1 \kappa_1 \kappa_2^{-1} \lambda_2^k + \mathcal{G}_1 f \quad \text{and} \quad u_2^{k+1} = H_2 u_{2|_T}^{k+1} + \mathcal{G}_2 f.$$

Thus equation (16) provides

$$\begin{aligned} a_1(H_1 \kappa_1 \kappa_2^{-1} \lambda_2^k, H_1 \mu) + a_2(H_2 u_{2|_T}^{k+1}, H_2 \mu) \\ = \sum_{i=1}^2 (f, H_i \mu)_{\Omega_i} - a_i(\mathcal{G}_i f, H_i \mu) \quad \forall \mu \in \Lambda, \end{aligned}$$

which is the same as

$$\langle S_2 u_{2|_T}^{k+1}, \mu \rangle = \langle -S_1 \kappa_1 \kappa_2^{-1} \lambda_2^k + \chi, \mu \rangle \quad \forall \mu \in \Lambda$$

and regarding (17) altogether yields

$$S_2(\lambda_2^{k+1} - \lambda_2^k) = \theta(\chi - (S_1 \kappa_1 \kappa_2^{-1} + S_2) \lambda_2^k) \quad \text{in } \Lambda. \quad (18)$$

Consequently the damped Dirichlet–Neumann algorithm applied to (7)–(9) is a preconditioned Richardson procedure for the nonlinear Steklov–Poincaré formulation (12) with S_2 as a preconditioner.

Note that an analogous formulation for the interface equation (11) can't be obtained due to the nonlinearity of $S_2 \kappa_2$. However, (18) can be treated just as in the linear case if we apply the following generalization of an abstract convergence result in [5, pp. 118/9]. Let X be a Hilbert space, $Q_1 : X \rightarrow X'$ a not necessarily linear operator and let $Q_2 : X \rightarrow X'$ be linear and invertible. With the definition $Q := Q_1 + Q_2$ and for given $G \in X'$ we consider the problem

$$\text{find } \lambda \in X : \quad Q\lambda = G \quad (19)$$

together with the corresponding Richardson iteration

$$\lambda^{k+1} = \lambda^k + \theta(G - Q\lambda^k). \quad (20)$$

Theorem 1. *Let Q_2 be continuous and coercive, i.e. there are positive constants β_2 and α_2 such that*

$$\langle Q_2 \eta, \mu \rangle \leq \beta_2 \|\eta\|_X \|\mu\|_X \quad \forall \eta, \mu \in X \quad (21)$$

and

$$\langle Q_2 \eta, \eta \rangle \geq \alpha_2 \|\eta\|_X^2 \quad \forall \eta \in X. \quad (22)$$

Let Q_1 be Lipschitz continuous, i.e. there is a constant $\beta_1 > 0$ such that

$$\langle Q_1 \eta - Q_1 \mu, \lambda \rangle \leq \beta_1 \|\eta - \mu\|_X \|\lambda\|_X \quad \forall \eta, \mu, \lambda \in X. \quad (23)$$

Suppose there exists a constant $\kappa^* > 0$ such that

$$\langle Q_2(\eta - \mu), Q_2^{-1}(Q\eta - Q\mu) \rangle + \langle Q\eta - Q\mu, \eta - \mu \rangle \geq \kappa^* \|\eta - \mu\|_X^2 \quad \forall \eta, \mu \in X. \quad (24)$$

Then (19) has a unique solution $\lambda \in X$. Furthermore for any given $\lambda^0 \in X$ and any $\theta \in (0, \theta_{\max})$ with

$$\theta_{\max} := \frac{\kappa^* \alpha_2^2}{\beta_2(\beta_1 + \beta_2)^2}$$

the sequence given by (20) converges in X to λ .

The proof is an application of Banach's fixed point theorem and can be carried out along the lines of the one given in [5, pp. 118/9], see also [2]. More specifically, one proves that with the conditions (21)–(24) the operator $T_\theta : \Lambda \rightarrow \Lambda$ defined by

$$T_\theta : \eta \mapsto \eta + \theta Q_2^{-1}(G - Q\eta)$$

is a contraction with respect to the Q_2 -norm $\|\cdot\|_{Q_2}$ on Λ induced by the Q_2 -scalar product $(\cdot, \cdot)_{Q_2}$ given by

$$(\eta, \mu)_{Q_2} := \frac{1}{2}(\langle Q_2\eta, \mu \rangle + \langle Q_2\mu, \eta \rangle) \quad \forall \eta, \mu \in \Lambda.$$

Remark 1. Note that condition (24) reduces to a much simpler expression if Q_2 is symmetric. In the linear case (24) is just the coerciveness of Q_1 . In our nonlinear case, (24) states a uniform monotonicity of Q_1 of the form

$$\langle Q_1\eta - Q_1\mu, \eta - \mu \rangle \geq \frac{\kappa^*}{2} \|\eta - \mu\|_X^2 \quad \forall \eta, \mu \in X. \quad (25)$$

Now, it is well known that in the particular situation of (12) and (18) both Steklov–Poincaré operators S_1 and S_2 are symmetric, continuous and coercive. Thus in order to apply Theorem 1 to the case $X = \Lambda$, $G = \chi$, $Q_2 = S_2$ and $Q_1 = S_1\kappa_1\kappa_2^{-1}$ we have to make sure that the conditions (23) and (25) are satisfied for $Q_1 = S_1\kappa_1\kappa_2^{-1}$. So we arrive at the following

Theorem 2. *The nonlinear Steklov–Poincaré equation (12) has a unique solution λ_2 in Λ to which the nonlinear Dirichlet–Neumann scheme (13)–(17) converges for sufficiently small θ and any $\lambda_2^0 \in \Lambda$ if the following two conditions are satisfied:*

$\kappa_1\kappa_2^{-1} : \Lambda \rightarrow \Lambda$ is Lipschitz continuous, i.e., there is a constant $L(\kappa_1\kappa_2^{-1}) > 0$ such that

$$\|\kappa_1\kappa_2^{-1}\eta - \kappa_1\kappa_2^{-1}\mu\|_\Lambda \leq L(\kappa_1\kappa_2^{-1})\|\eta - \mu\|_\Lambda \quad \forall \eta, \mu \in \Lambda, \quad (26)$$

and $S_1\kappa_1\kappa_2^{-1} : \Lambda \rightarrow \Lambda'$ is a uniformly monotone operator, i.e. there is a constant $\alpha_1 > 0$ such that

$$\langle S_1(\kappa_1\kappa_2^{-1}\eta - \kappa_1\kappa_2^{-1}\mu), \eta - \mu \rangle \geq \alpha_1\|\eta - \mu\|_\Lambda^2 \quad \forall \eta, \mu \in \Lambda. \quad (27)$$

Proposition 3. *The conditions (26) and (27) are satisfied in one space dimension.*

Proof. Let $\Omega_1 = [a, b]$, $\Omega_2 = [b, c]$, with $\Gamma = \{b\}$ and $a < b < c$. Then we have $\Lambda = H_{00}^{1/2}(\Gamma) = H^{1/2}(\Gamma) \cong (\mathbb{R}, |\cdot|)$ and condition (26) follows from Proposition 1.

Let $L(\kappa_1^{-1})$ and $L(\kappa_2)$ be the Lipschitz constants of the real functions κ_1^{-1} and κ_2 according to Proposition 1. In order to prove (27), let $\eta, \mu, \lambda \in \mathbb{R}$. The harmonic extension $H_1(\lambda)$ is the affine function

$$x \mapsto \frac{\lambda}{b-a} x - \frac{\lambda}{b-a} a.$$

Consequently we have

$$\begin{aligned} & \langle S_1(\kappa_1 \kappa_2^{-1} \eta - \kappa_1 \kappa_2^{-1} \mu), \eta - \mu \rangle \\ &= \int_a^b \nabla H_1(\kappa_1 \kappa_2^{-1} \eta - \kappa_1 \kappa_2^{-1} \mu) \nabla H_1(\eta - \mu) dx \\ &= \int_a^b \frac{\kappa_1 \kappa_2^{-1} \eta - \kappa_1 \kappa_2^{-1} \mu}{b-a} \cdot \frac{\eta - \mu}{b-a} dx \\ &= \frac{(\kappa_1 \kappa_2^{-1} \eta - \kappa_1 \kappa_2^{-1} \mu)(\eta - \mu)}{b-a} \\ &\geq \frac{1}{(b-a)L(\kappa_1^{-1})L(\kappa_2)} |\eta - \mu|^2. \end{aligned}$$

For the estimate note that both κ_1^{-1} and κ_2 are monotonically increasing.

Remark 2. Unfortunately, in higher dimensions condition (27) is violated to the extent that $\langle S_1(\kappa_1 \kappa_2^{-1} \eta - \kappa_1 \kappa_2^{-1} \mu), \eta - \mu \rangle$ can be negative. A counterexample in 2D is easily obtained by considering a harmonic function u with $\frac{\partial u}{\partial \mathbf{n}} \cdot u \leq c < 0$ on a subset of Γ with positive Hausdorff measure (see [2]).

As we have already pointed out, the damped Dirichlet–Neumann method applied to the original problem (4)–(6) or, equivalently, (11) produces a sequence $\tilde{\lambda}^k$ which differs from $\lambda^k = \kappa_2^{-1} \lambda_2^k$ in general. It is not clear whether this sequence also converges (for sufficiently small damping parameter) once (26) and (27) are satisfied. In 1D, however, one can prove that this is true. The proof is based on the above convergence result and the Lipschitz continuity of κ_2 and κ_2^{-1} (see [2]).

4 Numerical example

In this section we apply our nonlinear Dirichlet–Neumann method to a problem in two space dimensions: We consider the transmission problem (1)–(3) on the Yin Yang domain Ω as shown in Figure 2. We denote the white subdomain together with the grey circle B_1 by Ω_1 and the grey subdomain with the white circle B_2 by Ω_2 .

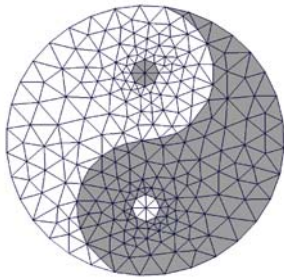


Fig. 2. Yin Yang domain Ω .

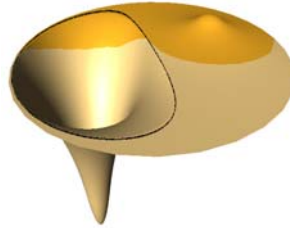


Fig. 3. Solution p on Ω with free boundary (black line).

We select the data

$$f(x) = \begin{cases} -1 & \text{on } B_1 \\ 1 & \text{on } B_2 \\ 0 & \text{on } \Omega \setminus (B_1 \cup B_2) \end{cases}$$

and the nonlinearities

$$k_i(p_i) = \begin{cases} K_h \max\{p_i^{3\lambda_i+2}, c\} & \text{for } p_i \leq -1 \\ 1 & \text{for } p_i \geq -1. \end{cases}$$

This choice is motivated by the well-known state equations of Brooks–Corey and Burdine (cf. [6]) for the hydraulic conductivity of a saturated/unsaturated porous media with different soils. In this way, our model problem can be regarded as a stationary Richards equation. Note that $p_i < -1$ characterizes the unsaturated region which is separated by a free boundary from the linear, saturated regime occurring for $p_i \geq -1$. The parameters λ_1 and λ_2 in Ω_1 resp. Ω_2 are called pore size distribution factors. We chose them in an extreme manner as $\lambda_1 = 1.0$ (very coarse sand) and $\lambda_2 = 0.1$ (fine clay). The factor $K_h = 0.002$ is a realistic hydraulic conductivity in the case of full saturation. The parameter $c = 0.1 > 0$ is introduced to enforce ellipticity.

The choice of the data f which result in a strong sink on B_1 and a strong source on B_2 and our special choice of $\bar{\mathcal{D}}_1$ and $\bar{\mathcal{D}}_2$ ensure that the free boundary has a nontrivial intersection with the interface $\Gamma = \bar{\mathcal{D}}_1 \cap \bar{\mathcal{D}}_2$ (see the numerical solution as shown in Figure 3). Since we apply the Dirichlet–Neumann scheme (13)–(17) we hereby make sure that step (14) is nonlinear.

We discretize the problem on the two subdomains using piecewise linear finite element spaces. The linear problems on the subdomains are solved by a linear multigrid method. Figure 4 shows the convergence rate ρ in the energy norm (for the transformed variables u_i^k) with respect to the damping parameter θ . We use a grid hierarchy of six levels as resulting from uniform mesh refinement of the coarse grid depicted in Figure 2. In this way, we obtain about 235.000 nodes on the finest mesh. Figure 5 shows the convergence rate

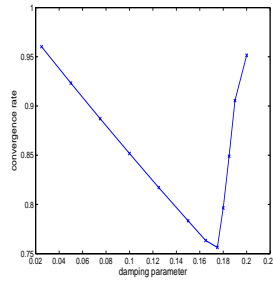


Fig. 4. ρ vs. damping parameter θ .

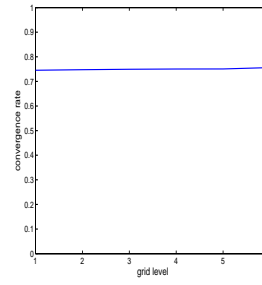


Fig. 5. ρ vs. mesh size h .

over the refinement levels. The damping parameter $\theta_{opt} = 0.175$ is obtained from Figure 4. For this optimal choice, we observe mesh-independence of the convergence speed.

References

1. H.W. Alt and S. Luckhaus. Quasilinear elliptic–parabolic differential equations. *Math. Z.*, 183:311–341, 1983.
2. H. Berninger. *Domain Decomposition Methods for Problems with Jumping Nonlinearities*. PhD thesis, Freie Universität Berlin, in preparation.
3. H. Berninger and M. Discacciati. Nonlinear robin algorithms for nonlinear transmission problems. In preparation.
4. R. Kornhuber. On constrained newton linearization and multigrid for variational inequalities. *Numer. Math.*, 91:699–721, 2002.
5. A. Quarteroni and A. Valli. *Domain Decomposition Methods for Partial Differential Equations*. Oxford Science Publications, 1999.
6. M.T. van Genuchten. A closed–form equation for predicting the hydraulic conductivity of unsaturated soils. *SOIL SCI. SOC. AM. J.*, 44:892–898, 1980.