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OPTIMAL CONTROL OF MULTISCALE SYSTEMS USING REDUCED-ORDER MODELS

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Abstract. We study the optimal control problem for diffusions with slow and fast variables and address a question raised by practitioners: is it possible to first eliminate the fast variables before solving the optimal control problem and then use the optimal control, computed from the reduced-order model to control the original, high-dimensional system? The strategy "first reduce, then optimize"—rather than "first optimize, then reduce"—is motivated by the fact that solving optimal control problems for high-dimensional multiscale systems is numerically challenging and often computationally prohibitive. We state sufficient and necessary conditions, under which the "first reduce, then control" strategy can be employed and discuss when it should be avoided. We further give numerical examples that illustrate the "first reduce, then optmize" approach and discuss possible pitfalls.

1. **Introduction.** Optimal control problems for diffusion processes have attracted a lot of attention in the last decades, both in terms of the development of the theory as well as in terms of concrete applications to problems in the sciences, engineering and finance [20, 39]. Stochastic control problems appear in a variety of applications, such as statistics [17, 16], financial mathematics [15, 54], molecular dynamics [56, 28] and materials science [58, 6], to mention just a few. A common feature of the models used is that they are high-dimensional and possess several characteristic time scales. For instance, in single molecule alignment experiments, a laser field is used to stabilize the slowly-varying orientation of a molecule in solution that is coupled to the fast internal vibrations of the molecule, but ideally the controller would like to base the control protocol only on the relevant slow degree of freedom, i.e. the orientation of the molecule [57].

If the time scales in the system are well separated, it is possible to eliminate the fast degrees of freedom and to derive low-order reduced models, using averaging and homogenization techniques [52]. Homogenization of stochastic control systems has been extensively studied by applied analysts using a variety of different mathematical tools, including viscosity solutions of the Hamilton-Jacobi-Bellman equation [8, 18, 1, 42], backward stochastic differential equations [11, 12, 31], Gamma convergence [41, 46] and occupation measures [37, 38, 36]. The latter has been also employed to analyse deterministic control systems, together with differential inclusion techniques [21, 59, 24, 5, 60]. The convergence analysis of multiscale control systems, both deterministic and stochastic, is quite involved and non-constructive, in that the limiting equations of motion are not given in explicit or closed form; see [35, 22, 33] for notable exceptions, dealing mainly with the case when the dynamics

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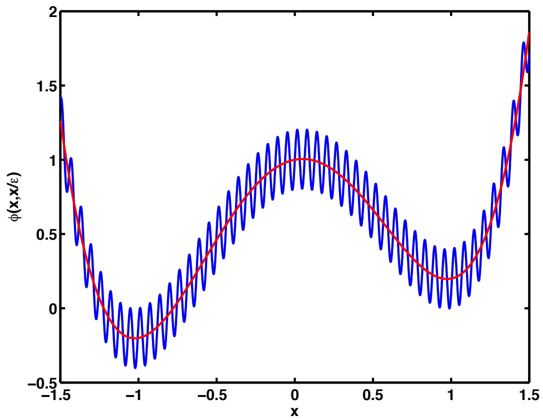


FIGURE 1. Bistable potential (shown in red) with superimposed small-scale oscillations of period ϵ (in blue).

is linear. We shall refer to all these approaches—without trying to be exhaustive—as “first optimize, then reduce”.

On the other side of the spectrum are model order reduction (MOR) techniques for large-scale linear and bilinear control systems that are based on tools from linear algebra and rational approximation. MOR aims at approximating the response of a controlled system to any given control input from a certain class, e.g., piecewise constant or square integrable functions; see, e.g., [25, 4] and the references given there. A very popular MOR method is balanced truncation that gives easily computable error bounds in terms of the Hankel norm of the corresponding transfer functions [44, 23], and which has recently been extended to deterministic and stochastic slow-fast systems, using averaging and homogenization techniques [29, 26, 27]. In applications MOR is often used to drastically reduce the system dimension, before a possibly computational expensive optimal control problem is solved. In most real-world applications, solving an optimal control problems on the basis of the unreduced large-scale model is prohibitive, which explains the popularity of MOR techniques. We will call this approach “first reduce, then optimize”.

1.1. **The first homogenization approach: first reduce, then optimize.** In this paper we focus on optimal control of diffusions with two characteristic time scales. As a representative example, we consider the diffusion of a driven Brownian particle in a two-scale energy landscape in one dimension

$$dx^\epsilon_s = (\sigma u^\epsilon_s - \nabla\Phi(x^\epsilon_s, x^\epsilon_s/\epsilon))\, ds + \sigma\beta^{-1/2} dw_s\,, \tag{1}$$

where u^ϵ is any time-dependent driving force (or control variable) and w_t is standard one-dimensional Brownian motion. The potential consists of a large metastable part with small-scale superimposed periodic fluctuations, $\Phi(x, y) = \Phi_0(x) + p(y)$ with $p(\cdot)$ a 1-periodic function. A typical potential is shown in Figure 1.

Now if u^ϵ is given as a function of time, say bounded and continuous, it is known that x^ϵ_s converges in distribution to a limiting process x_s as $\epsilon \to 0$, where x_s solves the homogenized equation [53]

$$dx_s = (\sigma A u_s - A\nabla\Phi_0(x_s))\, ds + \sqrt{A}\beta^{-1/2}\, dw_s\,. \tag{2}$$

Here $0 < A < 1$ is an effective diffusivity that accounts for the slowing down of the dynamics due to the presence of local minima in the two-scale potential. The property that x^ϵ weakly converges to x in the sense of probability measures will be referred to as *forward stability* of the homogenized equation. Now imagine a situation, in which u^ϵ depends on x_s^ϵ via a feedback law

$$u_s^\epsilon = c(x_s^\epsilon; \epsilon), \tag{3}$$

where $c(\cdot; \epsilon)$ is a measurable function of x. (For simplicity, we do not consider the case that c carries an explicit time-dependence.) Specifically, we choose u from an admissible class of feedback controls so that the cost functional

$$J(u^\epsilon) = \mathbf{E}\left(\int_0^\tau L(x_s^\epsilon, u_s^\epsilon)\, ds\right)$$

is minimized for some given running cost $L \geq 0$ associated with the sample paths of x_s^ϵ and u_s^ϵ up to a random stopping time τ of the process.

The aim of the paper is to study situations where the cost functional evaluated at u^ϵ, converges to $J(u)$, with u being the limit of u^ϵ (in some appropriate sense). Specifically, we are dealing with the situation that

$$\inf_u J(u^\epsilon) \to \inf_u J(u)\,,$$

a property that we will refer to as *backward stability.* If the homogenized equation is backward stable, it does not matter whether one first solves the optimal control problem and then sends ϵ to 0 or vice versa, in which case the control u is simply treated as a parameter. One of the implications then is that we can compute optimal controls from the homogenized model, such as (2), and use them in the original equation when ϵ is sufficiently small.

Unfortunately very few systems are backward stable in this sense, a notable exception being a system of the form (1) when the running cost L is quadratic in u, e.g. [38, Sec. 4.1]. It is reasonable to ask whether it is better to first reduce the equations **before** solving the optimal control problem, rather than the other way round. One possible answer is that solving optimal control problems for high-dimensional multiscale systems may be computationally too expensive; another answer is that there may be situations, in which a fully resolved model may not be explicitly available, and one only has a sufficiently accurate low-order model that captures the relevant dynamics of the system. In both cases one wants to make sure that the controls obtained from the low-order reduced model can be used in order to control the original system.

1.2. **Mathematical justification of the first homogenization approach.** In this article we consider the exceptional cases of backward stability and give necessary and sufficient conditions under which the reduced systems (disregarding the control) are indeed backward stable. It turns out that a class of optimal control problems that are backward stable are systems that are linear-quadratic in the control variable; they may be nonlinear in the state variables, though, and therefore cover many relevant applications in the sciences and engineering. Moreover we find that an additional requirement is that the controls of the multiscale system converge in a strong sense; an example of weak convergence, in which the systems fails to be backward stable due to lack of sequence continuity, is when the controls are oscillatory with rate $1/\epsilon$ around its homogenization limit, in case of which $J^\epsilon(u^\epsilon)$ does not converge to $J(u)$ unless J is linear in u. For a related discussion of weak convergence issues in optimal control, we refer to [2, 3]. Similar problems for parameter estimation and filtering are discussed in [22, 53, 51, 32, 50].

Strong convergence of the control is a necessary, but not sufficient condition for backward stability of the first homogenization (first reduce, then optimize) approach, in which the control variable is treated as a parameter during the homogenization procedure. The class of control problems, which can be homogenized in the above way are systems of SDEs that can be transformed to systems in which the controls are absent. The class of such systems are linear-quadratic in the controls (but possibly nonlinear in the states), and the transformation that does the job is a logarithmic transformation of the value function of the optimal control problem:

$$V^\epsilon(x) = \inf_{u^\epsilon} \mathbf{E}\left(\int_0^\tau L(x_s^\epsilon, u_s^\epsilon)\, ds \middle| x_0^\epsilon = x \right) .$$

It can be shown (see [20]) that the transformed value function solves a linear boundary value problem that does not involve any control variables and can be homogenized using standard techniques. Once the linear equation has been homogenized, it can be transformed back to an equivalent optimal control problem that is precisely the limiting equation of the original multiscale control problem. A nice feature of the logarithmic transformation approach is that the optimal control can be expressed in terms of the solution of the linear boundary value problem, which can be solved efficiently using Monte-Carlo methods. This approach is helpful when the dynamics are high-dimensional and any grid-based discretization of the above linear boundary value problem is prohibitive. (The case when the stopping time τ is deterministic and the log-transformed value function solves a linear transport PDE can be treated analogously.) Our approach is summarized in Table 1.

$$\begin{tikzcd}
V^\epsilon = \min_u J^\epsilon(u) \arrow[r, "\psi^\epsilon = \exp(-\beta V^\epsilon)"] \arrow[d, "\epsilon \to 0"'] & \text{linear PDE for } \psi^\epsilon \arrow[d, "\epsilon \to 0"] \\
V = \min_u J(u) & \text{linear PDE for } \psi \arrow[l, "V = -\beta^{-1} \log \psi"']
\end{tikzcd}$$

Table 1. Schematic approach of the homogenization procedure using logarithmic transformation.

This paper is organized as follows: In Section 2 the model reduction approach for the indefinite time-horizon control problem with multiple time scales is outlined, with a brief introduction to dynamic programming and logarithmic transformations in Section 2.1. The model reduction problem is illustrated in Section 3 with three different numerical examples: underdamped motion of Langevin-type (Sec. 3.1), diffusion in a highly-oscillatory potential (Sec. 3.2), and the Gaussian linear quadratic regulator (Sec. 3.3). The article contains three appendices: Appendix A discusses weak convergence under logarithmic transformations, Appendix B introduces the infinite time-horizon problem associated with the linear quadratic regulator example, Appendix C contains the proof of Theorem 3 and records various identities to bound the cost functional and the value function when using suboptimal controls.

2. Multiscale control problem.

We start by setting the notation which we will use throughout this article. We denote by $O \subset \mathbb{R}^n$ a bounded open set with sufficiently smooth boundary ∂O. Further let $(z_s^{\epsilon,u})_{s\geq 0}$ be a stochastic process assuming values in $\mathbb{R}^n$ that is the solution of

$$dz_s^{\epsilon,u} = (b(z_s^{\epsilon,u};\epsilon) + \sigma(z_s^{\epsilon,u};\epsilon)u_s^\epsilon)\,ds + \sigma(z_s^{\epsilon,u};\epsilon)\beta^{-1/2}dw_s\,, \tag{4}$$

where $u_s^\epsilon \in U \subseteq \mathbb{R}^n$ is the control applied at time s and $w = (w_s)_{s\geq 0}$ is n-dimensional Brownian motion and $\beta > 0$ is the (dimensionless) inverse temperature of the system. We assume that, for each $\epsilon > 0$, drift and noise coefficients, $b(\cdot;\epsilon)$ and $\sigma(\cdot;\epsilon)$, are continuous functions on O, satisfying the usual Lipschitz and growth conditions that guarantees existence and uniqueness of the process [48].

Cost functional. We want to control (4) in such a way that an appropriate cost criterion is minimized where the control is active until the process leaves the set O. Assuming $z_0^{\epsilon,u} = z \in O$, we define τ to be the stopping time

$$\tau = \inf\{s > 0\,;\, z_s^{\epsilon,u} \notin O\}\,, \tag{5}$$

i.e., τ is the first exit time of the process $z_s^{\epsilon,u}$ from O. Our cost criterion reads

$$J(u^\epsilon; z) = \mathbf{E}\left(\int_0^\tau L(z_s^{\epsilon,u}, u_s^\epsilon)\,ds \;\middle|\; z_0^{\epsilon,u} = z\right) \tag{6}$$

where L is the running cost that we assume to be of the form

$$L(z,u) = G(z) + \frac{1}{2}|u|^2\,, \tag{7}$$

with G being continuous on O. Note that the ϵ-dependence of the cost functional J comes only through the dependence of the control on $z_s^{\epsilon,u}$. We will omit the dependence on z in $J(u;z)$ and write it as $J(u)$ whenever there is no ambiguity.

2.1. Logarithmic transformation.

In order to pass to the limit in (4)–(7), we resort to the technique of logarithmic transformations that has been developed by Fleming and co-workers (see [20] and the references therein). We start by recalling the dynamic programming principle for stochastic control problems of the form (4)–(7). To this end we make the following assumptions (see [20, Secs. VI.3–5] for further details on the first two of the following assumptions) :

Assumption 1. For every $\epsilon > 0$, the matrices $a(\cdot;\epsilon) = \sigma(\cdot;\epsilon)\sigma(\cdot;\epsilon)^T$ are positive definite with uniformly bounded inverse $a(\cdot;\epsilon)^{-1}$.

Assumption 2. The running cost $G(z)$ is continuous, nonnegative, and $G(z) \leq M_1$ for all $z \in \bar{O}$ with bounded first order partial derivatives in z.

Assumption 3. There exist constants $\gamma, C_1 > 0$, which are independent of ϵ, such that $\mathbf{E}(\exp(\gamma\tau)|z_0^\epsilon = z) \le C_1 < +\infty$.

We define the generator of the dynamics $z_s^{\epsilon,u}$:

$$\mathcal{L}^\epsilon(u)\psi = \frac{1}{2\beta} a(z;\epsilon) : \nabla^2\psi + (\sigma(z;\epsilon)u^\epsilon + b(z;\epsilon)) \cdot \nabla\psi\,.$$

Notice that the generator depends on the control u. When the control is absent we will use the notation $\mathcal{L}^\epsilon = \mathcal{L}^\epsilon(0)$. The next result is standard (e.g., see [20, Sec. IV.2])) and stated without proof.

Theorem 1. *Let $V^\epsilon \in C^{1,2}(O) \cap C(\bar{O})$ be the solution of the Hamilton-Jacobi-Bellman (HJB) equation*

$$\begin{aligned} 0 &= \min_{c\in\mathbb{R}^n} \left\{\mathcal{L}^\epsilon(c)V^\epsilon + L(z,c)\right\}, \\ 0 &= V^\epsilon|_{\partial O}\,. \end{aligned} \tag{8}$$

Then

$$V^\epsilon(z) = \min_u J^\epsilon(z;u^\epsilon)\,,$$

where the minimum goes over all admissible feedback controls of the form $u_s^\epsilon = c(z_s^{\epsilon,u}, s\ ;\epsilon)$. The minimizer is unique and is given by the feedback law

$$\hat{u}^\epsilon = -\sigma(z;\epsilon)^T\nabla V^\epsilon(z) = \operatorname*{argmin}_{c\in\mathbb{R}^n} \left\{\mathcal{L}^\epsilon(c)V^\epsilon + L(z,c)\right\}. \tag{9}$$

The function V^ϵ is called *value function* or *optimal cost-to-go*. The homogenization problem for (4)–(7) can be studied using a multiscale expansion of the nonlinear PDE (8) in terms of the small parameter ϵ; see, e.g., [7, 38]. In this article we remove the nonlinearity from the equation by means of a logarithmic transformation of the value function. Specifically, let

$$\psi^\epsilon(z) = e^{-\beta V^\epsilon(z)}\,.$$

By chain rule,

$$\beta^{-1}e^{\beta V^\epsilon}\mathcal{L}^\epsilon e^{-\beta V^\epsilon} = -\mathcal{L}^\epsilon V^\epsilon + \frac{1}{2}|\sigma^T\nabla V^\epsilon|^2\,,$$

which, together with the relation

$$-\frac{1}{2}|\sigma^T\nabla V^\epsilon|^2 = \min_{c\in\mathbb{R}^n}\left\{\sigma c\cdot\nabla V^\epsilon + \frac{1}{2}|c|^2\right\},$$

implies that (8) is equivalent to the linear boundary value problem

$$\begin{aligned} (\mathcal{L}^\epsilon - \beta G)\,\psi^\epsilon &= 0\,, \\ \psi^\epsilon|_{\partial O} &= 1\,, \end{aligned} \tag{10}$$

for the function ψ^ϵ. By the Feynman-Kac formula, (10) has an interpretation as a control-free sampling problem (see [48, Thm. 8.2.1]):

$$\psi^\epsilon(z) = \mathbf{E}\left(\exp\left(-\beta\int_0^\tau G(z_s^\epsilon)\,ds\right)\,\middle|\, z_0^\epsilon = z\right), \tag{11}$$

where z_s^ϵ solves the control-free SDE

$$dz_s^\epsilon = b(z_s^\epsilon;\epsilon)\,ds + \sigma(z_s^\epsilon;\epsilon)\beta^{-1/2}dw_s\,.$$

Equations (8)–(11) express a Legrendre-type duality between the value of an optimal control problem and cumulant generating functions [14, 20]:

$$V^\epsilon = -\beta^{-1} \log \psi^\epsilon \,. \tag{12}$$

In other words,

$$\begin{aligned} -\beta^{-1} \log \mathbf{E} \left(\exp\left(-\beta \int_0^\tau G(z_s^\epsilon) ds \right) \;\middle|\; z_0^\epsilon = z \right) \\ = \inf_{u_s} \mathbf{E} \left(\int_0^\tau L(z_s^{\epsilon,u}, u_s) ds \;\middle|\; z_0^{\epsilon,u} = z \right), \end{aligned}$$

where $z_s^{\epsilon,u}$ satisfies the controlled SDE (4) and $z_s^\epsilon = z_s^{\epsilon,0}$.

By the above assumptions and the strong maximum principle for elliptic PDEs it follows that (10) has a classical solution $\psi^\epsilon \in C^{1,2}(O) \cap C(\bar{O})$. Moreover, combining Assumption 3, (11) and Hölder's inequality, we have that

$$\psi^\epsilon \geq \mathbf{E}(\exp(-\beta M_1 \tau) | z_0^\epsilon = z)$$

and

$$\mathbf{E}(\exp(-\beta M_1 \tau) | z_0^\epsilon = z)^{1/p} \mathbf{E}(\exp(\gamma \tau) | z_0^\epsilon = z)^{1/q} \geq 1$$

where $p = \beta M_1/\gamma + 1$ and $q = \gamma/(\beta M_1) + 1$, and thus

$$0 < C_2 \leq \psi^\epsilon \leq 1, \quad \epsilon > 0$$

for a constant $C_2 = C_1^{-\beta M_1/\gamma}$ that is independent of ϵ.

Remark 2. *In the course of the paper we will drop the assumption that the operator $\mathcal{L}^\epsilon$ is uniformly elliptic and instead require only that is hypoelliptic* [43]*. In this case the matrix $\sigma\sigma^T$ can be semidefinite, if the vector field b satisfies an additional controllability assumption, known as Hörmander's condition* [10]*, which guarantees that the transition probability has a strictly positive density with respect to Lebesgue measure, in which case (10) and therefore (8) have classical solutions; cf.* [20, Sec. IV].

2.2. **Homogenization problem.** We now specify the class of multiscale systems considered in this article. Specifically, we address slow-fast systems of the form

$$dx_s^\epsilon = \left(\frac{1}{\epsilon} f_0(x^\epsilon, y^\epsilon) + f_1(x^\epsilon, y^\epsilon) \right) ds + \beta^{-1/2} \alpha_1(x^\epsilon, y^\epsilon) dw_s^1, \tag{13a}$$

$$dy_s^\epsilon = \left(\frac{1}{\epsilon^2} g_0(x^\epsilon, y^\epsilon) + \frac{1}{\epsilon} g_1(x^\epsilon, y^\epsilon) \right) ds + \frac{\beta^{-1/2}}{\epsilon} \alpha_2(x^\epsilon, y^\epsilon) dw_s^2 \,, \tag{13b}$$

together with an exponential expectation

$$\psi^\epsilon(x, y) = \mathbf{E} \left(\exp\left(-\beta \int_0^\tau G(x_s^\epsilon, y_s^\epsilon)\; ds \right) \;\middle|\; x_0^\epsilon = x, y_0^\epsilon = y \right). \tag{14}$$

Letting $\mathcal{L}^\epsilon$ denote the infinitesimal generator of (13), it holds that

$$(\mathcal{L}^\epsilon - \beta G)\, \psi^\epsilon = 0, \tag{15}$$

where

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon^2} \mathcal{L}_0 + \frac{1}{\epsilon} \mathcal{L}_1 + \mathcal{L}_2 \,,$$

with

$$\begin{aligned}
\mathcal{L}_0 &= g_0 \cdot \nabla_y + \frac{1}{2}\beta^{-1}\alpha_2\alpha_2^T : \nabla_y^2,\\
\mathcal{L}_1 &= f_0 \cdot \nabla_x + g_1 \cdot \nabla_y,\\
\mathcal{L}_2 &= f_1 \cdot \nabla_x + \frac{1}{2}\beta^{-1}\alpha_1\alpha_1^T : \nabla_x^2\,.
\end{aligned}$$

Let us assume that ψ^ϵ admits the following perturbation expansion in powers of ϵ:

$$\psi^\epsilon = \psi_0 + \epsilon\psi_1 + \epsilon^2\psi_2 + \cdots\,.$$

By substituting the ansatz into (15) and comparing different powers of ϵ we obtain a hierarchy of equations, the first three of which are

$$\begin{aligned}
&\mathcal{L}_0\psi_0 = 0,\\
&\mathcal{L}_0\psi_1 = -\mathcal{L}_1\psi_0,\\
&\mathcal{L}_0\psi_2 = -\mathcal{L}_1\psi_1 - \mathcal{L}_2\psi_0 + \beta G\psi_0\,.
\end{aligned} \tag{16}$$

We suppose that for each fixed x, the dynamics (13b) of the fast variables are ergodic, with the unique invariant density $\rho_x(y)$. Then by construction ρ_x is the unique solution of the equation $\mathcal{L}_0^*\rho_x(y) = 0$, which together with the first equation of (16) implies that ψ_0 is independent of y. In order to proceed, we further assume that $f_0(x,y)$ satisfies the *centering condition*:

$$\int f_0(x,y)\rho_x(y)\,dy = 0\,.$$

The centering conditions, together with the strong maximum principle implies that the solution of the cell problem

$$\mathcal{L}_0\Theta(x,y) = -f_0(x,y)\,, \qquad \int \Theta(x,y)\rho_x(y)\,dy = 0 \tag{17}$$

is unique, with $\psi_1(x,y) = \Theta(x,y)\cdot\nabla_x\psi_0(x)$. Multiplying $\rho_x(y)$ on both sides of the third equation in (16) and integrating with respect to y, we obtain

$$\bar{\mathcal{L}}\psi_0 - \beta\bar{G}\psi_0 = 0, \tag{18}$$

where

$$\bar{\mathcal{L}} = \bar{f}(x)\cdot\nabla_x + \frac{1}{2}\beta^{-1}\bar{\alpha}\bar{\alpha}^T : \nabla_x^2, \tag{19}$$

with

$$\begin{aligned}
\bar{f}(x) &= \int \left[\nabla_x\Theta(x,y)f_0(x,y) + \nabla_y\Theta(x,y)g_1(x,y) + f_1(x,y)\right]\rho_x(y)\,dy,\\
\bar{G}(x) &= \int G(x,y)\rho_x(y)\,dy,\\
\bar{\alpha}(x)\bar{\alpha}(x)^T &= \int \Big[\beta\left(\Theta(x,y)f_0(x,y)^T + f_0(x,y)\Theta(x,y)^T\right)\\
&\quad + \alpha_1(x,y)\alpha_1(x,y)^T\Big]\rho_x(y)\,dy\,.
\end{aligned} \tag{20}$$

Homogenized control system. It follows using standard homogenization theory for linear elliptic equations (e.g. [49, 52]) that for $\epsilon \to 0$ the solution of (15) converges to the leading term of the asymptotic expansion:

$$\psi_0(x) = \mathbf{E}\left(\exp\left(-\beta \int_0^\tau \bar{G}(x_s)\, ds \right) \,\middle|\, x_0 = x \right), \tag{21}$$

where x_s is the solution of the homogenized SDE

$$dx_s = \bar{f}(x_s)ds + \bar{\alpha}(x_s)\beta^{-1/2}dw_s\,, \tag{22}$$

with coefficients as given in (20).

The corresponding asymptotic expansion of the value function V^ϵ for $\epsilon \to 0$ is obained by the logarithmic transformation (12):

$$V^\epsilon = -\beta^{-1}\log(\psi_0 + \epsilon\psi_1 + o(\epsilon)) = -\beta^{-1}\log\psi_0 - \beta^{-1}\frac{\psi_1}{\psi_0}\epsilon + o(\epsilon).$$

Therefore, using the ansatz $V^\epsilon = V_0 + \epsilon V_1 + \epsilon^2 V_2 + \cdots$ it follows that

$$V_0 = -\beta^{-1}\log\psi_0, \quad V_1 = -\beta^{-1}\frac{\psi_1}{\psi_0}.$$

Using the duality between value function and cumulant generating function, we conclude that V_0 is the value function of the optimal control problem

$$V_0(x) = \inf_u \mathbf{E}\left(\int_0^\tau \left[\bar{G}(x_s^u) + \frac{1}{2}|u_s|^2 \right] ds \,\middle|\, x_0^u = x \right),$$

where the minimization is subject to the homogenized dynamics

$$dx_s^u = (\bar{f}(x_s^u) + \bar{\alpha}(x_s^u)u_s)ds + \bar{\alpha}(x_s^u)\beta^{-1/2}dw_s\,. \tag{23}$$

According to (9), the optimal feedback law for the homogenized problem reads

$$\hat{u}_t = -\bar{\alpha}(x_t^u)^T \nabla V_0(x_t^u). \tag{24}$$

2.3. **Control of the full dynamics using reduced models.** Our goal is to find the optimal control policy $\hat{u}^\epsilon = (\hat{u}^{1,\epsilon}, \hat{u}^{2,\epsilon})$ for the fast/slow system (13) for $\epsilon \ll 1$. Using Theorem 1 and the asymptotic expansion of V^ϵ, we have

$$\begin{aligned} \hat{u}^{1,\epsilon} &= -\alpha_1^T \nabla_x V^\epsilon = -\alpha_1^T \nabla_x V_0 + \mathcal{O}(\epsilon), \\ \hat{u}^{2,\epsilon} &= -\frac{1}{\epsilon}\alpha_2^T \nabla_y V^\epsilon = -\alpha_2^T \nabla_y V_1 + \mathcal{O}(\epsilon) = -\alpha_2^T \nabla_y \Theta \nabla_x V_0 + \mathcal{O}(\epsilon). \end{aligned} \tag{25}$$

Notice that the leading terms in (25) are related to the value function of optimal control problem for the reduced SDE. This indicates that we may design the control policy from the reduced problem and use it to control the original multiscale equation. This assertion is justified by the following result for the general optimal control problem (4)–(7).

Theorem 3. *Let Assumptions 1,2 and 3 hold and, furthermore, suppose that $\epsilon < (\gamma/\beta)^{1/2}$ and $|u_t - \hat{u}_t| \le \epsilon$ uniformly in t. Then we have*

$$|J(u) - J(\hat{u}^\epsilon)| \le C\epsilon^2. \tag{26}$$

The proof of this result can be found in Appendix C.

Upon combining the above theorem with the formula for the optimal control policy in (25) we conclude that when the two time scales in the system are well separated, $\epsilon \ll 1$, the optimal control policy is well approximated by the leading order terms in (25) and result in a cost value that is nearly optimal.

Remark 4. *All considerations in this paper readily generalize to the averaging problem, i.e. when $f_0 = g_1 = 0$ in (13). This is not surprising since for averaging problems strong convergence is expected to hold (when the diffusion coefficient α_1 in (13) is independent of the fast variable y). This is similar to the convergence of the maximum likelihood function of the averaging problem, see* [50].

3. Three prototypical applications.

In this section we apply the results presented in the previous section to three typical multiscale models. For each model we first reformulate it as the optimal control problem studied in the previous section, then we consider the asymptotic limits of the value function and of the optimal control policy explicitly. The first two examples are taken from [50], while the third can be found in [25].

3.1. Overdamped Langevin equation.

We consider the second-order Langevin equation

$$\epsilon^2 \frac{d^2 x^\epsilon}{ds^2} = -\frac{dx^\epsilon}{ds} - \nabla\Phi(x^\epsilon) + \sqrt{2}\beta^{-1/2}\frac{dw}{ds}, \tag{27}$$

where $\epsilon \ll 1$, $x \in \mathbb{R}^n$, $\beta > 0$, and Φ being a smooth the potential energy function. Introducing the auxiliary variable y^ϵ we can recast (27) as

$$\frac{dx^\epsilon}{ds} = \frac{1}{\epsilon} y^\epsilon, \tag{28a}$$

$$dy^\epsilon = -\left(\frac{1}{\epsilon}\nabla\Phi(x^\epsilon) + \frac{1}{\epsilon^2} y^\epsilon\right) dt + \frac{\sqrt{2}}{\epsilon}\beta^{-1/2} dw\,. \tag{28b}$$

We see the solution of the optimal control problem

$$V^\epsilon(x,y) = \inf_{u^\epsilon} \mathbf{E}\left(\int_0^\tau \left[G(x_s^{\epsilon,u}) + \frac{1}{2}|u_s^\epsilon|^2\right] ds \;\middle|\; x_0^{\epsilon,u} = x, y_0^{\epsilon,u} = y\right) \tag{29}$$

under the controlled Langevin dynamics

$$\frac{dx_s^{\epsilon,u}}{ds} = \frac{1}{\epsilon} y_s^{\epsilon,u}, \tag{30a}$$

$$dy_s^{\epsilon,u} = \left(\frac{\sqrt{2}}{\epsilon} u_s^\epsilon - \frac{1}{\epsilon}\nabla\Phi(x_s^{\epsilon,u}) - \frac{1}{\epsilon^2} y_s^{\epsilon,u}\right) dt + \frac{\sqrt{2}}{\epsilon}\beta^{-1/2} dw. \tag{30b}$$

We notice that (28) is somewhat different to the form specified in Section 2, since there is no noise and hence no control term in the equation for x^ϵ. The infinitesimal generator correpsonding to (28) is hypoelliptic (rather than elliptic). Yet the standard homogenization arguments apply, for here the fast variable is y and the noise is acting uniformly in y. As a consequence the generator of the fast dynamics is uniformly elliptic, ans hence the standard theory applies. Let

$$\psi^\epsilon(x,y) = \mathbf{E}\left(\exp\left(-\beta\int_0^\tau G(x_s^\epsilon)\, ds\right) \;\middle|\; x_0^\epsilon = x, y_0^\epsilon = y\right).$$

Assuming that the linear boundary value problem (10) associated with ψ^ϵ has a classical solution, then the dual relation $V^\epsilon = -\beta^{-1}\log\psi^\epsilon$ holds and the results of the previous section carries over without alternations.

Homogenized control system. From the above and the considerations from the previous section we can conclude that the leading term of $V^\epsilon(x,y)$ satisfies the optimal control problem of the homogenized SDE, which is

$$V_0(x) = \inf_u \mathbf{E}\left(\int_0^\tau \left[G(x_s^u) + \frac{1}{2}|u_s|^2\right] ds \,\middle|\, x_0^u = x\right) \tag{31}$$

subject to the homogenized equation

$$dx_s^u = -\nabla\Phi(x_s^u)ds + \sqrt{2}u_s ds + \sqrt{2}\beta^{-1/2}dw_s. \tag{32}$$

Equation (32) is called the *overdamped Langevin equation* that is obtained from (27) by letting the inertial second-order term tend to zero [45].

What is missing is an asymptotic expression for the optimal feedback law

$$\hat{c}^\epsilon = -\sqrt{2}\epsilon^{-1}\nabla_y V^\epsilon(x,y)\,.$$

From (30) and the expansion $\psi^\epsilon(x,y) = \psi_0(x) + \epsilon\psi_1(x,y) + o(\epsilon)$ we find

$$\hat{u}^\epsilon = -\sqrt{2}\nabla_y V_1 + o(1) = -\sqrt{2}\nabla_y\Theta\nabla_x V_0 + o(1). \tag{33}$$

As before Θ is the solution to the associated cell problem. To solve it we notice that the infinitesimal generator of (28) has the form

$$\mathcal{L} = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1$$

with

$$\mathcal{L}_0 = -y\cdot\nabla_y + \beta^{-1}\Delta_y \tag{34}$$

$$\mathcal{L}_1 = y\cdot\nabla_x - \nabla\Phi\cdot\nabla_y\,. \tag{35}$$

Hence the cell problem for Θ reads

$$\mathcal{L}_0\Theta = -y\,,$$

with unique solution $\Theta(x,y) = y$. Combining it with (33), we the sought asymptotic expression for the optimal feedback law

$$\hat{c}^\epsilon = -\sqrt{2}\nabla_x V_0 + o(1)\,, \tag{36}$$

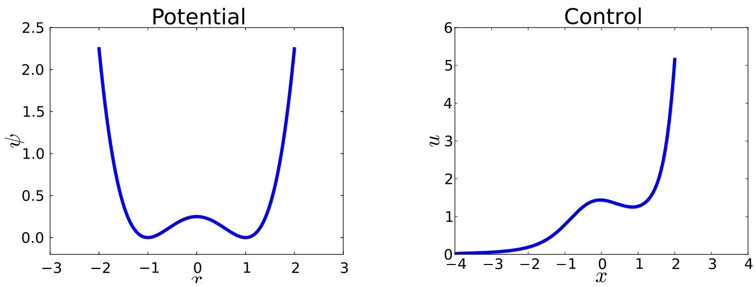
with V_0 as given in (31). We therefore conclude that the optimal control $\hat{u}^\epsilon$ for the Langevin equation (27) converges to the optimal control of the overdamped equation (32) as $\epsilon \to 0$. Moreover, by Theorem 3 guarantees that the control value is asymptotically correct if we replace $\hat{u}^\epsilon$ with the control $\hat{u} = -\sqrt{2}\nabla_x V_0$ in the multiscale dynamics (30). Hence the overdamped equation is backward stable.

Langevin dynamics in a double-well potential. Now consider the case $n = 1$ and running cost $G(x) = 1$ in (29), with random stopping time is defined by

$$\tau = \inf\{s > 0 : x_s^{\epsilon,u} > 2\}\,.$$

The dynamics are governed by the double-well potential

$$\Phi(x) = \frac{1}{4}(x^2-1)^2$$



(A) (B)

FIGURE 2. Overdamped Langevin equation. (A) Double-well potential $\Phi(x)$. (B) Leading term of optimal control in (36).

depicted in Figure 2A. As the homogenized problem is one-dimensional, the leading term V_0 of the value function V^ϵ can be computed by solving a two-point boundary value problem. The resulting leading term (36) for the optimal control

$$\hat{u}_t^\epsilon = \hat{c}^\epsilon(x_t^{\epsilon,u})$$

is shown in Figure 2B. We then computed the cost function $J^\epsilon = J(\hat{u}^\epsilon)$ starting from three different initial points $x_0 = 1.0, 1.2, 1.5$, and using the approximation

$$\hat{u}_t^\epsilon \approx -\sqrt{2}\nabla_x V_0(x_t^{\epsilon,u})$$

Figure 3 clearly shows that J^ϵ approaches its infimum $V_0(x_0)$ as $\epsilon \to 0$. A clear advantage of controlling the full dynamics using the optimal control obtained from the reduced model here is that the infinitesimal generator $\mathcal{L}^\epsilon$ of the original Langevin dynamics is not selfadjoint, whereas the infinitesimal generator $\bar{\mathcal{L}}$ of the reduced dynamics is essentially selfadjoint. That is, not only do we benefit from a lower dimensionality of the reduced-order model (by a factor of 2), but we also avoid solving a boundary value problem with a non-selfadjoint operator.

3.2. **Diffusion in a periodic potential.** Now we consider the SDE [16, 52]

$$dx_s^\epsilon = -\nabla\Phi^\epsilon(x_s^\epsilon)ds + \sqrt{2}\beta^{-1/2}dw_s \tag{37}$$

where $\beta > 0$ and $\Phi^\epsilon(x) = \Phi_0(x) + p(x/\epsilon)$, with $p(x)$ being a smooth, 1-periodic function (see Fig. 4 below). We consider the optimal control problem

$$V^\epsilon(x) = \inf_{u^\epsilon} \mathbf{E}\left(\tau + \frac{1}{2}\int_0^\tau |u_s^\epsilon|^2 ds \;\middle|\; x_0^{\epsilon,u} = x\right), \tag{38}$$

where

$$dx_s^{\epsilon,u} = -\nabla\Phi^\epsilon(x_s^{\epsilon,u})ds + \sqrt{2}u_s^\epsilon ds + \sqrt{2}\beta^{-1/2}dw_s. \tag{39}$$

and $\tau = \tau^{\epsilon,u}$ is the first hitting time of the set $\{x \geq 1.5\}$ (blue region in Fig. 4).

In order to relate this system with the homogenization problem studied in Section 2.2, we introduce the auxiliary variable $y^\epsilon = x^\epsilon/\epsilon$ and reformulate (37) as

$$dx_s^\epsilon = -\frac{1}{\epsilon}\nabla p(y_s^\epsilon)ds - \nabla\Phi_0(x_s^\epsilon)ds + \sqrt{2}\beta^{-1/2}dw_s, \tag{40a}$$

$$dy_s^\epsilon = -\frac{1}{\epsilon^2}\nabla p(y_s^\epsilon)ds - \frac{1}{\epsilon}\nabla\Phi_0(x_s^\epsilon)ds + \frac{\sqrt{2}}{\epsilon}\beta^{-1/2}dw_s, \tag{40b}$$

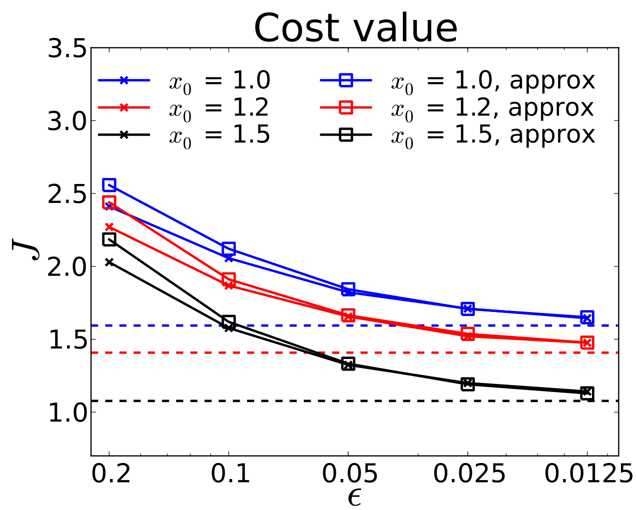


FIGURE 3. Overdamped Langevin dynamics. Cost function for different values of ϵ. Different colors correspond to different initial values x_0. Lines marked with "$\times$" are the value function V^ϵ computed from the exponential expectation using Monte-Carlo. Lines marked with "$\square$" are the cost function $J^\epsilon = J(\hat{u}^\epsilon)$, computed from the homogenized control with the original dynamics. We observe that the two values approach $V_0(x_0)$ as $\epsilon \to 0$ (horizontal line).

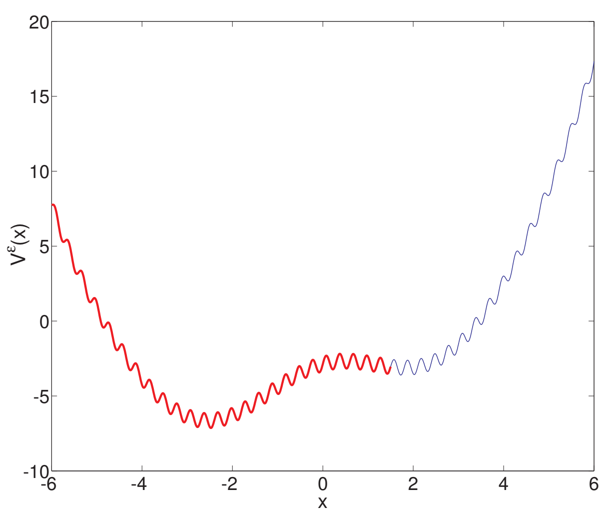


FIGURE 4. Controlled diffusion in a multiscale potential: minimize the transition time from the red to the blue region.

where $x_s^\epsilon, y_s^\epsilon$ are driven by the same noise w_s. The value function then reads

$$\widetilde{V}^\epsilon(x,y) = \inf_{u^\epsilon} \mathbf{E}\left(\tau + \frac{1}{2}\int_0^\tau |u_s^\epsilon|^2 ds \;\middle|\; x_0^{\epsilon,u} = x, y_0^{\epsilon,u} = y\right),$$

with

$$dx_s^{\epsilon,u} = -\frac{1}{\epsilon}\nabla p(y_s^{\epsilon,u})ds - \nabla\Phi_0(x_s^{\epsilon,u})ds + \sqrt{2}u_s^\epsilon\, ds + \sqrt{2}\beta^{-1/2}dw_s, \tag{41a}$$

$$dy_s^{\epsilon,u} = -\frac{1}{\epsilon^2}\nabla p(y_s^{\epsilon,u})ds - \frac{1}{\epsilon}\nabla\Phi_0(x_s^{\epsilon,u})ds + \frac{\sqrt{2}}{\epsilon}u_s^\epsilon\, ds + \frac{\sqrt{2}}{\epsilon}\beta^{-1/2}dw_s\,. \tag{41b}$$

Notice that the same noise and the same control are applied to both equations. Clearly $V^\epsilon(x) = \widetilde{V}^\epsilon(x, x/\epsilon)$ and the dual relation $\widetilde{V}^\epsilon(x,y) = -\beta^{-1}\log\psi^\epsilon(x,y)$ applies, where ψ^ϵ is defined as in Section 2.2. The generator of (40) now is

$$\mathcal{L}^\epsilon = \frac{1}{\epsilon^2}\mathcal{L}_0 + \frac{1}{\epsilon}\mathcal{L}_1 + \mathcal{L}_2,$$

with

$$\begin{aligned}
\mathcal{L}_0 &= -\nabla p\cdot\nabla_y + \beta^{-1}\Delta_y,\\
\mathcal{L}_1 &= -\nabla p\cdot\nabla_x - \nabla\Phi_0\cdot\nabla_y + 2\beta^{-1}\nabla_x\nabla_y,\\
\mathcal{L}_2 &= -\nabla\Phi_0\cdot\nabla_x + \beta^{-1}\Delta_x\,.
\end{aligned}$$

Homogenized system. Applying the results of Section 2, we conclude that the leading term of $V^\epsilon(x)$ is the value function of the optimal control problem

$$J(u) = \inf_u \mathbf{E}\left(\tau + \frac{1}{2}\int_0^\tau |u_s|^2 ds \,\middle|\, x_0^u = x\right), \tag{42}$$

subject to the homogenized dynamics

$$dx_s^u = -K\nabla\Phi(x_s^u)ds + \sqrt{2K}u_s ds + \sqrt{2K}\beta^{-1/2}dw_s, \tag{43}$$

with the effective diffusivity

$$K = \int (I + \nabla_y\Theta(y))(I + \nabla_y\Theta(y))^T\rho(y)\,dy\,.$$

In the above formula $\rho(y) = Z^{-1}\exp(-\beta p(y))$ denotes the invariant density of the fast variable y and $\Theta(y)$ is the solution of the Poisson equation

$$\mathcal{L}_0\Theta(y) = \nabla p(y).$$

Specifically, we have (cf. [53] for details)

$$K^{-1} = \int_0^1 \exp(-\beta p(y))\,dy\int_0^1\exp(\beta p(y))\,dy\,.$$

The value function of the homogenized control problem (42)–(43) and the corresponding optimal control satisfy

$$V_0(x) = -\beta^{-1}\log\psi_0(x)$$

and

$$\hat{u}_t = -\sqrt{2K}\nabla V_0(x_t^{\hat{u}})$$

where

$$\bar{\mathcal{L}}\psi_0(x) = K\mathcal{L}_2\psi_0(x) = \beta G(x)\psi_0(x), \quad \psi_0(x)\big|_{\partial O} = 0,$$

as given in (18).

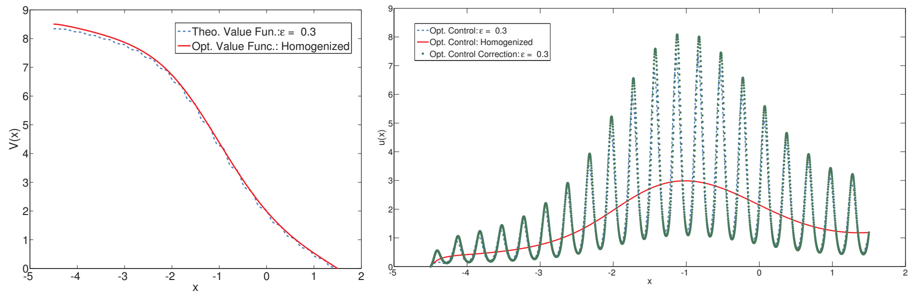


FIGURE 5. Value function and resulting optimal control (right panel).

Reduced model is not backward stable. In contrast to the previous example, however, the optimal control $\hat{u}$ obtained from the homogenized equation alone does meet the requirements of backward stability. This can be understood by noting that from (39) and the expansion $\psi^\epsilon(x, x/\epsilon) = \psi_0(x) + \epsilon\psi_1(x, x/\epsilon) + o(\epsilon)$, if follows that the optimal control the original dynamics is given by the feedback law

$$\begin{aligned} c^\epsilon(x) &= -\sqrt{2}\nabla V^\epsilon(x) = \sqrt{2}\beta^{-1}\frac{\nabla_x \psi^\epsilon(x, x/\epsilon)}{\psi^\epsilon(x, x/\epsilon)} \\ &= \sqrt{2}\beta^{-1}\frac{\nabla_x \psi_0(x) + \nabla_y \psi_1(x, x/\epsilon)}{\psi_0(x)} + o(1). \end{aligned} \tag{44}$$

After some manipulations we find that the asymptotic expression for c^ϵ reads

$$\begin{aligned} c^\epsilon(x) =& \sqrt{2}\beta^{-1}\frac{\exp(\beta p(x/\epsilon))}{\int_0^1 \exp(\beta p(z))\, dz}\frac{\psi_0'(x)}{\psi_0(x)} + \mathcal{O}(\epsilon) \\ =& \frac{\exp(\beta p(x/\epsilon))}{\sqrt{K}\int_0^1 \exp(\beta p(z))\, dz} c(x) + \mathcal{O}(\epsilon) \end{aligned} \tag{45}$$

where we used the shorthand $c(x) = -\sqrt{2K}\nabla V_0(x)$ in the last row. We conclude that c^ϵ is of the form

$$c^\epsilon(x) = \tilde{c}(x, x/\epsilon) + \mathcal{O}(\epsilon)\,,$$

yet $\tilde{c}(x, x/\epsilon)$ does not converge to $c(x)$ in norm for the x/ϵ part keeps oscillating as $\epsilon \to 0$. What does converge, however, is the *average*:

$$\int_0^1 \tilde{c}(x, y)\rho(y)dy = \int_0^1 \tilde{c}(x, y)\frac{e^{-\beta p(y)}}{\int_0^1 e^{-\beta p(z)}dz}dy = \sqrt{K}c(x)\,.$$

This fact is illustrated in Figure 5 that shows the oscillations of order one that are a consequence of the $\mathcal{O}(\epsilon)$ oscillations of the value function with period $\mathcal{O}(\epsilon)$; as the optimal control law involves the derivative of the value function, these little oscillations turn into $\mathcal{O}(1)$ contributions to the optimal control.

Remark 5. *The above case is an example, in which using a reduced-order models for optimal control is not recommended, for $J(\hat{u}^\epsilon)$ does not converge to $J(\hat{u})$ as $\epsilon \to 0$. Nonetheless, Theorem 3 suggests that we can use the leading term of c^ϵ in (45) as an approximation of the feedback law for the multiscale dynamics (39). The effect of the corrector estimate (45), is to enforce convergence of the* derivative *of the value function, which entails (weak) convergence of the optimal control and convergence of the optimal cost value (cf.* [16] *for an application in importance sampling).*

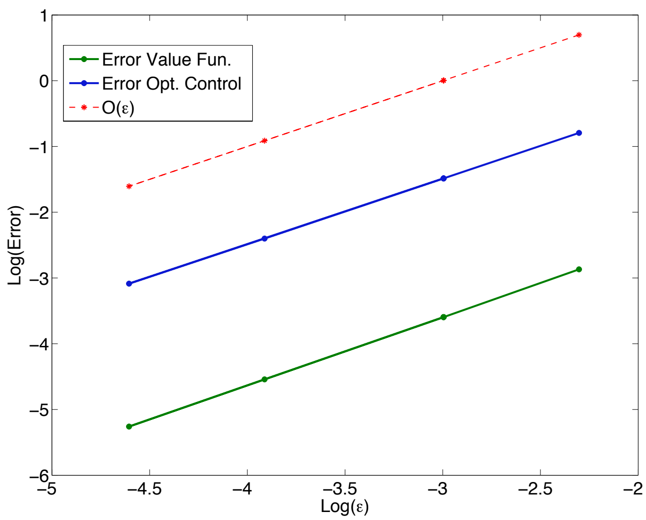


FIGURE 6. Convergence of value function and optimal control.

Convergence Tests. Now we present the results of numerical simulations. We employ the finite-volume numerical scheme developed in [40] for solving numerically PDEs of the type (10), (15) and for performing Markov Jump Monte Carlo (MJMC) simulations in order to sample the controlled processes (39), (43). We also investigate the convergence properties of the numerical solutions as well as show how the different approximations of the control can be used to sample the multiscale SDEs.

GP: DO WE GIVE THE FORMULA FOR THE FLUCTUATING PART OF THE POTENTIAL ANYWHERE IN THE PAPER?

In Figure 6 and Figure **??** we show the results of standard convergence test for the numerical solution of equations of the type (10, 15). For different values of ϵ we have solved (10) using a grid size $\Delta x = \epsilon/160$. Let us label this solution ψ_N^ϵ. We then compute solutions using grid sizes $\Delta x_n = \epsilon/n$, with $n = 10, 20, 40, 80$. Let us call these solutions ψ_n^ϵ The error E_n is computed as

$$\mathrm{E}_n = \max|\psi_N^\epsilon - \psi_n^\epsilon|.$$

In Figure 6 we observe the typical $\mathcal{O}(\Delta x_n^2)$ convergence of the solution. We then have computed the solution to the homogenized equation for different values of Δx proportional to ϵ, e.g. $\Delta x = \epsilon/20$, then compute the error E_n as,

$$\mathrm{E}_n = \max|\psi_N^\epsilon - \psi_n|.$$

We notice in Figure 6 that this error decreases as Δx showing a convergence of $\mathcal{O}(\epsilon)$ of the solution to the multiscale PDE to the solution of the homogenized PDE.

The optimal control is then computed using the formula $\hat{u}^\epsilon = -\sqrt{2}\beta^{-1}\nabla\log\psi^\epsilon$ as in (45) using centered finite differences for computing the derivative of ψ^ϵ. In Figure **??** we observe again the $\mathcal{O}(\epsilon)$ convergence of $\tilde{u}_n$ (from eq. (45)) to $\hat{u}_N^\epsilon$. This behavior is not observed however for the convergence of $\hat{u}_n^\epsilon$ to $\hat{u}_N^\epsilon$; rather, it is only observed for smaller values of Δx, implying that the $\mathcal{O}(\epsilon)$ terms in the numerical derivative are important for smaller values of Δx.

MFPT and Value Function. We now turn our attention to the sampling of functions for the controlled stochastic system. We have computed the optimal control problem (38), (39) for the mean first passage time ($G = 1$) for different values of ϵ as well as the optimal control problem for the homogenized system (42), (43). We have also performed MJMC simulations of the multiscale process with multiscale potential Φ^ϵ, but using the homogenized optimal control function $\hat{u}$[1],

$$dx_s^\epsilon = \left(-\nabla\Phi^\epsilon(x_s^\epsilon) + \sqrt{2K}\hat{u}(x_s^\epsilon)\right) ds + \sqrt{2}\beta^{-1/2} dw_s. \tag{46}$$

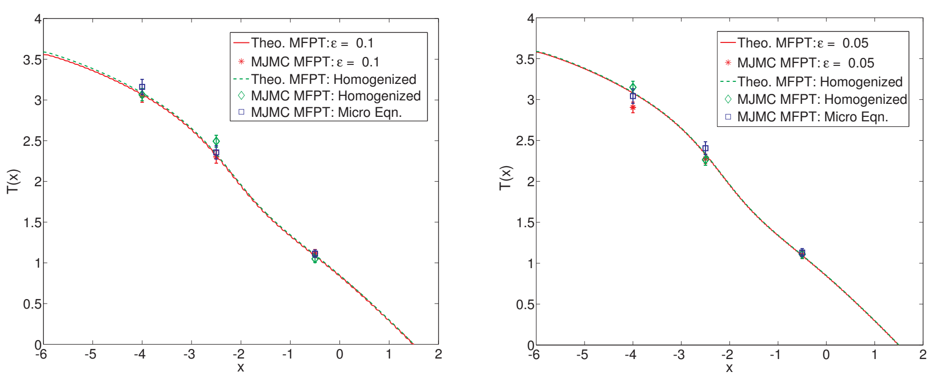
We then have computed averaged for both the MFPT of the three processes and the value corresponding value function. **GP: I DON'T UNDERSTAND THE PREVIOUS SENTENCE** For the system (46), we have sampled the value function,

$$\mathbf{E}\left(\int_0^\tau \left(G(x_s) + \frac{1}{2}|\hat{u}(x_s)|^2\right) ds \;\middle|\; x_0 = x\right). \tag{47}$$

In Figure 7A and Figure 7B we show the results for the MFPT, while in Figure 8A and Figure 8B we show the results for the value function. In these Figures we also show how the value function can be sampled using the asymptotic approximations of the control,

$$\mathbf{E}\left(\int_0^\tau \left(G(x_s) + \frac{1}{2}|\tilde{u}(x_s, x_s/\epsilon)|^2\right) ds \;\middle|\; x_0 = x\right). \tag{48}$$

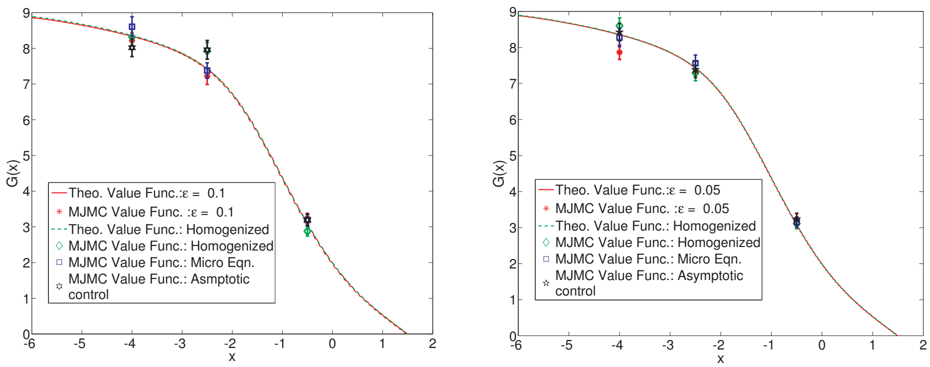
These numerical experiments indicate that we can use either the homogenized control or the asymptotic aproximation in order to *steer* efficiently the original (multiscale) SDE, without having to solve the boundary value problem for a PDE with a multiscale structure.



(A) $\epsilon = 0.1$ (B) $\epsilon = 0.05$

FIGURE 7. MFPT for different values of ϵ. Solid line: numerical solution of multiscale equation. $*$: MJMC sampling of (41). Dashed line: numerical solution of homogenized equation. $\diamond$: MJMC sampling of (43). $\square$: MJMC sampling of (46).

[1]The solution ψ for the log-transform PDE is computed using a *macro* grid . We then simply interpolate the values of ψ within the micro-grid in order to compute the optimal control



(A) $\epsilon = 0.1$ (B) $\epsilon = 0.05$

FIGURE 8. Optimal value function for different values of ϵ. Solid line: numerical solution of eq. (10). $*$: MJMC sampling of (41). Dashed line: numerical solution of eq. (18). $\diamond$: MJMC sampling of (43). $\square$: MJMC sampling of (46) sampling (47). $\star$: MJMC sampling of (46) sampling (48).

3.3. **Linear-quadratic regulator.** The third example is a multiscale linear quadratic regulator (LQR) problem that slightly falls out of the previous category. Specifically, we seek to minimize the time-averaged quadratic cost

$$J^\epsilon(u) = \limsup_{T\to\infty} \mathbf{E}\left(\frac{1}{T}\int_0^T \left\{|x_s^{\epsilon,u}|^2 + |y_s^{\epsilon,u}|^2 + \frac{1}{2}|u_s^\epsilon|^2\right\} ds\right) \tag{49}$$

subject to the linear dynamics

$$\begin{aligned} dx_s^{\epsilon,u} &= \left(A_{11}x_s^{\epsilon,u} + \frac{1}{\epsilon}A_{12}y_s^{\epsilon,u} + \sqrt{2}B_1u_s^\epsilon\right)ds + \sqrt{2\beta^{-1}}B_1dw_s \\ dy_s^{\epsilon,u} &= \left(\frac{1}{\epsilon}A_{21}x_s^{\epsilon,u} + \frac{1}{\epsilon^2}A_{22}y_s^{\epsilon,u} + \frac{1}{\epsilon}\sqrt{2}B_2u_s^\epsilon\right)ds + \frac{\sqrt{2\beta^{-1}}}{\epsilon}B_2dw_s \end{aligned} \tag{50}$$

where $x \in \mathbb{R}^k, y \in \mathbb{R}^{n-k}$, $u \in \mathbb{R}^l$, and A_{ij}, B_i are real matrices of appropriate size. Note that both slow and fast equations are driven by the same noise and control. Further let

$$A = \begin{pmatrix} A_{11} & \epsilon^{-1}A_{12} \\ \epsilon^{-1}A_{21} & \epsilon^{-2}A_{22} \end{pmatrix}, \quad B = \sqrt{2}\begin{pmatrix} B_1 \\ \epsilon^{-1}B_2 \end{pmatrix}.$$

We make the following additional assumptions (we suppress the ϵ in the matrix definition in order to keep the notation compact):

1. The initial values $(x_0^\epsilon, y_0^\epsilon) = (x_0, y_0)$ are independent of ϵ and satisfy
$$\mathbf{E}[|x_0|^2] < \infty\,, \quad \mathbf{E}[|y_0|^2] < \infty\,.$$
2. For all $\epsilon > 0$, the spectrum of A lies entirely in the open left half complex plane, i.e., all eigenvalues of A have strictly negative real part.
3. The spectrum of A_{22} lies entirely in the open left half complex plane.
4. For all $\epsilon > 0$, the matrix pair (A, B) is controllable, i.e., the matrix
$$K^\epsilon = (B\ AB\ A^2B\ \ldots\ A^{n-1}B^\epsilon)$$

has maximum rank n.

For the control problem (49)–(50), the analog of (8) for the case of an infinite-time horizon with time-averaged cost and unbounded domain reads [55, 56]

$$\eta^\epsilon = \min_{c\in\mathbb{R}^n}\left\{\mathcal{L}^\epsilon(c)V^\epsilon + |z|^2 + \frac{1}{2}|c|^2\right\} \tag{51}$$

where $z = (x, y)$ and

$$\mathcal{L}^\epsilon(u) = (2\beta)^{-1}BB^T : \nabla^2 + (Az + Bu)\cdot\nabla$$

The unknown parameter $\eta^\epsilon \in \mathbb{R}$ in the Hamiton-Jacobi-Bellman equation (51) needs to be determined along with the function $V^\epsilon = V^\epsilon(x, y)$, in fact (51) can be regarded as a nonlinear eigenvalue equation for the pair $(\eta^\epsilon, V^\epsilon)$; for details we refer to Appendix B.

LQR problems of this kind have quadratic value functions and admit an explicit solution in terms of an algebraic Riccati equation

$$A^TS^\epsilon + S^\epsilon A - 2S^\epsilon BB^TS^\epsilon + I_{n\times n} = 0\,, \tag{52}$$

where $I_{n\times n}$ denotes the $n \times n$ identity matrix. Specifically, plugging the ansatz

$$V^\epsilon(z) = z^TS^\epsilon z$$

into (51), it readily follows that S^ϵ solves (52). Hence the optimal control for the linear quadratic regulator (49)–(50) is given by the linear feedback law

$$\hat{u}_t^\epsilon = -B^TS^\epsilon z_t\,.$$

Under the above assumptions, the Riccati equation has a unique symmetric positive definite solution S^ϵ for all values of $\epsilon > 0$. Moreover, it follows that

$$\eta^\epsilon = BB^T : S^\epsilon\,,$$

which is the principal eigenvalue of the linear eigenvalue equation

$$(2\beta)^{-1}BB^T : \nabla^2\psi^\epsilon + (Az)\cdot\nabla\psi^\epsilon - \beta|z|^2\psi^\epsilon = -\beta\eta^\epsilon\psi^\epsilon \tag{53}$$

for the log-transformed eigenfunction $\psi^\epsilon = \exp(-\beta V^\epsilon)$. Notice that the eigefunction ψ^ϵ corresponding to the principal eigenvalue $-\beta\eta^\epsilon \le 0$ is strictly positive as a consequence of the Perron-Frobenius theorem, hence its log transformation is well defined.

Reduced Riccati equation. Given the above assumptions on the matrices A and B, the homogenized version of the linear eigenvalue equation (53) can be easily computed, since the cell problem has an explicit solution. We find

$$(2\beta)^{-1}\bar{B}\bar{B}^T : \nabla^2\psi + (\bar{A}z)\cdot\nabla\psi - \beta(|x|^2 + Q)\psi = -\beta\eta\psi \tag{54}$$

with the homogenized coefficients

$$\bar{A} = A_{11} - A_{12}A_{22}^{-1}A_{21}\,,\quad \bar{B} = \sqrt{2}\left(B_1 + A_{12}A_{22}^{-1}B_2\right)$$

and

$$Q = 2\beta^{-1}\mathrm{tr}\left(\int_0^\infty e^{A_{22}t}B_2B_2^Te^{A_{22}t}dt\right)\,,$$

denoting the sum of the eigenvalues of the asymptotic covariance matrix of the fast degrees of freedom. The limiting eigenpair (η, ψ) is given by

$$\eta = \bar{B}\bar{B}^T : S + Q\,,\quad \psi(x) = e^{-\beta x^TSx}$$

where S is the solution of the homogenized Riccati equation

$$\bar{A}^T S + S\bar{A} - 2S\bar{B}\bar{B}^T S + I_{k\times k} = 0\,, \tag{55}$$

in accordance with the solution of the algebraic Riccati equation of singularly-perturbed LQR problems that has been discussed in the literature; see [22] and the references therein. It can be shown by perturbation analysis of the Riccati equation (52) using the Chow transformation (see, e.g., [34] and the references therein) that S corresponds to the top left $k \times k$ block of the matrix S up to $\mathcal{O}(\epsilon^2)$. Moreover, for any open and bounded subset $\Omega \subset \mathbb{R}^n$ with smooth boundary, we have

$$\|V^\epsilon - V\|_{H^1(\bar{\Omega})} \le C_1\epsilon^2\,.$$

for $V = -\beta^{-1}\log\psi$ and some constant $0 < C_1 < \infty$. The latter implies that

$$|\hat{u}_s^\epsilon - \hat{u}_s| \le C_2\epsilon$$

uniformly on $[0,\tau_\Omega]$ where τ_Ω is the first exit time from $\Omega \subset \mathbb{R}^n$ and $0 < C_2 < \infty$. For large values of β the probability that the process exits from Ω is exponentially small in β, i.e., the exit from the domain is a rare event (see, e.g., [61]) and hence we can employ the approximation $\tau_\Omega \approx \infty$ for all practical purposes.

270-dimensional ISS model. We consider the 270-dimensional model of a component of the International Space Station (ISS) that is taken from the SLICOT benchmark library [13]. In this case, $n = 270$ and $l = 3$ in equation (49); the dimension of the slow subspace is set to $k = 4$, because the spectrum of dimensionless Hankel singular values of the full system shows a significant spectral gap at $k = 4$ when the slow variables are chosen as the observed variables; see [26] for details. The original system is Hamiltonian, but we pay no attention to the specific geometric structure of the equations here; cf. [29] for related work. The corresponding control task for the 4-dimensional reduced system thus is to minimize

$$\bar{J}(u) = \limsup_{T\to\infty} \mathbf{E}\left(\frac{1}{T}\int_0^T \left\{|x_s^u|^2 + \frac{1}{2}|u_s|^2\right\} ds\right) \tag{56}$$

subject to the dynamics

$$dx_s^u = \left(\bar{A}x_s^u + \bar{B}u_s\right) ds + \beta^{-1/2}\bar{B}dw_s\,, \tag{57}$$

with $\bar{A}$ and $\bar{B}$ as in (55). Without loss of generality, we have ignored the additive constant Q in the cost term that appears in the homogenized eigenvalue equation (54). As before the optimal control is given by the linear feedback law

$$\hat{u}_s = -\bar{B}^T S x_s\,.$$

where S denotes the solution of (52). To verify the convergence of the value function numerically, we have computed eigenvalues of S and S^ϵ, the matrix norms of $S - S_{11}^\epsilon$ and the norm of the matrix S^ϵ with the S_{11}^ϵ block set to zero, called S_r^ϵ. Here S_{11}^ϵ refers to the upper left $k\times k$ block of the matrix S^ϵ, in accordance with the notation in (50). Figure 9 shows this comparison for $\beta = 0.01$, which, given the parameters of the ISS model, amounts to the small noise regime; the plots clearly show that the convergence is of $\mathcal{O}(\epsilon^2)$. We refrain from testing the convergence $\eta^\epsilon \to \eta$ of the corresponding nonlinear eigenvalue since the $1/\epsilon^2$ singularity makes the evaluation of the trace term $BB^T : S^\epsilon$ numerically unstable for all interesting values of ϵ.

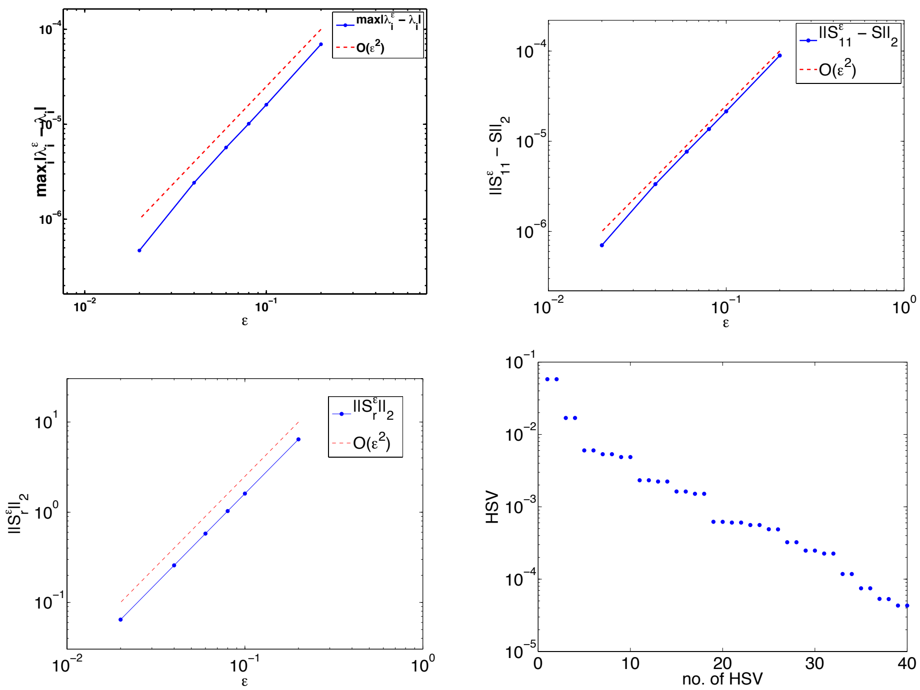


FIGURE 9. Hankel singular values and quadratic convergence of the matrix S^ϵ in terms of the k dominant eigenvalues (upper left panel), the 1-1 matrix block (upper right panel) and the residual matrix S^ϵ_r (lower left panel); for smaller values of ϵ the numerical solution of the Riccati equation is dominated by round-off errors, hence the results are not shown. The lower right panel shows the first 40 Hankel singular values (out of 270) when the slow variables are observed; the Hankel singular values are independent of ϵ.

Appendix A. **Weak convergence under logarithmic transformations.** Weak convergence is mainly an issue for homogenization problems with periodic coefficients. Here we distinguish the case $T < \infty$ (evolution problem) from the case that $T = \infty$ (exit problem or ergodic control problem). In the evolution case a well-known result (e.g., see [49, Sec. 3] or [52, Sec. 20]) that is based on the maximum principle states that the convergence is uniform on bounded time intervals under fairly weak assumptions.

In the elliptic case, to which the exit problem belongs, the lowest order perturbation expansion gives only weak convergence. In general, however, weak convergence is not preserved under nonlinear transformation. In general, given a weakly convergent sequence ψ^ϵ on $\mathbb{R}$ and a nonlinear continuous function $F\colon \mathbb{R} \to \mathbb{R}$, we have

$$\psi^\epsilon \rightharpoonup \psi \quad \not\Rightarrow \quad F(\psi^\epsilon) \rightharpoonup F(\psi)\,.$$

In our case, however, weak convergence follows from the properties of the logarithm and the fact that ψ^ϵ is bounded away from 0. Let ψ^ϵ be the solution of the elliptic boundary value problem (10) for $T \to \infty$ and recall that

$$\psi^\epsilon \to \psi \quad \text{strongly in} \quad L^2(\bar{O})$$

and

$$\psi^\epsilon \rightharpoonup \psi \quad \text{weakly in} \quad H^1(\bar{O})\,.$$

Moreover, we have that

$$0 < C \le \psi^\epsilon \le 1 \quad \epsilon \in (0,1)$$

for some $C \in (0,1)$.

Lemma 6. *We have*

$$\log \psi^\epsilon \to \log \psi \quad \textit{strongly in} \quad L^2(\bar{O})$$

Proof. Since $C \le \psi^\epsilon \le 1$ the monotony of the logarithm entails that

$$\log C \le \log \psi^\epsilon \le 0\,.$$

Since $\log C > -\infty$ and $O \subset \mathbb{R}^n$ is bounded it follows that $\log \psi^\epsilon \in L^2(\bar{O})$ and, by the same argument, $\log \psi \in L^2(\bar{O})$. Convergence now follows from the fact that $\log(x)$ is Lipschitz continuous with a Lipschitz constant $L < \infty$ if $x \le C > 0$:

$$\begin{aligned}\|\log \psi^\epsilon - \log \psi\|^2_{L^2(\bar{O})} &= \int_{\bar{O}} |\log \psi^\epsilon - \log \psi|^2 dz \\ &\le L^2 \int_{\bar{O}} |\psi^\epsilon - \psi|^2 dz\,,\end{aligned}$$

which vanishes in the limit $\epsilon \to 0$ as $\psi^\epsilon \to \psi$ in $L^2(\bar{O})$. □

This implies strong convergence of the value function. For the optimal control, the above conditions give only weak convergence, which is implied by:

Lemma 7. *We have*

$$\log \psi^\epsilon \rightharpoonup \log \psi \quad \textit{weakly in} \quad H^1(\bar{O})$$

Proof. It suffices to show that $\nabla \log \psi^\epsilon \rightharpoonup \nabla \log \psi$ in $L^2(\bar{O})$. To this end recall that $\nabla \psi^\epsilon \rightharpoonup \nabla \psi$ in $L^2(\bar{O})$ since ψ^ϵ converges weakly in $H^1(\bar{O})$. Then, for all test functions $\phi \in L^2(\bar{O})$, using again that $\psi^\epsilon \ge C > 0$ pointwise and uniformly in ϵ,

$$\begin{aligned}\int_{\bar{O}} \left(\frac{\nabla \psi^\epsilon}{\psi^\epsilon} - \frac{\nabla \psi}{\psi}\right) \phi\, dz &= \int_{\bar{O}} (\psi \nabla \psi^\epsilon - \psi^\epsilon \nabla \psi) \frac{\phi}{\psi^\epsilon \psi}\, dz \\ &\le \frac{1}{C^2} \int_{\bar{O}} (\psi \nabla \psi^\epsilon - \psi^\epsilon \nabla \psi)\, \phi\, dz \\ &\le \frac{1}{C^2} \underbrace{\int_{\bar{O}} (\psi \nabla \psi^\epsilon - \psi \nabla \psi)\, \phi\, dz}_{I_1} + \frac{1}{C^2} \underbrace{\int_{\bar{O}} (\psi \nabla \psi - \psi^\epsilon \nabla \psi)\, \phi\, dz}_{I_2}\end{aligned}$$

We look at the two integrals separately. Using that $0 < \psi \le 1$ it follows that

$$|I_1| \le \left| \int_{\bar{O}} (\nabla \psi^\epsilon - \nabla \psi)\, \phi\, dz \right| \to 0$$

since $\phi \in L^2(\bar{O})$ and $\nabla \psi^\epsilon \rightharpoonup \nabla \psi$ weakly in $L^2(\bar{O})$. Now for the second integral: since the weakly convergent sequence ψ^ϵ and its limit ψ are bounded in $H^1(\bar{O})$ we conclude that $\nabla\psi \in L^2(\bar{O})$, which together with the boundedness of $|\psi^\epsilon - \psi|$ implies that $(\psi^\epsilon - \psi)\nabla\psi \in L^2(\bar{O})$. So, by the Cauchy-Schwarz inequality,

$$\begin{aligned}|I_2|^2 &\le \left(\int_{\bar{O}} |(\psi^\epsilon - \psi)\nabla\psi|^2\,dz\right)\left(\int_{\bar{O}} |\phi|^2\,dz\right)\\ &= \|\phi\|^2_{L^2(\bar{O})}\int_{\bar{O}} |(\psi^\epsilon - \psi)\nabla\psi|^2\,dz\\ &\le M\|\phi\|^2_{L^2(\bar{O})}\int_{\bar{O}} |(\psi^\epsilon - \psi)\nabla\psi|\,dz\end{aligned}$$

for some constant $0 < M < \infty$. Reiterating the preceding argument it follows that

$$|I_2|^2 \le M\|\phi\|^2_{L^2(\bar{O})}\|\psi^\epsilon - \psi\|^2_{L^2(\bar{O})}\|\nabla\psi\|^2_{L^2(\bar{O})} \to 0$$

as $\psi^\epsilon \to \psi$ in $L^2(\bar{O})$ and $\nabla\psi \in L^2(\bar{O})$. Hence

$$\left|\int_{\bar{O}} (\nabla \log\psi^\epsilon - \nabla\log\psi)\,\phi\,dz\right| \to 0$$

which, together with the last Lemma yields the assertion. □

Appendix B. **Ergodic control problem.** We briefly discuss the ergodic control problem of Section 3.3 that is known to be related to an elliptic eigenvalue problem [30, 9, 19]. In principle, the equivalence of (53) and (51) directly follows from the logarithmic transformation. We will moreover motivate the use of HJB equation more, starting from the linear eigenvalue problem. To this end let

$$\eta^\epsilon = -\limsup_{T\to\infty}\frac{1}{\beta T}\log\mathbf{E}\left(\exp\left(-\beta\int_0^T G(z_t^\epsilon)\,dt\right)\right). \tag{58}$$

for a continuous bounded function $G\colon \mathbb{R}^n \to [0,\infty)$ Further let $\varphi(z,t)$ be given by

$$\varphi^\epsilon(z,t) = \mathbf{E}\left(\exp\left(-\beta\int_0^t G(z_s^\epsilon)\,ds\right)\,\middle|\, z_0^\epsilon = z\right). \tag{59}$$

By the Feynman-Kac formula $\varphi^\epsilon(z,t)$ is the solution of

$$\begin{gathered}\left(\frac{\partial}{\partial t} - \mathcal{L}^\epsilon\right)\varphi^\epsilon = -\beta G\varphi^\epsilon\\ \varphi^\epsilon(z,0) = 1\,.\end{gathered} \tag{60}$$

Here

$$\mathcal{L}^\epsilon = \frac{1}{2}\beta^{-1}\sigma(z;\epsilon)\sigma(z;\epsilon)\colon\nabla^2 + b(z;\epsilon)\cdot\nabla$$

denotes the infinitesimal generator of our generic diffusion process. Setting $V^\epsilon = -\beta^{-1}\log\varphi^\epsilon$, we can rewrite Equation (58) in the form

$$\eta^\epsilon = \lim_{t\to\infty}\frac{V^\epsilon(z,t)}{t}\,,$$

assuming that the limit exists. This motivates the following asymptotic ansatz for large t:

$$\varphi^\epsilon(z,t) \sim \psi^\epsilon(z)\exp(-\eta^\epsilon\beta t)\,,\quad \psi^\epsilon > 0\,.$$

Plugging the separation ansatz into (60) it follows that ψ^ϵ solves the eigenvalue equation

$$\left(G - \beta^{-1}\mathcal{L}^\epsilon\right)\psi^\epsilon = \eta^\epsilon\psi^\epsilon\,,$$

or, equivalently,

$$(\mathcal{L}^\epsilon - \beta G)\,\psi^\epsilon = -\beta\eta^\epsilon\psi^\epsilon\,,$$

As a consequence of the Perron-Frobenius theorem the eigenfunction ψ^ϵ corresponding to the principal eigenvalue $-\beta\eta^\epsilon$ is strictly positive. The equivalent nonlinear eigenvalue problem for the log-transformed eigenfunction $V^\epsilon = -\beta^{-1}\log\psi^\epsilon$ reads

$$\mathcal{L}^\epsilon V^\epsilon - \frac{1}{2}|\sigma^T\nabla V^\epsilon|^2 + G = \eta^\epsilon\,.$$

which, as before, can be rewritten in the form

$$\min_{c\in\mathbb{R}^n}\left\{(\mathcal{L}^\epsilon V^\epsilon + (\sigma c)\cdot\nabla V^\epsilon + G + \frac{1}{2}|c|^2\right\} = \eta^\epsilon\,.$$

The last equation is recognized as the dynamic programming equation of the ergodic optimal control problem, of which (49)–(50) is a special case: minimize

$$J^\epsilon(u) = \limsup_{T\to\infty}\mathbf{E}\left(\frac{1}{T}\int_0^T\left(G(z_s^\epsilon) + \frac{1}{2}|u_s|^2\right)ds\right)$$

subject to

$$dz_s^{\epsilon,u} = (b(z_s^{\epsilon,u};\epsilon) + \sigma(z_s^{\epsilon,u};\epsilon)u_s^\epsilon)\,ds + \sigma(z_s^{\epsilon,u};\epsilon)\beta^{-1/2}dW_s\,.$$

B.1. **Homogenized ergodic control problem.** Let $z = (x, y)$ and consider the expansion $\psi^\epsilon = \psi_0 + \epsilon\psi_1 + \cdots$ and $\eta^\epsilon = \eta_0 + \epsilon\eta_1 + \cdots$, as in the previous subsections. The leading term in the expansion ψ_0 is independent of y and satisfies

$$(\bar{\mathcal{L}} - \beta\bar{G})\psi_0 = -\beta\eta_0\psi_0\,,$$

with $\bar{\mathcal{L}}, \bar{G}$ defined in (20). Now suppose $V^\epsilon = V_0 + \epsilon V_1 + \cdots$, then again

$$V_0 = -\beta^{-1}\log\psi_0, \quad V_1 = -\beta\frac{\psi_1}{\psi_0}.$$

This indicates that the leading nonlinear eigenpair (η_0, V_0) satisfies

$$\eta_0 = \limsup_{T\to\infty}\mathbf{E}\left(\frac{1}{T}\int_0^T\left(\bar{G}(x_s) + \frac{1}{2}|\bar{\alpha}(x_s)^T\nabla V_0(x_s)|^2\right)ds\right)\,,$$

where x_s solves the optimally controlled SDE

$$dx_s = \left(\bar{f}(x_s) - \bar{\alpha}(x_s)\bar{\alpha}(x_s)^T\nabla V_0(x_s)\right)ds + \bar{\alpha}(x_s)\beta^{-1/2}dw_s\,.$$

By ergodicity of the controlled process, the above expectation is independent of the distribution of the initial values; see [56] and the references therein.

Appendix C. **Entropy bounds for the cost function.** In this section we study the cost function of the optimal control problem from the point of view of change of measure. Consider the SDE

$$\begin{aligned} dz_s &= b(z_s)\,ds + \beta^{-1/2}\sigma(z_s)\,dw_s \\ z_0 &= z \end{aligned} \tag{61}$$

and the controlled SDE

$$\begin{aligned} dz_s &= (b(z_s) + \sigma(z_s)u_s)\,ds + \beta^{-1/2}\sigma(z_s)\,dw_s \\ z_0 &= z, \end{aligned} \tag{62}$$

where u_s is any bounded measurable control that is adapted to z_s. Let μ and μ_u denote the path measures generated by (61) and (62), respectively. Then by Girsanov's theorem [47], we have that

$$\frac{d\mu_u}{d\mu} = \exp\left(-\beta^{1/2}\int_0^\tau u_s\,dw_s - \frac{\beta}{2}\int_0^\tau |u_s|^2\,ds\right). \tag{63}$$

Let a cost functional be given by

$$J(u) = \mathbf{E}_{\mu_u}\left(\int_0^\tau \left(G(z_s) + \frac{1}{2}|u_s|^2\right) ds \;\middle|\; z_0 = z\right), \tag{64}$$

where G satisfies Assumption 2 from Section 2.1. Here we use the notation $\mathbf{E}_{\mu_u}$ to indicate that the expectation is understood with respect to the probability measure μ_u. Moreover the dependence of J on the initial value z is omitted.

Let $\hat{u} = \operatorname{argmin} J(u)$, then from Theorem 1 we know $\hat{u}_s$ only depends on z_s. Let $\hat{\mu}$ denote the measure $\mu_{\hat{u}}$ for simplicity. Our purpose here is to estimate $|J(u) - J(\hat{u})|$ when $||u - \hat{u}||_{L^\infty}$ is small. We will make use of the following definition.

Definition 8. *For two probability measures μ_u, μ with $\mu_u \ll \mu$, the Kullback-Leibler divergence of μ_u relative to $\hat{\mu}$ is defined as*

$$I(\mu_u \,|\, \hat{\mu}) = \int \log\left(\frac{d\mu_u}{d\hat{\mu}}\right) d\mu_u. \tag{65}$$

We also assume that Assumption 3 from Section 2.1 holds: there exists $\gamma > 0$, such that $\mathbf{E}_\mu(e^{\gamma\tau}) = C_1 < +\infty$. As in Section 2.1, we have that

$$\mathbf{E}_\mu\left(\exp\Big(-\beta\int_0^\tau G(z_s)\,ds\Big)\right) \geq C_1^{-\beta M_1/\gamma}$$

The conditioning on the initial value is omitted here and also in the following.

We also need two technical estimates in order to study the convergence of the cost functional. We start with the following estimate.

Lemma 9. $\mathbf{E}_{\hat{\mu}}(e^{\gamma\tau}) \leq C_1^{1+\beta M_1/\gamma}$.

Proof. we have $\mathbf{E}_{\hat{\mu}}(e^{\gamma\tau}) = \mathbf{E}_\mu(e^{\gamma\tau}\frac{d\hat{\mu}}{d\mu})$. Using the dual relation

$$-\beta^{-1}\log\mathbf{E}_\mu\left(\exp\Big(-\beta\int_0^\tau G(z_s)\,ds\Big)\right) = \inf_u J(u) = J(\hat{u})$$

and Jensen's inequality, we know that

$$\exp\Big(-\beta\int_0^\tau G(z_s)\,ds\Big)\frac{d\mu}{d\hat{\mu}} = \mathbf{E}_\mu\left(\exp\Big(-\beta\int_0^\tau G(z_s)\,ds\Big)\right) \geq C_1^{-\beta M_1/\gamma}, \quad \mu - a.s. \tag{66}$$

where we have assumed the equivalence of μ and $\hat{\mu}$. Since G is nonnegative,

$$\mathbf{E}_{\hat{\mu}}(e^{\gamma\tau}) = \mathbf{E}_\mu\left(e^{\gamma\tau}\frac{d\hat{\mu}}{d\mu}\right) \leq C_1^{\beta M_1/\gamma}\mathbf{E}_\mu(e^{\gamma\tau}) = C_1^{1+\beta M_1/\gamma}.$$

□

The following lemma provides us with an estimate on the relative entropy when the control u is close to $\hat{u}$.

Lemma 10. *Suppose there is an $\epsilon > 0$, such that $|u_s - \hat{u}_s| \le \epsilon$, for all $s > 0$, and let $\epsilon < (\gamma/\beta)^{1/2}$. Then*

$$I(\mu_u \,|\, \hat{\mu}) \le \beta C_3 \epsilon^2 \,, \quad \mathbf{E}_{\mu_u}(\tau) \le 2C_3 \,,$$

with the constant $C_3 = \gamma^{-1}(1 + \beta M_1/\gamma) \log C_1$.

Proof. From (63), we know

$$I(\mu_u \,|\, \hat{\mu}) = \int \log \left(\frac{d\mu_u}{d\hat{\mu}}\right) d\mu_u = \frac{\beta}{2}\, \mathbf{E}_{\mu_u} \left(\int_0^\tau |u_s - \hat{u}_s|^2 ds\right) \le \frac{\beta}{2}\epsilon^2 \mathbf{E}_{\mu_u}(\tau).$$

On the other hand, by Jensen's inequality,

$$\log \mathbf{E}_{\hat{\mu}}(e^{\gamma\tau}) \ge \gamma \mathbf{E}_{\mu_u}(\tau) - I(\mu_u \,|\, \hat{u}).$$

The conclusion follows from the last two inequalities. $\square$

Now we are ready to prove Theorem 3, which is restate here more precisely.

Theorem 11. *Let Assumption 1,2 and 3 from Section 2.1 hold. Further suppose that $\epsilon < (\gamma/\beta)^{1/2}$ and $|u_s - \hat{u}_s| \le \epsilon$, for all $s > 0$. Then it holds that*

$$J(u) = J(\hat{u}) + \beta^{-1} I(\mu_u|\, \hat{\mu}) \le J(\hat{u}) + C_3\epsilon^2. \tag{67}$$

Proof.

$$J(u) = \mathbf{E}_{\hat{\mu}} \left\{ \Big[\int_0^\tau \Big(G(z_s) + \frac{1}{2}|u_s|^2\Big)\, ds\Big] \frac{d\mu_u}{d\hat{\mu}} \right\}$$

It follows from (66) that we can write the above as

$$J(u) = J(\hat{u}) + \mathbf{E}_{\hat{\mu}}\Big[\Big(\beta^{-1} \log \frac{d\mu}{d\hat{\mu}} + \int_0^\tau \frac{1}{2}|u_s|^2 \, ds\Big) \frac{d\mu_u}{d\hat{\mu}}\Big] \tag{68}$$

Combining this with (63), we get

$$J(u) = J(\hat{u}) + \beta^{-1} I(\mu_u|\, \hat{\mu})\,.$$

The conclusion now readily follows from Lemma 10. $\square$

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